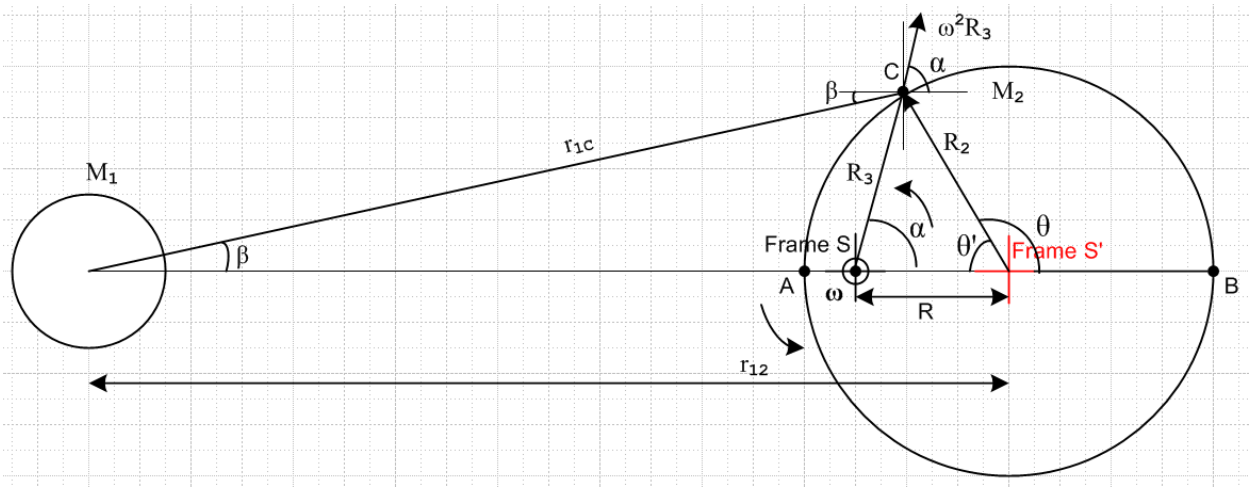
Facing Planet Problem PhL 8.13.12

This is a basic outline of how you might do this problem. This is tides on the Earth if the earth faced the moon during its rotation.

Consider this picture



Before doing anything, we have a lot of geometry to figure out. Assume that r_{12}, M_1, M_2, R_2 and θ are given quantities. Then here is how we figure out everything else shown:

$\theta' = \pi - \theta$ // determines θ'

$R = r_{12}\,[M_1/(M_1+M_2)]$ // determines R

$r_{1c}^2 = r_{12}^2 + R_2^2 - 2r_{12}R_2\cos\theta'$ // determines r_{1c}

$\sin\beta = \sin\theta(R_2/r_{1c})$ // determines $\sin\beta$

$R_2^2 = r_{1c}^2 + r_{12}^2 - 2r_{1c}r_{12}\cos\beta$ // determines $\cos\beta$

$R_3^2 = R^2+R_2^2 - 2RR_2\cos\theta'$ // determines R_3

$\sin\alpha = (R_2/R_3)\sin\theta'$ // determines $\sin\alpha$

$R_2^2 = R^2+R_3^2 - 2RR_3\cos\alpha$ // determines $\cos\alpha$

So now we know all quantities in the picture.

Next, we consider a particle at the center of object 1. We will have there this obvious force balance

$\mathbf{F}_{grav1}(\text{center}) = -(mM_1G/r_{12}^2)\hat{\mathbf{x}}$ real

$\mathbf{F}_{\mathbf{cent}}(\text{center}) = m\omega^2 R\hat{\mathbf{x}}$ fict

Since a particle at the center has $\mathbf{a}' = 0$, these must cancel and we get

$M_1 G/r_{12}^2 = \omega^2 R$ // determines ω

Next, we compute the total force on the Particle at C. The magnitude of the upper vector is $\omega^2 R_3$ and this is a fictitious force, the angle is α as shown. The grav1 vector has magnitude $mM_1 G/r_{1c}^2$ and it is at angle β as shown. Now stare at the picture to conclude that, at point C,

$\mathbf{F}_{\mathbf{grav1}}(C) = (mM_1 G/r_{1c}^2)\,[\cos(\pi/2-\theta'-\beta)\hat{\boldsymbol{\theta}} + \sin(\pi/2-\theta'-\beta)\,\hat{\mathbf{r}}\,]$ real

$\mathbf{F}_{\mathbf{cent}}(C) = (m\omega^2 R_3)\,[-\cos(\alpha - \pi/2 +\theta')\,\hat{\boldsymbol{\theta}} + \sin(\alpha - \pi/2 +\theta')\,\hat{\mathbf{r}}\,]$ real

$\mathbf{F}_{\mathbf{grav2}}(C) = -(mM_2 G/R_2^2)\,\hat{\mathbf{r}}$ fict

$\mathbf{F}_{\mathbf{surf}}(C) = F_{\mathbf{surf}}\,\hat{\mathbf{r}}$ real

The total force $\mathbf{F}'_{\mathbf{eff}}$ acting on Particle C is the sum of these four forces

$$\mathbf{F}'_{\mathbf{eff}} = [\,(mM_1 G/r_{1c}^2)\cos(\pi/2-\theta'-\beta) - (m\omega^2 R_3)\cos(\alpha - \pi/2 +\theta')]\,\hat{\boldsymbol{\theta}}$$

$$+ [(mM_1 G/r_{1c}^2)\sin(\pi/2-\theta'-\beta) + (m\omega^2 R_3)\sin(\alpha - \pi/2 +\theta') - (mM_2 G/R_2^2) + F_{\mathbf{surf}}\,]\,\hat{\mathbf{r}}$$

I think this is the tidal force at point C. Now simplify by using

$\cos(\pi/2-x) = \sin(x)$
$\sin(\pi/2-x) = \cos(x)$

Then

$\cos(\pi/2-\theta'-\beta) = \cos(\pi/2 - [\theta'+\beta]) = \sin(\theta'+\beta)$

$\cos(\alpha - \pi/2 +\theta') = \cos(\pi/2 -\alpha - \theta') = \cos(\pi/2-[\alpha+\theta']) = \sin(\alpha+\theta')$

$\sin(\pi/2-\theta'-\beta) = \sin(\pi/2 - [\theta'+\beta]) = \cos(\theta'+\beta)$

$\sin(\alpha - \pi/2 +\theta') = -\sin(\pi/2-\alpha - \theta') = -\sin(\pi/2- [\alpha+\theta']) = -\cos(\alpha+\theta')$

$$\mathbf{F}'_{\mathbf{eff}} = [\,(mM_1 G/r_{1c}^2)\sin(\theta'+\beta) - (m\omega^2 R_3)\sin(\alpha+\theta')]\,\hat{\boldsymbol{\theta}}$$

$$+ [(mM_1 G/r_{1c}^2)\cos(\theta'+\beta) - (m\omega^2 R_3)\cos(\alpha+\theta') - (mM_2 G/R_2^2) + F_{\mathbf{surf}}]\,\hat{\mathbf{r}}$$

Now we would like to know the direction of this tidal force.

We finally simplify by assuming that object 1 is far away so $r_{12} >> R_2$. Then our list above becomes

$\theta' = \pi - \theta$ // determines θ'

$R = r_{12}\,[M_1/(M_1 + M_2)]$ // determines R

$r_{1c}^2 = r_{12}^2$ // determines r_{1c}

$\sin\beta = \sin\theta(R_2/r_{12})$ // determines $\sin\beta$

$0 = r_{12}^2 + r_{12}^2 - 2r_{12}r_{12}\cos\beta$ // determines $\cos\beta \approx 1$ so $\beta \approx 0$ but see above

$R_3^2 = R^2 + R_2^2 - 2RR_2\cos\theta'$ // determines R_3 , no simplification

$\sin\alpha = (R_2/R_3)\sin\theta'$ // determines $\sin\alpha$, no simplification

$R_2^2 = R^2 + R_3^2 - 2RR_3\cos\alpha$ // determines $\cos\alpha$

$M_1G/r_{12}^2 = \omega^2 R$ // fact also known

Then we have

$$\mathbf{F'_{eff}} \approx [\,(mM_1G/r_{12}^2)\sin(\theta' + \beta) - (m\omega^2 R_3)\sin(\alpha + \theta')]\,\hat{\boldsymbol{\theta}}$$

$$+ [(mM_1G/r_{12}^2)\cos(\theta' + \beta) - (m\omega^2 R_3)\cos(\alpha + \theta') - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R)\sin(\theta' + \beta) - (m\omega^2 R_3)\sin(\alpha + \theta')]\,\hat{\boldsymbol{\theta}}$$

$$+ [(m\,\omega^2 R)\cos(\theta' + \beta) - (m\omega^2 R_3)\cos(\alpha + \theta') - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

What happens at $\theta' = 0$? At that point $\alpha = \pi$ and $R_3 = R_2 - R$ and $\beta = 0$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R)\sin(0) - (m\omega^2 R_3)\sin(\pi)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\,\omega^2 R)\cos(0) - (m\omega^2 R_3)\cos(\pi) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx + [(m\,\omega^2 R) + (m\omega^2 R_3) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx + [(m\,\omega^2 R) + (m\omega^2 [R_2 - R]) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}}(A) \approx + [\,m\omega^2 R_2 - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

and since A is at rest, this adds up to 0.

Normally we would have $F_{surf} = (mM_2G/R_2^2)$ for something at rest on the surface, But now we have'

$$F_{surf} = (mM_2G/R_2^2) - m\omega^2 R_2$$

so the earth presses less than normal, so something is pulling to the left, and

$$F_{tid}(A) = m\omega^2 R_2 \qquad \text{to the left}$$

This is the main term I got in my version 2 document. My approx here killed off the other term which recall is 41x smaller for moon-earth. The centrifugal effect dominates.

What about point B? There $\theta' = \pi$, $\alpha = 0$, $R_3 = R+R_2$, $\beta = 0$. So

$$\mathbf{F'}_{eff} \approx [\, (m\,\omega^2 R)\sin(\theta'+\beta) - (m\omega^2 R_3)\sin(\alpha+\theta')]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\,\omega^2 R)\cos(\theta'+\beta) - (m\omega^2 R_3)\cos(\alpha+\theta') - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'}_{eff} \approx [\, (m\,\omega^2 R)\sin(\pi+0) - (m\omega^2 R_3)\sin(0+\pi)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\,\omega^2 R)\cos(\pi+0) - (m\omega^2 R_3)\cos(0+\pi) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'}_{eff} \approx + [-(m\,\omega^2 R) + (m\omega^2 R_3) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'}_{eff} \approx + [-(m\,\omega^2 R) + (m\omega^2 [R+R_2]) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'}_{eff}(B) \approx + [m\omega^2 R_2 - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

Now we have

$$F_{surf}(B) = (mM_2G/R_2^2) - m\omega^2 R_2$$

and again our particle is "lighter" by the same amount $m\omega^2 R_2$

Now let's try a point at the top, so $\theta' = \pi/2$. Then we have

$\sin\alpha = R_2/R$ $\quad$ $\sin\beta \approx R_2/r_{12}$ = very small $\quad$ $R_3 = [\, R^2 + R_2^2]^{1/2}$
$\cos\alpha = R/R_3$

so:

$$\mathbf{F'}_{eff} \approx [\, (m\,\omega^2 R)\sin(\theta'+\beta) - (m\omega^2 R_3)\sin(\alpha+\theta')]\,\hat{\boldsymbol{\theta}}$$

$$+ [(m\,\omega^2 R)\cos(\theta'+\beta) - (m\omega^2 R_3)\cos(\alpha+\theta') - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R)\sin(\pi/2) \; - (m\omega^2 R_3)\sin(\alpha+\pi/2)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\,\omega^2 R)\cos(\pi/2) - (m\omega^2 R_3)\cos(\alpha+\pi/2) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R)\,1 \; - (m\omega^2 R_3)\cos(\alpha)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\,\omega^2 R)\,0 + (m\omega^2 R_3)\sin(\alpha) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R) \; - (m\omega^2 R_3)\cos(\alpha)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\omega^2 R_3)\sin(\alpha) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx [\,(m\,\omega^2 R) \; - (m\omega^2 R_3)\,(R/R_3)]\,\hat{\boldsymbol{\theta}}$$
$$+ [(m\omega^2 R_3)(R_2/R) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

$$\mathbf{F'_{eff}} \approx \; + [(m\omega^2 R_3)(R_2/R) - (mM_2G/R_2^2) + F_{surf}]\,\hat{\mathbf{r}}$$

Now we find that

$$F_{surf} = (mM_2G/R_2^2) - (m\omega^2 R_3)(R_2/R)$$

and again we are lighter. The amount lighter is $m\omega^2 R_2(R_3/R)$ so we have an extra factor. This is a fact I have never computed before. the tidal force is outward at the top and it is larger than at A and B ! This suggests to me that the bulge will be around the belt, and we have dips at the A and B. Maybe I have made an error.

OK, that is enough for now.