Note that page numbering is turned on in this template.

**(d) Comparison with Marion (2nd Ed 1970) and Thornton & Marion (5th Ed 2004)**

(*Classical Dynamics of Particles and Systems*) ( Jerry B. Marion died in 1981)

Marion uses the a hybrid notation (a combination of subscripts and primes) which has S↔S' relative to our world (so **r**↔**r'**). He then refers to *his* frame S' as the fixed frame (subscript f), and *his* frame S as the rotating frame (subscript r). He also uses **R** in place of our **b**. The translation from us to him is then:

$$\mathbf{v'_{S'}} \rightarrow \mathbf{v_r} \qquad \mathbf{v_S} \rightarrow \mathbf{v_f} \qquad \mathbf{r} \leftrightarrow \mathbf{r'} \qquad \mathbf{b} \rightarrow \mathbf{R} \qquad \dot{\mathbf{b}}_S \rightarrow \dot{\mathbf{R}}_f = (d\mathbf{R}/dt)_{fixed} \equiv \mathbf{V}$$

and then our (6.6a) becomes

$$\mathbf{v_S} = \mathbf{v'_{S'}} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S \qquad (6.6a)$$

$$\mathbf{v_f} = \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}$$

// Marion 1970 p 343 (11.12) [ with (11.11) , picture p 341]
// Marion & Thornton 2004 p 391 (10.17) [ with (10.16) , picture p 388]

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**(g) Comparison with Goldstein (1950) and Goldstein, Poole and Safko (3rd Ed 2001)**

(*Classical Mechanics*) ( Herbert Goldstein died in 2005)

In the Earth application, one prefers to measure velocities based on **r'** rather than **r** (which goes all the way to the center of the earth!). The two velocities of interest are then

$$\mathbf{v'_{S'}} = (d\mathbf{r'}/dt)_{S'} \qquad // = \text{Goldstein } \mathbf{v_r} \text{ (velocity in rotating frame = body frame)}$$

$$\mathbf{v'_S} = (d\mathbf{r'}/dt)_S \qquad // = \text{Goldstein } \mathbf{v_s} \text{ (velocity in space or fixed frame)} \qquad (6.9)$$

These are related by (6.8a)

$$\mathbf{v'_S} = \mathbf{v'_{S'}} + \boldsymbol{\omega} \times \mathbf{r'} \qquad (6.8a)$$

Like Marion, Goldstein also has S↔S' relative to our world (so **r**↔**r'**), but there is no drawing. The translation from us to Goldstein is this

$$\mathbf{v'_{S'}} \rightarrow \mathbf{v_r} \qquad \mathbf{v'_S} \rightarrow \mathbf{v_s} \qquad \mathbf{r} \leftrightarrow \mathbf{r'}$$

So that (6.8a) above becomes

$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$ // Goldstein 1970 p 135 (4-104)
// Goldstein, Poole and Safko 2001 p 175 (4.88)

Goldstein's only hint as to the meaning of his **r** is this

> The initial step is to apply Eq. (4.86) to the radius vector, **r**, from the origin of the terrestrial system to the given particle:

and this terrestrial system is our rotating Frame S' in which *our* Particle position vector is **r'**.

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**(d) Continuation of comparison with Goldstein and Marion ( from Section 6 (g))**

As noted earlier, in the Earth application, one prefers to measure velocities based on **r'** rather than **r** (which goes all the way to the center of the earth!). The two velocities of interest are then

$\mathbf{v}'_{S'} = (d\mathbf{r}'/dt)_{S'}$ // = Goldstein $\mathbf{v}_r$ (velocity in **r**otating frame = body frame)
$\mathbf{v}'_S = (d\mathbf{r}'/dt)_S$ // = Goldstein $\mathbf{v}_s$ (velocity in **s**pace or fixed frame) *(6.9)*

and we now add the corresponding accelerations of interest

$\mathbf{a}'_{S'} = (d\mathbf{v}'_{S'}/dt)_{S'} = (d^2\mathbf{r}'/dt^2)_{S'}$ // = Goldstein $a_r$ (acceleration in **r**otating frame = body frame)
$\mathbf{a}'_S = (d\mathbf{v}'_S/dt)_S = (d^2\mathbf{r}'/dt^2)_S$ // = Goldstein $a_s$ (acceleration in **s**pace or fixed frame)

Like Marion, Goldstein also has S↔S' relative to our world (so **r**↔**r'**), but there is no drawing. The translation from us to Goldstein is this where we now add the second line,

$\mathbf{v}'_{S'} \rightarrow \mathbf{v}_r$ $\mathbf{v}'_S \rightarrow \mathbf{v}_s$ $\mathbf{r} \leftrightarrow \mathbf{r}'$
$\mathbf{a}'_{S'} \rightarrow \mathbf{a}_r$ $\mathbf{a}'_S \rightarrow a_s$

Consider then our equation (7.5) above

$$\mathbf{a}'_S = \mathbf{a}'_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}'_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.5)$$

becomes under translation to Goldstein notation

$$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (7.6)$$

// Goldstein, Poole and Safko 2001 p175 (4.89) with $\dot{\boldsymbol{\omega}}=0$

Solving this for $\mathbf{a}_r$ and multiplying by mass m of the Particle gives

$$m\mathbf{a}_r = m\mathbf{a}_s - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (7.7)$$

If the space frame is an inertial frame, Newton's Law $\mathbf{F} = m\mathbf{a}_s$ is applicable there. Then define $\mathbf{F}_{eff} \equiv m\mathbf{a}_r$ as an "effective Newton's Law" in the rotating frame. Then(7.7) becomes,

$$\mathbf{F} = m\mathbf{a}_s \quad \text{and} \quad \mathbf{F}_{eff} \equiv m\mathbf{a}_r \tag{7.8}$$

$$\mathbf{F}_{eff} = m\mathbf{a}_r = m\mathbf{a}_s - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.9}$$

// Marion 1970 p 344 (11.19)

$$\mathbf{F}_{eff} = m\mathbf{a}_r = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.10}$$

// Goldstein (1970) p 135 (4-106) and (4-107) with $\dot{\boldsymbol{\omega}}=0$.

// Goldstein, Poole and Safko 2001 p175 (4.90) and (4.91) with $\dot{\boldsymbol{\omega}}=0$.

$$\mathbf{F}_{fict} \equiv - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.11}$$

$$\mathbf{F}_{eff} = \mathbf{F} + \mathbf{F}_{fict} \tag{7.12}$$

Solving (7.10) for $\mathbf{F}$ gives

$$\mathbf{F} = m\mathbf{a}_s = m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{7.13}$$

// Marion 1970 p 344 (11.18)

We can now connect the above equations to those in Goldstein and Marion

Equation (7.6) = Goldstein, Poole and Safko 2001 p175 (4.89) with $\dot{\boldsymbol{\omega}}=0$

Equation (7.10) = Goldstein (1970) p 135 (4-106) and (4-107) with $\dot{\boldsymbol{\omega}}=0$.
Equation (7.10) = Goldstein, Poole and Safko 2001 p175 (4.90) and (4.91) with $\dot{\boldsymbol{\omega}}=0$.

Equation (7.9) = Marion 1970 p 344 (11.19)
Equation (7.13) = Marion 1970 p 344 (11.18)

// Marion & Thornton 2004 p 391 (10.17) [ with (10.16) , picture p 388]

$$m\mathbf{a}_r = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

// Goldstein (1970) p 135 (4-106) with $\dot{\boldsymbol{\omega}}$=0

// Goldstein, Poole and Safko 2001 p 175 (4.90) with $\dot{\boldsymbol{\omega}}$=0

If we want to have a "private Newton's Law" in the rotating frame, we can do so as long as we add to any real forces **F** the fictitious forces shown:

$$\mathbf{F}_{fict} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

With the simple of subscript s (Goldstein space) to f (Marion fixed), the above become

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**(d) Fictitious Forces**

We can now restate all this back in our notation. Start from (7.5),

$$\mathbf{a}'_s = \mathbf{a}'_{s'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}'_{s'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.5)$$

Solve for $\mathbf{a}'_{s'}$

$$\mathbf{a}'_{s'} = \mathbf{a}'_s - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}'_{s'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.6)$$

$$m\mathbf{a}'_{s'} = m\mathbf{a}'_s - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}'_{s'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

$$= \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}'_{s'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.7)$$

$$\mathbf{F}_{fict} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}'_{s'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.8)$$

continue here, and try to interpret the "centripetal force" term!!

**(e) Continuation of Comparison with Marion (from Section ...)**

To get an alternate form for **a**, we can solve (4.5) for $\dot{\mathbf{v}}_1$

$$\dot{\mathbf{v}}_1 = -\dot{\boldsymbol{\omega}} \times \mathbf{b} - \boldsymbol{\omega} \times \mathbf{v}_1 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) + \ddot{\mathbf{b}} \qquad (7.6)$$

and then use this to replace $\dot{\mathbf{v}}_1$ in (7.5),

$$\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathbf{v}_1 + [-\dot{\boldsymbol{\omega}} \times \mathbf{b} - \boldsymbol{\omega} \times \mathbf{v}_1 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) + \ddot{\mathbf{b}}]$$

**or**

$$\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathbf{v}_1 - \dot{\boldsymbol{\omega}} \times \mathbf{b} - \boldsymbol{\omega} \times \mathbf{v}_1 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) + \ddot{\mathbf{b}}$$

**or**

$$\mathbf{a} = \mathbf{a}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}} \qquad (7.7)$$

where $\boldsymbol{\omega} \times \mathbf{v}_1$ cancelled and then two pairs of terms were combined using $\mathbf{r}' = \mathbf{r} - \mathbf{b}$. If Frame S is an inertial frame so that Newton's Law there reads $\mathbf{F} = m\mathbf{a}$, multiplication of (7.7) by mass m gives

$$\mathbf{F} = m\mathbf{a}' + 2\,m\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + m\ddot{\mathbf{b}} \qquad (7.8)$$

Comment: The earth has $\omega = 7.29 \times 10^{-5}$ sec$^{-1}$ and $r = 6.4 \times 10^{6}$ m. At the equator which is the location of the maximum of $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ one has $|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = \omega^2 r = .034$ m sec$^{-2}$. For an object going 5 m/sec, the second term is at most $7.29 \times 10^{-4}$ m sec$^{-2}$.

Comment: In his notation, Marion (2nd Ed. 1970) states equation (7.8) above as equation (11.17) on page 344 of his book. Recall that he uses S↔S' and $\mathbf{b} = \mathbf{R}$ and f = fixed and r=rotating, so his version of (7.8) reads

$$\mathbf{F} = m\mathbf{a}_r + 2\,m\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + m\ddot{\mathbf{R}}$$

or

$$\mathbf{F} = m\ddot{\mathbf{R}} + m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,m\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion (11.17)}$$

except he sets $\dot{\boldsymbol{\omega}} = 0$. He then claims (erroneously I feel) that in the Earth application $\ddot{\mathbf{R}} = \ddot{\mathbf{b}}$ can be ignored because it is small relative to other terms, and he ends up with

$$\mathbf{F} = m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,m\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion (11.18)}$$

and then, since $\mathbf{F} = m\mathbf{a}_f$,

$$\mathbf{F}_{eff} \equiv m\mathbf{a}_r = m\mathbf{a}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\,m\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion (11.19)}$$

He then claims that this is the equation which shows the fictitious forces. But in his notation, $\mathbf{r}$ is the location of a Particle in the fixed frame which lies on the surface of the Earth, and therefore his $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is in fact not the centrifugal force as he claims in the paragraph following (11.19). His centrifugal force would in fact be $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ in his notation. The only possible resolution of this mystery would

be if the rotating frame were placed at the center of the Earth so that **b** = **R** = 0 and then **r** = **r'** , but this is not how Earth Coriolis problems are handled.

In Marion and Thornton (5th Ed 2004) this mystery remains. The frames are defined by this picture on page 388 (showing the S↔S' swap from our notation, including the **r**↔**r'** swap).

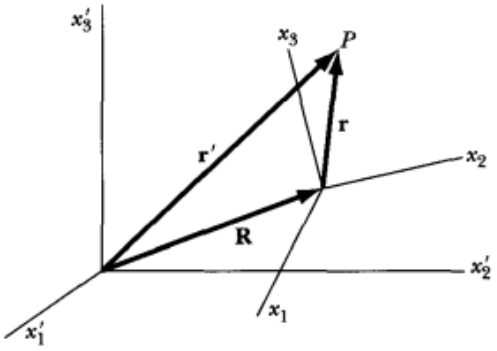



**FIGURE 10-1** The $x_i'$ are coordinates in the fixed system, and $x_i$ are coordinates in the rotating system. The vector **R** locates the origin of the rotating system in the fixed system.

and then we quote from page 392 (in this edition, the $\dot{\boldsymbol{\omega}}$ term is maintained)

$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r \quad \textbf{(10.23)}$$

To an observer in the rotating coordinate system, however, the effective force on a particle is given by*

$$\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r \quad \textbf{(10.24)}$$

$$= \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \quad \textbf{(10.25)}$$

The first term, **F**, is the sum of the forces acting on the particle as measured in the fixed inertial system. The second ($-m\ddot{\mathbf{R}}_f$) and third ($-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$) terms result because of the translational and angular acceleration, respectively, of the moving coordinate system relative to the fixed system.

The quantity $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the usual *centrifugal force* term and reduces to $m\omega^2 r$ for the case in which $\boldsymbol{\omega}$ is normal to the radius vector. Note that the

So the comment that $\ddot{\mathbf{R}}$ can be ignored no longer appears, but the claim that - m**ω** x **(ω** x **r)** is the centrifugal force remains.

(but in his notation where S↔S' and **b** → **R**). He then (I feel erroneously) argues that the $\ddot{\mathbf{b}}$ can be ignored in practice (relative to the other terms) because it is small and this leads to his equation (11.19) which is just (7.7) above with $\ddot{\mathbf{b}} = 0$. But **ω** x **(ω** x **r')** is not the centripetal acceleration because this is the wrong **r**. Goldstein gets it right in his (4-107) p 135.

It is possible that Marion has both frames with a common origin so that then **b** = 0 and **r** = **r'** , but that is not what the figure on page 341 suggests!

I see that my Marion book is nowadays called "Marion and Thornton", and the franchise is still going (5th edition 2003). I will download a copy just to see if they have repaired this problem.

Well, it is changed, but still not very clean. The comments about $\ddot{\mathbf{b}} \approx 0$ are gone, but the confusion exists until page 395 where **ω** = constant and $\dot{\mathbf{b}} = \boldsymbol{\omega} \times \mathbf{b}$ is actually used (and $\ddot{\mathbf{b}} = \boldsymbol{\omega} \times \dot{\mathbf{b}}$ ). Marion has died and Thornton kept the original section and then wrote his own about earth motions where repairs are made.

Comment: I think my Berkeley mechanics book has problems as well. It starts on page 84 suggesting that the rotating frame has the same origin as the static frame (which would mean **b** = 0 for me). In this case we have **r** = **r'** and I agree with their result (I would set $\ddot{\mathbf{b}} = 0$ ) . But then on page 86 they put their little rotating frame on the surface of the earth which does not have **b** = 0. It is an "advanced topic" in the book and is a bit of a hack job.

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**(**) Translation of Marion 1970 into our Notation**

We pick up the Marion discussion on page 342. Marion use the reverse convention to us for where the frame primes go, and a simple translation rule is this

S ↔ S' which includes: **r** ↔ **r'**

In addition, we provide this table of translations:

| Marion | Us |
|---|---|
| **R** | **b** |
| $\dot{\mathbf{R}}_f = \mathbf{V}$ | $\dot{\mathbf{b}}_S$ |
| $\ddot{\mathbf{R}}_f$ | $\ddot{\mathbf{b}}_S$ |
| **r' = R + r** | **r = b + r'** |
| $(d/dt)_{fixed}$ | $(d/dt)_S$ |
| $(d/dt)_{rotating}$ | $(d/dt)_{S'}$ |
| $\mathbf{v}_f$ | $\mathbf{v}_S$ |
| $\mathbf{a}_f$ | $\mathbf{a}_S$ |
| $\mathbf{v}_r$ | $\mathbf{v}'_{S'}$ |
| $\mathbf{a}_r$ | $\mathbf{a}'_{S'}$ |

With the S ↔ S' rule and the translation table above, we may now proceed to translate our certain of equations into Marion notation:

$\mathbf{v}_S = \mathbf{v}'_{S'} + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$ (6.6a)

$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}$ // Marion (11.12)

$\mathbf{F} = m\mathbf{a}_s$

$\mathbf{F} = m\mathbf{a}_f$ // Marion (11.13)

$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\,\boldsymbol{\omega} \times \mathbf{v}_f$ // Marion (11.14)

$\mathbf{a}_f = \ddot{\mathbf{R}}_f + \mathbf{a}_r + \boldsymbol{\omega} \times \mathbf{v}_f$

$\mathbf{a}_S = \ddot{\mathbf{b}}_S + \mathbf{a}'_{S'} + \boldsymbol{\omega} \times \mathbf{v}_S$

$(d\mathbf{v}'_{S'}/dt)_S = \mathbf{a}'_{S'} + \boldsymbol{\omega} \times \mathbf{v}'_{S'}$ // G rule for vector $\mathbf{v}'_{S'}$

$(d\mathbf{v}_r/dt)_{fixed} = \mathbf{a}_r + \boldsymbol{\omega} \times \mathbf{v}_r$ // Marion (11.15)

$\boldsymbol{\omega} \times \mathbf{v}_f = \boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

**********

*(8.5)*

$\mathbf{F}_{fict} \approx -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}'_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ *(8.6)approx*

We first quote from above

$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ // Goldstein p 135 (4-105)
// GPS p 175 (4.89)

Using $\mathbf{F} \approx m\mathbf{a}_s$ this then says

$\mathbf{F} \approx m\mathbf{a}_s = m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

**********8

**Translations into notation of Goldstein (1970) and Goldstein, Poole and Safko ( GPS, 3rd Ed 2001))**

Goldstein also has S↔S' including **r**↔**r'** relative to us, and here is the translation table (us → Goldstein/GPS)

$$\mathbf{r}' \rightarrow \mathbf{r} \qquad \mathbf{v}_S \rightarrow \mathbf{v}_s \qquad \mathbf{v'}_{S'} \rightarrow \mathbf{v}_r$$
$$\mathbf{a}_S \rightarrow \mathbf{a}_s \qquad \mathbf{a'}_{S'} \rightarrow \mathbf{a}_r \tag{8.8}$$

So things are the same as with Marion, except Goldstein uses subscript s for space instead of s for fixed. Goldstein says on p 135 " $\mathbf{v}_s$ and $\mathbf{v}_r$ are the velocities of the Particle relative to the space and rotating axes respectively."

Consider our velocity result (6.6a) and its translation into Goldstein notation

$$\mathbf{v}_S = \mathbf{v'}_{S'} + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \tag{6.6a}$$
$$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_S$$

But here is what we see in Goldstein's books

$$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$

// Goldstein p 135 (4-104)
// GPS p 175 (4.88)

He makes no comments about any approximations, but he must be assuming at this point that $\dot{\mathbf{b}}_S$ is small relative to the other terms in the equation so it can be neglected. This is similar to Marion's assumption stated explicitly that $\ddot{\mathbf{b}}_S$ can be neglected. In fact, Goldstein also neglects $\ddot{\mathbf{b}}_S$ and we can then obtain several more of his equations from ours :

$$\mathbf{a}_S = \mathbf{a'}_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v'}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \tag{7.6}$$
$$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_S$$

Consider our velocity result (6.8a) and then translate it into Goldstein notation,

$$\mathbf{v}'_S = \mathbf{v}'_{S'} + \boldsymbol{\omega} \times \mathbf{r}' \qquad \textit{(6.8a)}$$

$$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{// Goldstein p 135 (4-104)}$$

$$\text{// GPS p 175 (4.88)}$$

The fact that we obtain Goldstein's velocity equation is how we know the meaning of his $\mathbf{v}_s$ and $\mathbf{v}_r$.

Now consider our acceleration equation from above, and its translation into Goldstein notation

$$\mathbf{a}'_S = \mathbf{a}'_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}'_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad \textit{(7.4)}$$

$$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \text{// Goldstein p 135 (4-105)}$$

$$\text{// GPS p 175 (4.89)}$$

We know that Newton's Law says

$$\mathbf{F} = m\mathbf{a}_S \qquad \textit{(8.1)}$$

Quoting from above,

$$\mathbf{a}_S = \mathbf{a}'_S + \ddot{\mathbf{b}}_S \quad . \qquad \textit{(7.5)}$$

so Newton's Law can be written as

$$\mathbf{F} = m\mathbf{a}'_S + m\,\ddot{\mathbf{b}}_S \qquad (8.11)$$

which translates into

$$\mathbf{F} = m\mathbf{a}_s + m\,\ddot{\mathbf{b}}_S \qquad (8.12)$$

But what appears in Goldstein is this

$$\mathbf{F} = m\mathbf{a}_s \qquad \text{// Goldstein p 135, no equation number}$$

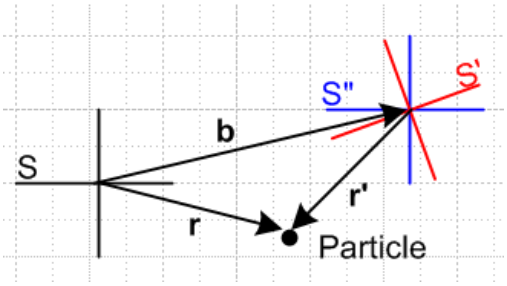
We can compare

$$\mathbf{v}_f = \mathbf{v}_S$$

$$\mathbf{v}_r = \mathbf{v}'_S = \mathbf{v}_S - \dot{\mathbf{b}}_S$$

**(b) Approximate Scheme B based on r'**

Recall that Frame S' is separated from Frame S by the vector **b** and that Frame S' is rotating. Suppose we take Frame S and translate it so its origin aligns with that of Frame S'. Call this new translated frame S". As **b** and therefore S' moves, the S" origin stays attached to the S' origin and the S" axes stay aligned with the S axes. Therefore frame S" accelerates relative to frame S by amount $\ddot{\mathbf{b}}_S$ . Here is a picture:




In Frame S" our particle has

$$\mathbf{r}'' = \mathbf{r}'$$

$$\mathbf{v}'' = \mathbf{v}'_S + \dot{\mathbf{b}}_S \qquad \text{// that is,} \quad \mathbf{v}'' = (d\mathbf{r}''/dt)_{S''} = (d\mathbf{r}''/dt)_S + \dot{\mathbf{b}}_S = (d\mathbf{r}'/dt)_S + \dot{\mathbf{b}}_S = \mathbf{v}'_S + \dot{\mathbf{b}}_S$$

$$\mathbf{a}'' = \mathbf{a}'_S + \ddot{\mathbf{b}}_S \qquad \text{// that is,} \quad \mathbf{a}'' = (d\mathbf{v}''/dt)_{S''} = (d\mathbf{v}'_S/dt)_S + \ddot{\mathbf{b}} = \mathbf{a}'_S + \ddot{\mathbf{b}}$$

Therefore, from *** we find that

$$\mathbf{a}'' = \mathbf{a}_S$$

Newton's Law in the inertial Frame S says

$$\mathbf{F} = m\mathbf{a}_S$$

If for some reason $\ddot{\mathbf{b}}_S$ can be neglected compared to $\mathbf{a}'_S$,

$$\ddot{\mathbf{b}}_S \ll \mathbf{a}'_S$$

we can say

$$\mathbf{F} \approx m\mathbf{a}'_S$$

Then from (7.4),

$$\mathbf{a}'_S = \mathbf{a}'_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}'_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (7.4)$$

we may write

$$\mathbf{F} \approx m\mathbf{a}'_S = m\mathbf{a}'_{S'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2m\,\boldsymbol{\omega} \times \mathbf{v}'_{S'} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

Meanwhile, in Frame S our particle has

$$\mathbf{r} = \mathbf{b} + \mathbf{r}'$$
$$\mathbf{v} = \mathbf{v}_S$$
$$\mathbf{a} = \mathbf{a}_S$$

Newton's Law in the inertial Frame S says

$$\mathbf{F} = m\mathbf{a} = m\mathbf{a}_S$$

Contradiction: this seems to say that $\mathbf{a}'_S = \mathbf{a}_S$, but we found above that

$$\mathbf{a}_S = \mathbf{a}'_S + \ddot{\mathbf{b}}_S$$

One scheme is based on the vector r which has its tail at the Frame S origin, and the other is based

If Frame S is an inertial frame, then Newton's 2nd Law in that frame, $\mathbf{F} = m\mathbf{a}$, is valid in frame S and we can write, using (7.5),

$$\mathbf{F} = m\mathbf{a} = m\mathbf{a}' + 2\,m\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + m\boldsymbol{\omega} \times \mathbf{v}_1 + m\dot{\mathbf{v}}_1 \quad . \qquad (8.1)$$

Our Observer in Frame S' would like to use his own personal "Newton's Law",

$$\mathbf{F}' = m\,\mathbf{a}' \ . \tag{8.2}$$

Solving (8.1) for m**a**' we get

$$\mathbf{F}' = \mathbf{F} - 2\,m\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\boldsymbol{\omega} \times \mathbf{v}_1 - m\dot{\mathbf{v}}_1$$

$$= \mathbf{F} + \mathbf{F}_{fict} \tag{8.3}$$

where

$$\mathbf{F}_{fict} = -2\,m\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\boldsymbol{\omega} \times \mathbf{v}_1 - m\dot{\mathbf{v}}_1 \tag{8.4}$$

So our Observer in Frame S' is allowed to use his own version of Newton's Law, (8.2), *as long as* he includes in his sum of forces the "fictitious" forces shown in (8.4).