scraps3

Maple tells us that

From this equation we can compute f_B

Question: What is a' for this Particle at point B? It is at rest in Frame S', so a' = 0.

STOP RIGHT HERE. Particle at B is at rest in Frame S', so it must be true that

$$-GmM_1/(r_{12}+R_2)^2 - GmM_2/R_2^2 + m\omega^2(R+R_2) = 0 \qquad (3)$$

Is this consistent with (1) and (2) above? Set masses equal so that

$$GM_1/r_{12}^2 = \omega^2 R \qquad (1)$$
$$R = r_{12}/2 \qquad (2)$$
$$-GmM_1/(r_{12}+R_2)^2 - GmM_1/R_2^2 + m\omega^2 R(1+R_2/R) = 0 \qquad (3)$$

In last line use (1) to get

$$-GmM_1/(r_{12}+R_2)^2 - GmM_1/R_2^2 + m\, GM_1/r_{12}^2\,(1+R_2/R) = 0 \qquad (3)$$

Cancel GmM_1 to get

$$-1/(r_{12}+R_2)^2 - 1/R_2^2 + 1/r_{12}^2\,(1+R_2/R) = 0 \qquad (3)$$

This cannot be true for the following reason: there are four terms, three depend on R_2 and one does not. If you were to increase the density of material in M^2_2 such that R_2 gets smaller, but M_2 stays the same, then r_{12} would not change, and R would not change. Thus, three terms change in value and the fourth does not.

If you don't buy that argument, just have Maple compute the LHS for Earth Moon:

$G M_1$ LHS = -.12 $\neq 0$

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nothing else matters until the above is cleared up.

$$\mathbf{F'}_{\mathbf{eff}}(A) = [\, -GmM_1/(r_{12}-R_2)^2 + GmM_2/R_2^2 + m\omega^2(R-R_2)]\,\hat{\mathbf{r}}$$

We can combine these where upper sign refers to location B

$$\mathbf{F'_{eff}} = [\, -GmM_1/(r_{12} \pm R_2)^2 \mp GmM_2/R_2^2 + m\omega^2(R \pm R_2)]\, \hat{\mathbf{r}}$$

or

$$\mathbf{F'_{eff}} = [\, -GmM_1/(r_{12} \pm R_2)^2 \mp GmM_2/R_2^2 + m\omega^2 R(1 \pm R_2/R)]\, \hat{\mathbf{r}}$$

Now use (1) in the last term to get

$$\mathbf{F'_{eff}} = [\, -GmM_1/(r_{12} \pm R_2)^2 \mp GmM_2/R_2^2 + m\, GM_1/r_{12}^2\, (1 \pm R_2/R)]\, \hat{\mathbf{r}}$$

$$= (GmM_1/r_{12}^2)\, [-\, r_{12}^2/(r_{12} \pm R_2)^2 \mp (M_2/M_1)(\, r_{12}^2/R_2^2) + (1 \pm R_2/R)]\, \hat{\mathbf{r}}$$

Now we make our first approximation, which is that $R_2 << r_{12}$. This would be valid for the two Earth system mentioned in the Example above. Then

$$r_{12}^2/(r_{12} \pm R_2)^2 = 1/(1 \pm [R_2/r_{12}])^2 = (1 \pm [R_2/r_{12}])^{-2} \approx 1 \pm [R_2/r_{12}]\, (-2) = 1 \mp 2\, (R_2/r_{12})$$

Then we get

$$\mathbf{F'_{eff}} = (GmM_1/r_{12}^2)\, [-1 \pm 2\, (R_2/r_{12}) \mp (M_2/M_1)(\, r_{12}^2/R_2^2) + (1 \pm R_2/R)]\, \hat{\mathbf{r}}$$

The 1's cancel and we then have

$$\mathbf{F'_{eff}} = (GmM_1/r_{12}^2)\, [\, \pm 2\, (R_2/r_{12}) \mp (M_2/M_1)(\, r_{12}^2/R_2^2) \pm R_2/R)]\, \hat{\mathbf{r}}$$

$$= \pm\, (GmM_1/r_{12}^2)\, [2\, (R_2/r_{12}) - (M_2/M_1)(\, r_{12}^2/R_2^2) + R_2/R]\, \hat{\mathbf{r}}$$

At this point, use (2) which says $R = r_{12} * [M_1/(M_1 + M_2)]$ or

$$(1/R) = (1/r_{12})[\, (M_1 + M_2)/M_1] \quad \Rightarrow \quad (R_2/R) = (R_2/r_{12})[\, (M_1 + M_2)/M_1]$$

so then

$$\mathbf{F'_{eff}} = \pm\, (GmM_1/r_{12}^2)\, [2\, (R_2/r_{12}) - (M_2/M_1)(\, r_{12}^2/R_2^2) + (R_2/r_{12})[\, (M_1 + M_2)/M_1]]\, \hat{\mathbf{r}}$$

We can now combine the first and last terms

$$(R_2/r_{12})\, \{\, 2 + (M_1 + M_2)/M_1\, \} = (R_2/r_{12})\, \{\, 2M_1 + (M_1 + M_2)\, \}/M_1$$

$$= (R_2/r_{12})\, (3M_1 + M_2)/M_1$$

and we then get our "final answer"

$$\mathbf{F'_{eff}} = \pm (GmM_1 / r_{12}^2) [(R_2/r_{12}) (3M_1+M_2)/M_1 - (M_2/M_1)(r_{12}^2/R_2^2)] \hat{\mathbf{r}}$$

Now let's apply this to a two-moon system, say, so $M_2 = M_1$ and we get

$$\mathbf{F'_{eff}} = \pm (GmM_1 / r_{12}^2) [(R_2/r_{12}) 4 - (r_{12}^2/R_2^2)] \hat{\mathbf{r}}$$

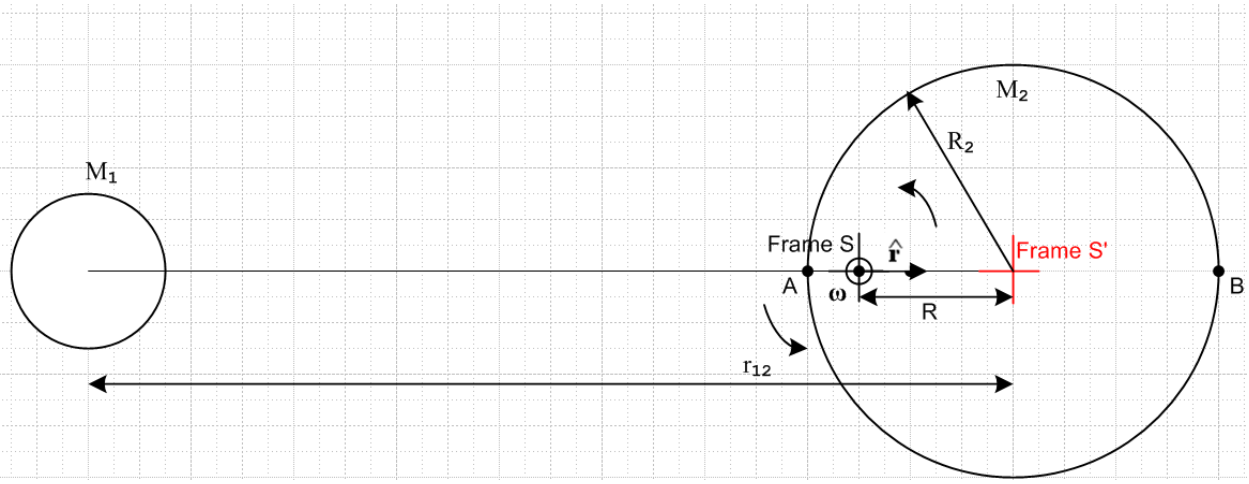
For such a system, the second term wins by many miles, so we have

$$\mathbf{F'_{eff}} = \pm (GmM_1 / r_{12}^2) [- (r_{12}^2/R_2^2)] \hat{\mathbf{r}} = \mp (GmM_1 / R_2^2) \hat{\mathbf{r}}$$

[But

What this says is that, when all is said and done, it is as if the other Earth were not even present. The two terms are just the gravity contributions for the isolated Earth on the right. A water drop at either A or B would be squashed into the surface and we would NOT get tidal bulges!

11. Now back to our picture shown in Fig 2. We will treat Fig 2 separately below.



In our calculations, is anything different now? Equations (1) and (2) are still the same. What about the forces at A and B in Frame S' ?

$$\mathbf{F'_{eff}}(B) = [-GmM_1/(r_{12}+R_2)^2 - GmM_2/R_2^2 + m\omega^2(R+R_2)] \hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}}(A) = [-GmM_1/(r_{12}-R_2)^2 + GmM_2/R_2^2 + m\omega^2(R-R_2)] \hat{\mathbf{r}}$$

At point A, the centrifugal force is $-m\omega^2(R_2-R)$ pointing to the left, but the equations from the last section are then exactly the same and are as quoted above. Everything is the same (no surprise really). The approximation $R_2 << r_{12}$ is still valid, and we get the same "final answer"

$$\mathbf{F'_{eff}} = \pm (GmM_1/r_{12}^2)\,[(R_2/r_{12})\,(3M_1+M_2)/M_1 - (M_2/M_1)(r_{12}^2/R_2^2)]\,\hat{\mathbf{r}}$$

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OK, so both pictures give the same equations. Let's go back then to this earlier point

$$\mathbf{F'_{eff}} = [\,-GmM_1/(r_{12}\pm R_2)^2 \mp GmM_2/R_2^2 + m\,GM_1/r_{12}^2\,(1\pm R_2/R)]\,\hat{\mathbf{r}}$$

and let's separate out the obvious local gravity term which is in the middle above

$$\mathbf{F'_{eff}} = [\,-GmM_1/(r_{12}\pm R_2)^2 + m\,GM_1/r_{12}^2\,(1\pm R_2/R)]\,\hat{\mathbf{r}} \mp GmM_2/R_2^2\hat{\mathbf{r}}$$

This dominant term is never going to go away, unlike what happened in the tether problem. Now do the approximation step as above to get

$$\mathbf{F'_{eff}} = (GmM_1/r_{12}^2)\,[-1 \pm 2\,(R_2/r_{12}) + (1\pm R_2/R)]\,\hat{\mathbf{r}} \mp GmM_2/R_2^2\hat{\mathbf{r}}$$

The ones cancel and we get

$$\mathbf{F'_{eff}} = (GmM_1/r_{12}^2)\,[\pm 2\,(R_2/r_{12}) \pm R_2/R)]\,\hat{\mathbf{r}} \mp GmM_2/R_2^2\hat{\mathbf{r}}$$

$$= \pm (GmM_1/r_{12}^2)\,[2\,(R_2/r_{12}) + R_2/R)]\,\hat{\mathbf{r}} \mp GmM_2/R_2^2\hat{\mathbf{r}}$$

Now write this as

$$\mathbf{F'_{eff}} = \mathbf{F_{tidal}} + \mathbf{F_{grav}}$$

so we regard the tidal force as the extra force in addition to the local gravitational force

$$\mathbf{F_{tidal}} \pm (GmM_1/r_{12}^2)\,[2\,(R_2/r_{12}) + R_2/R)]\,\hat{\mathbf{r}}$$

For a Particle at point B, this is an extra force to the right.
For a Particle at point A, this is an extra force to the left.

What happens at points C and D? First, the $\hat{\mathbf{r}}$ component of the local gravity force vanishes. Second, the remaining terms are about the same, assuming $R_2 << r_{12}$, but these terms are now tangent to the surface and this are not tidal forces.

So the upshot is that there will be bulges on the expected surfaces, and water will run to these bulges from the band of surface centered at the equator.

Size of the tide? Let's try some numbers. For the earth-moon situation we can keep only the second term to get

$$\mathbf{F}_{tidal}/m \approx \pm (GM_1/r_{12}^2)\,[\,(R_2/R)]\,\hat{\mathbf{r}}$$

Maple says this is

.0000454 Nt/kg or m/sec^2

Now assume that Mass M_2 is an all-water world and R_2 is the radius in the absence of M_1.

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1. $[\dot{\mathbf{a}}_s]_j = (\dot{a}_j)_s$

2. $[\dot{\mathbf{a}}_{s'}]_j = (\dot{a}_j)_{s'} + (a)_j(\dot{e}_i)_{s'}$

3. $[(d\mathbf{a}'/dt)_{s'}]_j = (d(a')_i/dt)_{s'}\, R_{ij}$

4. $[(d\mathbf{a}'/dt)_s]_j = (d(a')_i/dt)_s\, R_{ij} + \varepsilon_{jab}\omega_a R_{ib}(a')_i$ (1.38)

1. $[(d\mathbf{a}/dt)_s]_j = (d(a)_j/dt)_s$ // $[(d[a_i e_i]/dt)_s]_j$

2. $[(d\mathbf{a}/dt)_{s'}]_j = (d(a)_j/dt)_{s'} + (a)_j(de_i/dt)_{s'}$

3. $[(d\mathbf{a}'/dt)_{s'}]_j = (d(a')_i/dt)_{s'}\, R_{ij}$

4. $[(d\mathbf{a}'/dt)_s]_j = (d(a')_i/dt)_s\, R_{ij} + \varepsilon_{jab}\omega_a R_{ib}(a')_i$ (1.38)

Footnote: Here are details of the above four calculations which the reader is invited to ignore :

1. $(d\mathbf{a}/dt)_s = (d\,[(a)_i e_i]/dt)_s = \dot{a}_i\, e_i + (a)_i\,(de_i/dt)_s = \dot{a}_i\, e_i$

   $\Rightarrow [(d\mathbf{a}/dt)_s]_j = \{\dot{a}_i\, e_i\} \bullet e_j = \dot{a}_i\, \delta_{i,j} = \dot{a}_j = (\dot{a}_j)_s$

2. $(d\mathbf{a}/dt)_{s'} = (d\,[(a)_i e_i]/dt)_{s'} = (d(a)_i/dt)_{s'}\, e_i + (a)_i\,(de_i/dt)_{s'} = (d(a)_i/dt)\, e_i - (a)_i\, \omega \times e_i$

In the above we used (1.27) and (1.28).

Now since $(\mathbf{e}_i)_j = \delta_{i,j}$ and $[\boldsymbol{\omega} \times \mathbf{e}_i]_j = \varepsilon_{jab}\omega_a(e_i)_b = \varepsilon_{jab}\omega_a\delta_{i,b} = \varepsilon_{jai}\omega_a$, this last line is

$$[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_j/dt) - \varepsilon_{jai}\,\omega_a(a)_i \ .$$

For the other two derivatives we do copy paste and edit on the above:

3. $(d\mathbf{a}'/dt)_{S'} = (d\,[\,(a')_i\mathbf{e}'_i]/dt)_{S'} = (d(a')_i/dt)_{S'}\,\mathbf{e}'_i + (a')_i\,(d\mathbf{e}'_i/dt)_{S'} = (d(a')_i/dt)\,\mathbf{e}'_i$

$$\Rightarrow [(d\mathbf{a}'/dt)_{S'}]_j = (d(a')_i/dt)\,\mathbf{e}'_i \bullet \mathbf{e}_j = (d(a')_i/dt)\,R_{ij}$$

4. $(d\mathbf{a}'/dt)_S = (d\,[\,(a')_i\mathbf{e}'_i]/dt)_S = (d(a')_i/dt)_S\,\mathbf{e}'_i + (a')_i\,(d\mathbf{e}'_i/dt)_S = (d(a')_i/dt)\,\mathbf{e}'_i + (a')_i\,\boldsymbol{\omega} \times \mathbf{e}'_i$

In the above we used (1.26) and (1.7) and (1.25).

Now since $(\mathbf{e}'_i)_j = R_{ij}$ (see 1.7) and $[\boldsymbol{\omega} \times \mathbf{e}'_i]_j = \varepsilon_{jab}\omega_a(e'_i)_b = \varepsilon_{jab}\omega_a R_{ib}$ this last line is

$$[(d\mathbf{a}'/dt)_S]_j = (d(a')_i/dt)\,R_{ij} + \varepsilon_{jab}\omega_a R_{ib}(a')_i \ .$$

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Lemma 1: For any vector $\mathbf{v}$,

$$[(d\mathbf{v}/dt)_S]_k = (dv_k/dt). \qquad 1a$$
$$[(d\mathbf{v}/dt)_{S'}]'_k = (d(v)'_k/dt) \qquad 1b \qquad (2.3)$$

The operations "compute d/dt" and "take a component" commute as in Section 1 (i). For example, to prove 1b, just insert the expansion $\mathbf{v} = (v)'_k\mathbf{e}'_k$

Lemma 2: For any vector $\mathbf{v}$ and scalar u,

$$(d[u\mathbf{v}]/dt)_S = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_S \qquad 2a$$
$$(d[u\mathbf{v}]/dt)_{S'} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{S'} \qquad 2b \qquad (2.4)$$

This is Leibnitz Rule when one object is a vector and the other a scalar. Applying the above to a sum,

$$(d[u_i\mathbf{v}^{(i)}]/dt)_S = (du_i/dt)\,\mathbf{v}^{(i)} + u_i\,(d\mathbf{v}^{(i)}/dt)_S \qquad a$$
$$(d[u_i\mathbf{v}^{(i)}]/dt)_{S'} = (du_i/dt)\,\mathbf{v}^{(i)} + u_i\,(d\mathbf{v}^{(i)}/dt)_{S'} \qquad b \qquad (2.5)$$

Lemma 3:

$$(d[(a)'_i\mathbf{e}'_i]/dt)_S = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_S \qquad 3a$$
$$(d[(a)'_i\mathbf{e}'_i]/dt)_{S'} = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_{S'} \qquad 3b \qquad (2.6)$$

Lemma 3:

$$(d[(a)'_i\mathbf{e}'_i]/dt)_S = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_S \qquad 3a$$
$$(d[(a)'_i\mathbf{e}'_i]/dt)_{S'} = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_{S'} \qquad 3b \qquad (2.6)$$

Before proving the G Rule, we start off with a few miniscule Lemmas to make sure we are not running off the track somewhere. These items are usually glossed over as being totally obvious.

Lemma 1: For any vector **w**, $[(d\mathbf{w}/dt)_S]_k = (dw_k/dt)$. 1a

$[(d\mathbf{w}/dt)_{S'}]'_k = (d(w)'_k/dt)$ 1b (2.3)

proof: These two equations are examples of the Commutation Theorem of Section 1 (j). For each equation, the frame of the time derivative matches the frame of the components, so the two operations "take the time derivative" and "take the k^{th} component" commute. Yes, the results are obvious. If one parks oneself in some Cartesian frame, one knows that Lemma 1 is true because the Cartesian unit vectors are constants.

Lemma 2: For any vector **v** and scalar u, $(d[u\mathbf{v}]/dt)_S = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_S$ 2a

$(d[u\mathbf{v}]/dt)_{S'} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{S'}$ 2b (2.4)

Again, if we just park ourselves in some particular Cartesian frame and work there and delete the frame subscripts, we *know* this Lemma 2 is true as a "Leibnitz product rule" where one object is a vector and the other is a scalar. But let's prove the claims anyway to exercise the notation:

proof of claim 2a: Start with

$[(d[u\mathbf{v}]/dt)_S]_k = (d[uv_k]/dt)$ // Lemma 1a applied to vector $\mathbf{w} = u\mathbf{v}$

$= (du/dt)v_k + u\,(dv_k/dt)$ // normal Leibnitz product rule for scalar functions

$= (du/dt)v_k + u\,[(d\mathbf{v}/dt)_S]_k$ // Lemma 1a applied to $\mathbf{w} = \mathbf{v}$ in second term

Take both sides times $\mathbf{e}_k$ and sum over k to get

$(d[u\mathbf{v}]/dt)_S = (du/dt)\mathbf{v} + u\,(d\mathbf{v}/dt)_S$ QED Lemma 2a

proof of claim 2b: Start with

$[(d[u\mathbf{v}]/dt)_{S'}]'_k = (d[u(v)'_k]/dt)$ // Lemma 1b applied to vector $\mathbf{w} = u\mathbf{v}$

$= (du/dt)\,(v)'_k + u\,(d(v)'_k/dt)$ // normal Leibnitz product rule for scalar functions

$= (du/dt)\,(v)'_k + u\,[(d\mathbf{v}/dt)_{S'}]'_k$ // Lemma 1b applied to $\mathbf{w} = \mathbf{v}$ in second term

Multiply both sides times $\mathbf{e}'_k$ and sum over k to get (for example, $\mathbf{v} = (v)'_k\,\mathbf{e}'_k$)

$(d[u\mathbf{v}]/dt)_{S'} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{S'}$ QED Lemma 2b

It seems clear that Lemma 2 is true for any linear combination of vectors $\mathbf{v}^{(i)}$ having some label (i),

$$(d[u_i \mathbf{v}^{(i)}]/dt)_S = (du_i/dt)\,\mathbf{v}^{(i)} + u_i\,(d\mathbf{v}^{(i)}/dt)_S \qquad a$$
$$(d[u_i \mathbf{v}^{(i)}]/dt)_{S'} = (du_i/dt)\,\mathbf{v}^{(i)} + u_i\,(d\mathbf{v}^{(i)}/dt)_{S'} \qquad b \qquad (2.5)$$

In particular, we could apply the above line to the case $u_i = (a)'_i$ and $\mathbf{v}^{(i)} = \mathbf{e}'_i$. The $(a)'_i$ are just a set of scalar functions and the $\mathbf{e}'_i$ are just a set of labeled vectors. Thus, we have proven

Lemma 3: $(d[(a)'_i \mathbf{e}'_i]/dt)_S = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_S$ 3a
$(d[(a)'_i \mathbf{e}'_i]/dt)_{S'} = (d(a)'_i/dt)\,\mathbf{e}'_i + (a)'_i\,(d\mathbf{e}'_i/dt)_{S'}$ 3b (2.6)

Lemma 3 is the basis of the following proof.

Proof of the G Rule

We shall now derive the G Rule based on Hunt[11].

Expand the vector **a** in Frame S' components :

$$\mathbf{a} = a'_i\,\mathbf{e}'_i \quad . \qquad (1.5)$$

Differentiate **a** in Frame S

$(d\mathbf{a}/dt)_S = (da'_i/dt)\,\mathbf{e}'_i + a'_i(d\mathbf{e}'_i/dt)_S$ // Lemma 3a

$= (da'_i/dt)\,\mathbf{e}'_i + a'_i\,\boldsymbol{\omega} \times \mathbf{e}'_i$ // using (1.25) which says $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$

$= (da'_i/dt)\,\mathbf{e}'_i + \boldsymbol{\omega} \times (a'_i\,\mathbf{e}'_i)$ // move in scalar factor a'_i

$= (da'_i/dt)\,\mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}$ // recognize expansion of **a**

so we have shown that

$$(d\mathbf{a}/dt)_S = (da'_i/dt)\,\mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a} \quad . \qquad (2.7)$$

Now go back to the start

$$\mathbf{a} = a'_i\,\mathbf{e}'_i \qquad (1.5)$$

and differentiate this time in Frame S' to get,

$(d\mathbf{a}/dt)_{S'} = (da'_i/dt)_{S'}\,\mathbf{e}'_i + a'_i(d\mathbf{e}'_i/dt)_{S'}$ // Lemma 3b

$= (da'_i/dt)_{S'}\,\mathbf{e}'_i + a'_i * 0$ // using (1.26) because the $\mathbf{e}'_i$ are frozen in S'

$$= (da'_i/dt)_{S'}\ \mathbf{e}'_i \ . \qquad (2.8)$$

We have then shown that

$$(d\mathbf{a}/dt)_S = (da'_i/dt)\ \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a} \qquad (2.7)$$
$$(d\mathbf{a}/dt)_{S'} = (da'_i/dt)\ \mathbf{e}'_i \ . \qquad (2.8)$$

Insert the second line into the first and the G Rule has been derived,

$$(d\mathbf{a}/dt)_S = (d\mathbf{a}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{a} \quad . \qquad (2.1)$$

This rule relates the time derivatives of a vector taken in the two frames. If vector **a** is static in Frame S', then all of the Frame S time derivative comes from the (1.23) conical rotation term $\boldsymbol{\omega} \times \mathbf{a}$. If there is no frame rotation ($\boldsymbol{\omega} = 0$), the two time derivatives are equal.

Next, write this out in components

$$[(dA/dt)_S]_{abc} = [(dA_{abc}/dt)_S] = (dA'_{ijk}/dt)\ (\mathbf{e}'_i)_a(\mathbf{e}'_j)_b(\mathbf{e}'_k)_c \qquad (C.10)$$

$$+ A'_{ijk}\,[\boldsymbol{\omega} \times \mathbf{e}'_i]_a(\mathbf{e}'_j)_b(\mathbf{e}'_k)_c + A'_{ijk}\,(\mathbf{e}'_i)_a\,[\boldsymbol{\omega} \times \mathbf{e}'_j]_b\,(\mathbf{e}'_k)_c + A'_{ijk}\,(\mathbf{e}'_i)_a\,(\mathbf{e}'_j)_b\,[\boldsymbol{\omega} \times \mathbf{e}'_k]_c$$