About Torque PhL 9.18.12

Motivation: While trying to add torque to my frames doc, I got involved in the "point of reference" for both torque and angular momentum, and now I am totally confused about how torques work for even simple problems. This is enhanced by my reading of Lai concerning elastostatics where he has all those couples and moments. At least I did learn that Halliday and Resnick used τ for torque, and so does the wiki site, whereas Goldstein and Marion both use N for torque (the old books). Here is an example of my confusion. Take a bar and apply torques at the two ends. Is each torque applied with a different reference point? Are you allowed to add torques referred to different reference points? What about the notion of "axis", where does it fit in? I will do a little review of what books have to say.

1. Review of Goldstein, Marion and Halliday/Resnick

Goldstein. Page 144, he writes **L** for a system of particles as sum of **r** x **p**. The context here is motion of a rigid body which has "one point stationary", and that point is then used as the origin for **r**. He then does some developments such as **v** = **ω** x **r** and he ends up with **L** = I**ω** where I is the inertia tensor. As I page through the rest of his rigid body motion chapter, there is just not much mention of torque. So this might be one reason I am fuzzy on this, G did not do much in this area.

Marion. Chapter 12 is on rigid body dynamics. He starts by showing that $T = (1/2)\, I_{ij}\omega_i\omega_j$ (KE) and that is how the inertia tensor appears in his development. Then on p 365 we have angular momentum. Right off the bat we has a sum of r x p "with respect to some point O that is fixed in the body system". For me, that is the rotating system of Frame S', and I would call that point **c**'. Marion then says you could choose origin O (he does not call it an origin) to be a fixed point in your rotating "body" if possible. If there are no fixed points, you choose it at the center of mass! Then I guess if you do a boost, that CM point will become a fixed point. So this is good info. He then obtains **L** = I**ω .** Page 367 says T = (1/2) **ω** • **L** which would be like (1/2) **v** • **p** = $(1/2)mv^2$ in the linear world. He is then off on details of the inertia tensor. Then come the Euler angles on page 384. He then gets the "Euler equations" on page 389 "for a rigid body in a force field", which involve **ω**, I and torque **N**. This does start with **N** = $\dot{\mathbf{L}}$ on p 388. We then have the top and nutation and the chapter ends. There are no torque static problems here! But on page 62 he does state that L and N have the same "origin from which r is measured", as I state in my doc update. So Marion is much like Goldstein in what he is doing, and simple mechanics problems do not appear. Remember that mechanics covers lots of topics and you can't do everything in one book.

Halliday & Resnick. [I thought this was a high school book, but it says Winthrop F-31 1967. This was in fact a text used in fall semester of Sophomore year, Physics 13a. Also used in this course was the Berkeley book and Feynman Vol I.]

Chap 11 is "**rotational kinematics**" p 241. Comparison chart of linear and rot on page 246. Spinning disk on a rotating turntable page 250 so show addition of $\boldsymbol{\omega}_i$ vectors. On page 255 we see our friend **v** = **ω** x **r** for circular motion, then that chapter ends and ...

Chap 12 starts on **Rot Dyn I.** Torque introduced on page 260. Nice picture of example, and "with respect to the origin O" is stated. They say torque is called the "moment of force" and r_{perp} is the "moment arm" (not r). $r_{perp} = r\sin\theta$ is the projection of r that is perp to F. The direction of the force is called the "line of action" and if it goes through your reference origin, there is no torque. They also write $F_{perp} = F\sin\theta$. H&R on page 264 claim that angular momentum is called "the moment of (linear)

momentum" and again r_{perp} is the moment arm. So far everything is with this fixed origin O as reference. He finds L and τ for a falling particle relative to an origin O on p 265, at least a problem and a picture. We get $\tau = \dot{L}$ p 265. Page 266 then talks about more than one particle, a "system" of same. They use $\boldsymbol{\ell}_i$ as L of one particle, but non script so hard to parse, p 266 bottom. You must use the same reference point for all these $\boldsymbol{\ell}_i$ contributions to L. They are then off into "inertia" with lots of nice examples.

Then p 274 we have a torque acting on a rigid body, but first footnote page 268: We are supposed to choose O at the center of mass. Then "axis" means a line going through that CMS point and it is allowed even to translate, but that is all. It does not have to translate uniformly at constant v, but it has to stay parallel (this is the ω axis). Hints of the future here some 45 years later.

Now back to that rigid body: it is pinned on an axis, so "constrained to rotate about a fixed axis". The constraining thin bar will of course put some force on the object, but no torque since the torque origin is at that point (frictionless?), this is a 2D thing p 274. Now in this picture, force F is applied to a particular point P (at r) in the rigid body. And $\tau = r \times F$ acting then on the entire rigid body. Page 278 shows the table of Analogy of linear and rot. There are then lots of problems.

Chap 13 is **Rot Dyn II** on p 295. The top is treated, then more problems.

Then p 320 is The Equilibrium of Rigid Bodies which is "**statics**", maybe something will be lurking here. First, $\Sigma F_i = 0$ as expected for a rigid body. Here comes a theorem:

Theorem: If $\Sigma F_i = 0$, then $\tau = \Sigma \tau_i$ is independent of selection of reference point for the τ_i. Here for the first time we have point P as my point **c**, and he shows three forces in the figure, and he then writes the torque about O, and then the torque about P. He shows that indeed $\boldsymbol{\tau}^{(c)} = \boldsymbol{\tau}^{(0)}$ in my notation. Then a corollary is that if one is zero, so is the other. This is a rigid body in uniform linear motion or at rest. The point of this corollary is this: if a rigid body is fully "at rest" (equilibrium linearly and rotationally, boosting uniformly or rotating uniformly is OK), then $\Sigma F_i = 0$ AND also we have $\Sigma \tau_i = 0$ with respect to any point we want! So these are the statics equations. A simple proof is given.

This is interesting, let's try to restate. Take a rigid object at rest. Apply some forces F_i at various points in or on this rigid body. If $\Sigma F_i = 0$, then the torque acting on this body is the same with respect to any reference point. Perhaps this is what is meant by a "couple". The proof is trivial.

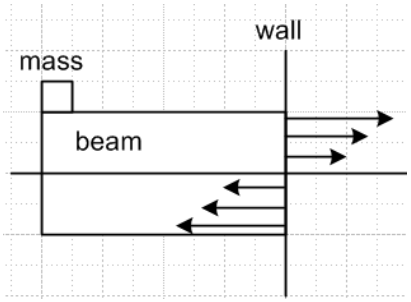
Page 323 talks about "**center of gravity**". First, a rigid body can be thought of as a set of particles. If you have a uniform g field, then effect is same as if rigid body were a point mass M at the center of mass with g acting on that point. So we have this effective equality of rigid body = point at CMS. We know that the4 torque τ on the second object will be 0 since no moment arm (O = **CMS**). But they prove directly that τ = 0 if you choose CMS. So a uniform g field makes no torque about the CMS, and then by the theorem above, there can be no torque about any point c in the rigid body (due to uniform g). They then discuss a case where center of mass and gravity are different, a miles long stick with masses on the end.

They then solve some "statics problems" but these seem to involve only F_i and torques are not mentioned. Then problems and then we are done! θ

Paradox A. Consider a massless very thin beam attached to a wall of length $\boldsymbol{\ell}$ with mass m on the end. There are two forces on this beam. We have m**g** pulling down say at the right end, and we have the wall pushing up at the left end with an upwards force -m**g**. The total force on the beam is then $\Sigma F_i = 0$ and so the beam does not accelerate. This is consistent with the fact that it is at rest. All this is fine.

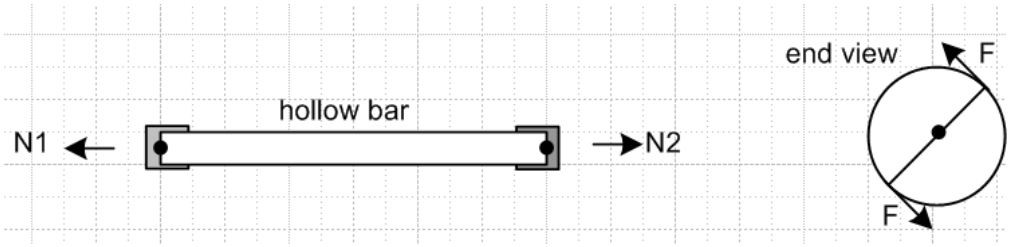
Now what is the total torque on the beam relative to the wall attachment point? The torque at the wall end on the beam is 0 because it is r x p relative to this attachment point and r = 0 there. The torque due to the mass is r x mg. Thus, the total torque on the beam is $\boldsymbol{\tau}^{(wall)} = \mathbf{r} \times m\mathbf{g}$. The paradox is this: since the beam has a non-zero total torque, it should be doing angular acceleration. But it is at rest!

Resolution of paradox: although the beam might be very thin, it still has some finite cross sectional area where it attaches to the ball. Assume it is glued to the wall. Here is the picture as in Lai



Each of my 6 force arrows shows a force of the wall acting on the beam end. The top forces are where the glue acts and the wall pulls the beam to the right. The lower forces are an area of compression so the wall pushes to the left and the glue there could be missing with no effect. Each of the 6 force arrows creates a torque about the beam center line intersection with the wall which points into the plane of paper, so the total points into paper. The torque on the left end points out of the plane of paper. Both torques are referenced to the wall center point. The total torque is 0 on the beam, so it is happy to be at rest.

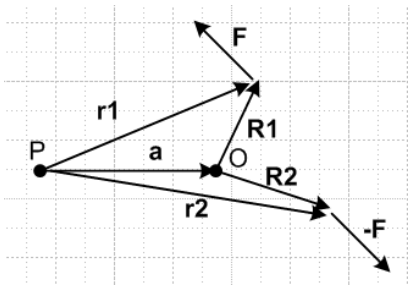
Paradox B. Consider two equal and opposite torques applied to a hollow bar,



The end view shows that the applied twisting CCW torque arises from a large number of forces acting along the perimeter of the end grabber cap (I show only two such forces). Each force contributes the same direction to the end torque. So if you think of pairwise forces, you find that $\Sigma \mathbf{F}_i = 0$ on the bar, in fact separately for each grabber cap. So the bar is at linear rest, all is well. Now here is the paradox: we want to say that the total torque on the bar is the $\mathbf{N}_t = \mathbf{N}_1 + \mathbf{N}_2 = 0$ and that is why the bar is also at rotational rest. But something is amiss, because we are then adding torques which are referenced to different points. For each endcap (think very thin endcap!) all torques are referenced to the center of that endcap. So why is it that we are apparently allowed to add these two torques and make the conclusion quoted?

The resolution of this dilemma is contained in the following theorem.

Theorem 1: Consider this picture:



We have two equal and opposite forces which cause this total torque about point O

$$\mathbf{N}^{(O)} = \mathbf{R}_1 \times \mathbf{F} + \mathbf{R}_2 \times (-\mathbf{F}) = (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}$$

Now using these same two forces, let's compute the torque about point P:

$$\mathbf{N}^{(P)} = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F})$$

Now we know that $\mathbf{a} + \mathbf{R}_1 = \mathbf{r}_1$ and also $\mathbf{a} + \mathbf{R}_2 = \mathbf{r}_2$. Therefore,

$$\mathbf{N}^{(P)} = (\mathbf{R}_1 - \mathbf{a}) \times \mathbf{F} + (\mathbf{R}_2 - \mathbf{a}) \times (-\mathbf{F})$$

$$= \mathbf{R}_1 \times \mathbf{F} + \mathbf{R}_2 \times (-\mathbf{F}) = \mathbf{N}^{(O)}$$

So the theorem says this: If you have a pair of torques which involve equal and opposite forces, the total torque of the pair is independent of the point of reference!

Theorem 2: Imagine a similar picture with n different torques, each with some F_i. Suppose the forces are such that $\Sigma_{i=0}^{n} \mathbf{F}_i = 0$. Then we could find that

$$\mathbf{N}^{(O)} = \Sigma_{i=1}^{n} \mathbf{R}_i \times \mathbf{F}_i$$

$$\mathbf{N}^{(P)} = \Sigma_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i$$

For every single pair of position vectors we have

$$\mathbf{a} + \mathbf{R}_i = \mathbf{r}_i \qquad i = 1,2...n$$

Therefore

$$\mathbf{N}^{(P)} = \Sigma_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = \Sigma_{i=1}^{n} (\mathbf{a} + \mathbf{R}_i) \times \mathbf{F}_i = \Sigma_{i=1}^{n} \mathbf{R}_i \times \mathbf{F}_i + \mathbf{a} \times \Sigma_{i=1}^{n} \mathbf{F}_i$$

$$= \Sigma_{i=1}^{n} \mathbf{R}_i \times \mathbf{F}_i + \mathbf{0} = N^{(O)}$$

So the theorem here is that if all the forces add to zero, the total torque is the same for any reference point you pick. This is the theorem in H&R quoted above.

Application to Paradox B. The forces around one of the coupling rings add to zero. You can think of them doing this in a pairwise fashion, or you can do it by calculus. For example, let $\mathcal{F}$ be the force per unit length of circle perimeter on the rod. Then

$$\oint \mathcal{F} \bullet d\ell = \oint \mathcal{F}\hat{\theta} \bullet d\ell\hat{\theta} = \mathcal{F} \oint d\ell = \mathcal{F}\, 0 = 0$$

This proof works for a circular boundary with mag $\mathcal{F}$ the same at every point.

Therefore, we can take the total torque about the right endcap center point and it will be the same if we move that to the left endcap center point. Then we are adding torques relative to the same reference point, and then the paradox goes away!

Theorem 2 restated: If the sum of a set of forces acting on a rigid body is 0, then the sum of the torques associated with those forces, acting on that body, is independent of the point **c** you choose to as the reference point for all the torques.

An example of this is the twisted bar above. Wiki makes this claim

	classical physics	continuum mechanics
L	angular momentum	momentum of momentum
r x **F**	torque	moment
$\Sigma\, \mathbf{r}_i$ x $\mathbf{F}_i$ with $\Sigma \mathbf{F}_i = 0$	torque	a couple, a torque, a pure moment

Imagine a cylindrical steel bar under twisting strain due to equal and opposite torques applied to the ends of the bar through small grabber chucks, one at each end, no gravity. The effect of a chuck on the bar can be represented as a continuous sum of tangential forces around the surface of the bar under the chuck, which forces add up to zero (think pairwise). Therefore, since the force of a chuck on the bar is zero, the torque of a chuck on the bar is independent of reference point. Since the same is true for each end, one can pick some arbitrary point **c** and reference both torques to that point, and then add them "legally" to conclude that the total torque on the bar is 0. The bar thus shows no angular acceleration.