Earth-Moon Tide Calculation PhL 8.11.12

This is the first attempt using these pictures. This stuff is *better presented* in the next version of this doc which is v2. This assumes M_2 always faces M_1. So probably just ignore this doc!

1. I draw this picture of two balls orbiting about their center of mass

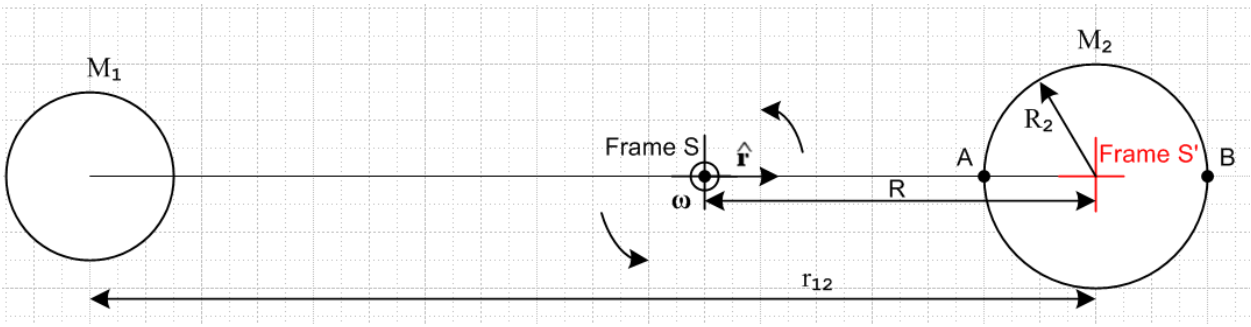


Fig 1

It might be that things come out looking like this if the center of mass lies inside M_2 :

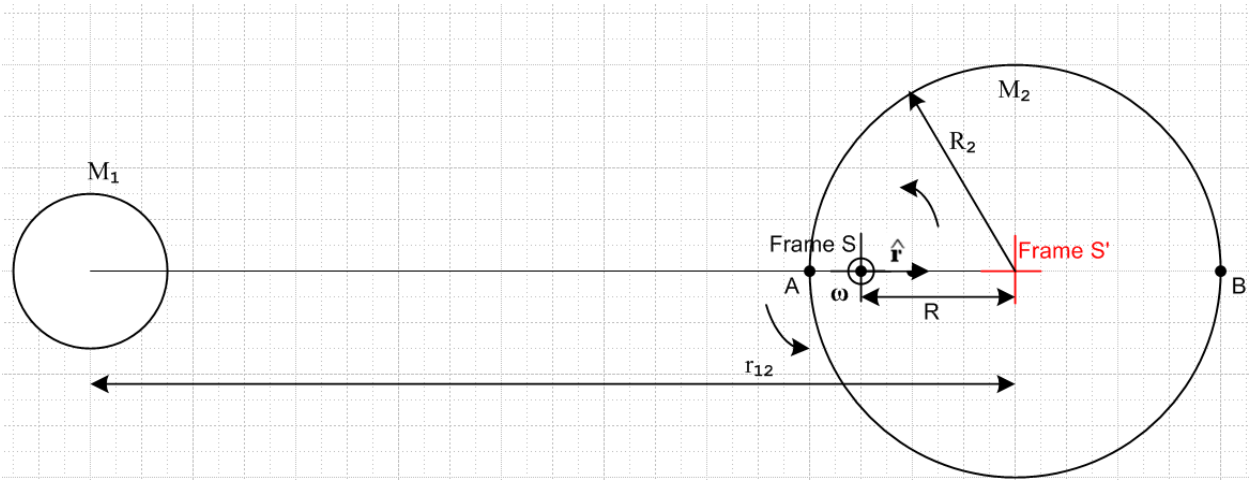


Fig 2

2. Since the center of mass is the point of rotation of this system, and ω passes through the Frame S origin, this is a Special Case #1 situation where (8.12) applies,

$$\mathbf{F'_{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{Special Case \#1} \qquad (8.12)$$

3. Frame S' is placed at the center of M_2 and rotates with it at ω.

4. We imagine a Particle of mass m sitting in a little hollow cavity at the center of M_2 and we ask, in Frame S', what forces does this mass feel? Since it is at rest in Frame S', $\mathbf{v'} = 0$. Also, the rotation rate is a constant (I think...). So answer is

$\mathbf{F'_{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$

5. The only "real" force acting on our particle is the grav pull from M_1. There is no grav from M_2 because we are at its center. Therefore

$\mathbf{F} = -GmM_1/r_{12}^2\ \hat{\mathbf{r}}$

6. We know that

$-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = \omega^2 R\ \hat{\mathbf{r}}$

and so we then have

$\mathbf{F'_{eff}} = -GmM_1/r_{12}^2\ \hat{\mathbf{r}} + m\omega^2 R\ \hat{\mathbf{r}} = m[-GM_1/r_{12}^2 + \omega^2 R]\ \hat{\mathbf{r}}$

7. Since our Particle is at rest in rotating Frame S', it has $\mathbf{a}' = 0$. Since $\mathbf{F'_{eff}} = m\mathbf{a}'$, we know that $\mathbf{F'_{eff}} = 0$. Therefore

$GM_1/r_{12}^2 = \omega^2 R$ (1)

8. The center of mass in Frame S is located at the origin of Frame S, so

$\mathbf{0} = [M_2 R\hat{\mathbf{r}} - M_1(r_{12}-R)\ \hat{\mathbf{r}}]/(M_1+M_2)$

which says

$M_2R - M_1(r_{12}-R) = \mathbf{0} \quad \Rightarrow \quad M_2R + M_1R = M_1r_{12} \quad \Rightarrow \quad (M_1+M_2)R = M_1r_{12}$

so we have

$R = r_{12} * [M_1/(M_1+M_2)]$ (2)

Example: Suppose we have two Earths orbiting around each other. We then have from (2) that $R = r_{12}/2$ so (1) then says

$GM_2/r_{12}^2 = \omega^2 R = \omega^2\ r_{12}/2 \quad \Rightarrow \quad r_{12}^3 = 2GM_2/\omega^2$

If we arbitrarily set the period to 27.3 days (the Earth-Moon rotation period) we find that

$r_{12}^3 = 2GM_2/\omega^2$

$r_{12} = [2GM_2/\omega^2]^{1/3} = .48 \times 10^9\ \text{m} = 4.8 \times 10^8\ \text{m} = 4.8 \times 10^5\ \text{km} = 480{,}000\ \text{km}$

Since $R_2 = 6378$ km and $R = r_{12}/2 = 240{,}000$ km, certain Figure 1 would apply.

9. We now quote these numbers from three wiki for 1 = moon, 2 = sun

$M_1 = 7.3477 \times 10^{22}$ kg moon
$M_2 = 5.9736 \times 10^{24}$ kg[3] earth
$r_{12} = 384{,}399$ km $= 3.84399 \times 10^8$ m
T = 27.321582 d = 27.312 * 24 * 3600 = 2359756.8 $= 2.356 \times 10^6$ sec
$\omega = 2\pi f = 2\pi/T = (2*3.14159/2.356)\times 10^{-6} = 2.66688455 \times 10^{-6}$ rad/sec
$G = 6.67384 \times 10^{-11}$ m³ /(kg sec²)

wiki says: $G = 6.67384(80) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$

We install these numbers into Maple with 3 decimal places only

```
M1 := 7.35e22;
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$$M1 := .735\ 10^{23}$$

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M2 := 5.97e24;
```

$$M2 := .597\ 10^{25}$$

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r12 := 3.84e8;
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$$r12 := .384\ 10^{9}$$

```
T := 2.36e6;
```

$$T := .236\ 10^{7}$$

```
omega := 2.67e-6;
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$$\omega := .267\ 10^{-5}$$

```
G := 6.67e-11;
```

$$G := .667\ 10^{-10}$$

Maple can then compute R according to (2) above,

```
R := r12*M1/(M1+M2);
```

$$R := .4670141474\ 10^{7}$$

We now compute the left and right sides of (1) to make sure we are OK:

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LHS := G*M1/(r12^2);
     LHS := .00003324686686
RHS := (omega^2)*R;
     RHS := .00003329297155
```

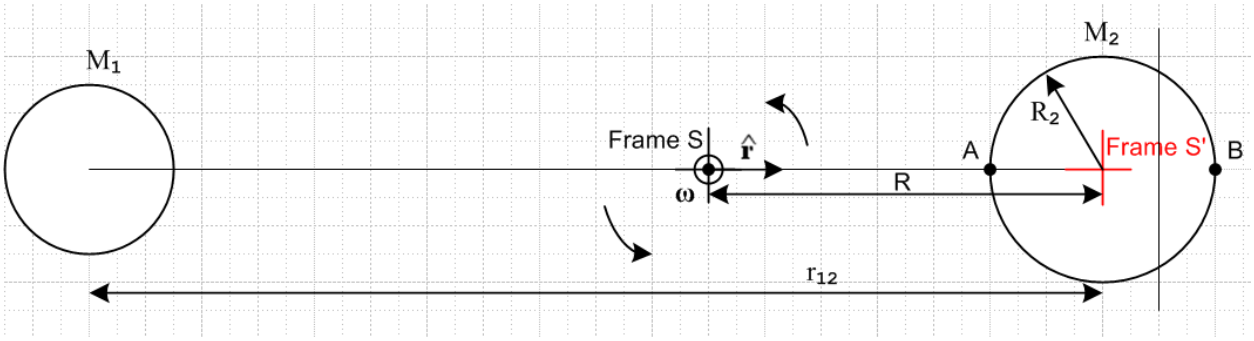
One thing we learn from the above calculation is that

R = 4670 km

R_2 = 6378 km

so we must use Figure 2 for the Moon-Earth system since the CMS is inside the earth.

10. Now back to our picture shown in Fig 1. We will treat Fig 2 separately below.



(a) Consider a particle at location B. What are the real forces acting on this particle?

- gravity 1 pulling left
- gravity 2 pulling left
- force of ground pushing to the right, call this f_B (an unknown, like tether tension T)

What are the fictional forces"

- centrifugal force pushing to the right

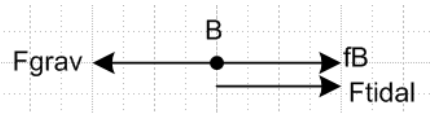
Therefore we can write

$$\mathbf{F'_{eff}}(B) = [\ -GmM_1/(r_{12}+R_2)^2 - GmM_2/R_2^2 + m\omega^2(R+R_2) + f_B]\ \hat{\mathbf{r}}$$

Since Particle at B is at rest in Frame S', we have [..] = 0 and therefore

$$f_B = GmM_2/R_2^2 + m[\ GM_1/(r_{12}+R_2)^2 - \omega^2(R+R_2)]$$

The first term is the normal gravity force on particle B, while the remaining terms are the other forces involved. For Earth Moon Maple says that [..] = -.000046591 < 0. Thus, the ground pushes on the particle with a force less than normal gravity and that is because in this system, the other forces are flinging particle B a little to the right.



So we can then say

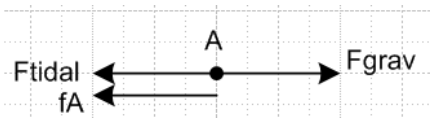
$$F_{tidal}(B) = F_{grav} - f_B = GmM_2/R_2^2 - f_B = -m[\, GM_1/(r_{12}+R_2)^2 - \omega^2(R+R_2)]$$

$$= m[\omega^2(R+R_2) - GM_1/(r_{12}+R_2)^2] = m\,[.00004659] > 0$$

As you move away from the rotation axis, centrifugal increases, and gravity 1 decreases. At M_2 center we had exact matching of these two terms.

(b) Now consider particle at point A.

$$\mathbf{F'}_{eff}(A) = [\, -GmM_1/(r_{12}-R_2)^2 + GmM_2/R_2^2 + m\omega^2(R-R_2) - f_A]\,\hat{\mathbf{r}}$$



In this case we have Particle A at rest so

$$f_A = GmM_2/R_2^2 - GmM_1/(r_{12}-R_2)^2 + m\omega^2(R-R_2)$$

$$= mg_0 - GmM_1/(r_{12}-R_2)^2 + m\omega^2(R-R_2) = mg$$

This time we have

$$F_{tidal}(A) = F_{grav} - f_A = GmM_2/R_2^2 - f_A = GmM_1/(r_{12}-R_2)^2 - m\omega^2(R-R_2)$$

$$= m[GM_1/(r_{12}-R_2)^2 - \omega^2(R-R_2)] = m\,[\,.00004655\,]$$

So first thing to notice: The tidal forces are slightly different on the two sides, about 1/1000 different. Also

$$f_A = mg = mg_0 - F_{tidal} \qquad g = g_0 - F_{tidal}/m$$

Suppose now we move a little away from the surface so w are distance r from Earth center.
Then just replace R_2 by r in the above to get

$$F_{tidal}(A,r) = F_{grav} - f_A = GmM_2/r^2 - f_A = GmM_1/(r_{12}-r)^2 - m\omega^2(R-r)$$

$$= m[GM_1/(r_{12}-r)^2 - \omega^2(R-r)]$$

Then

$$g_A(r) = g_0(r) - F_{tidal}(A,r)/m = g_0(r) - [GM_1/(r_{12}-r)^2 - \omega^2(R-r)]$$

and now we have something that varies with r. Similarly

$$g_B(r) = g_0(r) - F_{tidal}(A,r)/m = g_0(r) - [\omega^2(R+r) - GM_1/(r_{12}+r)^2]$$

Attempt to calculate the tide height.

I just don't know how to even approach this. I try but fail to write little force balance things.

Subquestion: Imagine a water world where gravity is some g(θ) where θ is latitude. How would a layer of water distribute itself on such a world? Forget tidal forces, forget all the above, just answer this question. Nothing rotates, forget frames. *This* is the problem I don't know how to solve!

OK, take a Taylor peek.

Consider a little column of water of area A. This column is bulging height h above the nominal surface of our little water planet. This column of water feels a force down that is

$$F_{down} = (\rho hA)g = mg$$

The earth pushes up on the bottom of the column with $f_A = mg$ as shown above. So $f_A = F_{down}$ and we learn zippo.

g as above

What is the tidal force acting on this column? Column has mass ρhA so

$$F_{tidal}(A) = m [.00004655] = \rho hA [.00004655]$$

Attempt to calculate the tide height.

Consider first a Particle of water at point B sitting at the top of the water bulge. The distance from the center is R_2 + dr where we want to compute dr. This particle is at rest in Frame S' so $\mathbf{F'}_{ext}$ on this force should be 0. But we know that

$$\mathbf{F'}_{eff}(B+dr) = [-GmM_1/(r_{12}+r)^2 - GmM_2/r^2 + m\omega^2(R+r)] \hat{\mathbf{r}} \qquad \text{where } r = R_2 + dr$$

// this must be right

Now we do some approximations.

$$(r_{12}+r)^{-2} = r_{12}^{-2}(1+r/r_{12})^{-2} \approx r_{12}^{-2}(1 - 2\, r/r_{12}) = r_{12}^{-2}(1 - 2\, R_2/r_{12} - 2dr/r_{12})$$

$$r^{-2} = (R_2+dr)^{-2} = R_2^{-2}(1 + dr/R_2)^{-2} = R_2^{-2}(1-2\, dr/R_2)$$

Then we have

$$\mathbf{F'_{eff}}(B+dr) \approx$$
$$[\, -GmM_1 r_{12}^{-2}(1 - 2\, R_2/r_{12} - 2dr/r_{12}) - GmM_2 R_2^{-2}(1-2\, dr/R_2) + m\omega^2(R+R_2+dr)]\, \hat{\mathbf{r}}$$

or

$$\mathbf{F'_{eff}}(B+dr) \approx$$
$$m[\, -GM_1 r_{12}^{-2}(1 - 2\, R_2/r_{12} - 2dr/r_{12}) - GM_2 R_2^{-2}(1-2\, dr/R_2) + \omega^2 R(1+R_2/R+dr/R)]\, \hat{\mathbf{r}}$$

Now in the last term use

$$GM_1/r_{12}^2 = \omega^2 R$$

so

$$\mathbf{F'_{eff}}(B+dr) \approx$$
$$m[\, -GM_1 r_{12}^{-2}(1 - 2\, R_2/r_{12} - 2dr/r_{12}) - GM_2 R_2^{-2}(1-2\, dr/R_2) + GM_1/r_{12}^2\,(1+R_2/R+dr/R)]\, \hat{\mathbf{r}}$$

Now just shuffle things in a rewrite to get

$$= m\, GM_1 r_{12}^{-2}\, [\, -\, (1 - 2\, R_2/r_{12} - 2dr/r_{12}) - (M_2/M_1)(r_{12}/R_2)^2(1-2\, dr/R_2) + (1+R_2/R+dr/R)]\, \hat{\mathbf{r}}$$

$$= m\, GM_1 r_{12}^{-2}\, [\, -1 + 2\, R_2/r_{12} + 2dr/r_{12}) - (M_2/M_1)(r_{12}/R_2)^2(1-2\, dr/R_2) + 1+R_2/R+dr/R]\, \hat{\mathbf{r}}$$

$$= m\, GM_1 r_{12}^{-2}\, [\, 2\, R_2/r_{12} + 2dr/r_{12}) - (M_2/M_1)(r_{12}/R_2)^2(1-2\, dr/R_2) + R_2/R+dr/R]\, \hat{\mathbf{r}}$$

Now since our Particle atop the B side bulge is at rest in Frame S', we have

$$[\, 2\, R_2/r_{12} + 2dr/r_{12}) - (M_2/M_1)(r_{12}/R_2)^2(1-2\, dr/R_2) + R_2/R+dr/R] = 0$$

and we want to solve this for dr. So put the dr terms on the LHS

$$2dr/r_{12} + 2\,(M_2/M_1)(r_{12}/R_2)^2(dr/R_2) + dr/R$$

$$= -\, 2\, R_2/r_{12} + (M_2/M_1)(r_{12}/R_2)^2 - R_2/R$$

and then

$$dr[\ 2/r_{12} + \ 2\ (M_2/M_1)(r_{12}/R_2)^2(1/R_2) + (1/R)]$$

$$= \ (M_2/M_1)(r_{12}/R_2)^2 - 2\ R_2/r_{12} - R_2/R$$

and we hope the RHS is positive! We can eliminate R using (2) which says

$$(1/R) = (1/r_{12}) * \ (M_1 + M_2)/M_1$$

Then

$$dr[\ 2/r_{12} + \ 2\ (M_2/M_1)(r_{12}/R_2)^2(1/R_2) + (1/r_{12}) * \ (M_1 + M_2)/M_1]$$

$$= \ (M_2/M_1)(r_{12}/R_2)^2 - 2\ R_2/r_{12} - \ (R_2/r_{12}) * \ (M_1 + M_2)/M_1$$

or

$$(dr/r_{12})[\ 2 + \ 2\ (M_2/M_1)(r_{12}/R_2)^2(r_{12}/R_2) + \ (M_1 + M_2)/M_1]$$

$$= \ (M_2/M_1)(r_{12}/R_2)^2 - 2\ (R_2/r_{12}) - \ (R_2/r_{12}) * \ (M_1 + M_2)/M_1$$

or

$$(dr/r_{12})[\ 2 + \ 2\ (M_2/M_1)(r_{12}/R_2)^3 + \ 1 + (M_2/M_1)]$$

$$= \ (M_2/M_1)(r_{12}/R_2)^2 - \ (R_2/r_{12})\ \{\ 2 + 1 + M_2/M_1\ \}$$

or

$$(dr/r_{12})[\ 3 + \ 2\ (M_2/M_1)(r_{12}/R_2)^3 \ + (M_2/M_1)]$$

$$= \ (M_2/M_1)(r_{12}/R_2)^2 - \ (R_2/r_{12})\ (3 + M_2/M_1)$$

$$dr = r_{12} * [(M_2/M_1)(r_{12}/R_2)^2 - \ (R_2/r_{12})\ (3 + M_2/M_1)]\ /\ [\ 3 + \ 2\ (M_2/M_1)(r_{12}/R_2)^3 \ + (M_2/M_1)]$$

Since I have all this in Maple I can just compute it as

$$dr = r_{12} * \text{top/bottom}$$

This gives $r_{12}\ 10^7$ m which I am afraid is not correct, my little "theory" has failed. You can see how this work. In top, the first term wins by a mile, and in bot the middle term wins by a mile, so

$$dr \approx r_{12} * [(M_2/M_1)(r_{12}/R_2)^2/[2\ (M_2/M_1)(r_{12}/R_2)^3]$$

$$= r_{12} * [(r_{12}/R_2)^{-1}/[2]\ = (1/2)\ r_{12} * R_2/r_{12} = R_2/2$$

So our little tide is half the radius of the earth!