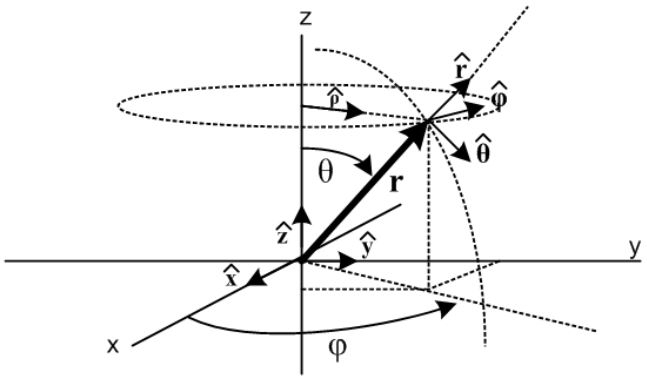
Appendix A: Derivation of R(ξ) and Properties of Rotation Matrices

Here we derive equations (14.4) through (14.6) for spherical coordinates. First, just for reference, here are the three active rotation matrices used below :

$$R_{\mathbf{x}}(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\quad R_{\mathbf{y}}(\theta)=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\quad R_{\mathbf{z}}(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\ . \qquad (A.1)$$

These are called "active" since they rotate a vector forward (counterclockwise) relative to fixed axes by amount θ according to the right hand rule when the thumb is aligned with the axis in question.

From the usual picture of spherical coordinates,



(A.2)

one can see (by staring hard enough) that

$$\begin{aligned}
\hat{\mathbf{r}} &= R_{\mathbf{z}}(\varphi)\,R_{\mathbf{y}}(\theta)\,\hat{\mathbf{z}} &&= \mathcal{R}\hat{\mathbf{z}} && \mathcal{R}\equiv R_{\mathbf{z}}(\varphi)\,R_{\mathbf{y}}(\theta)\\
\hat{\boldsymbol{\theta}} &= R_{\mathbf{z}}(\varphi)\,R_{\mathbf{y}}(\theta)\,\hat{\mathbf{x}} &&= \mathcal{R}\hat{\mathbf{x}} &&\\
\hat{\boldsymbol{\varphi}} &= R_{\mathbf{z}}(\varphi)\,R_{\mathbf{y}}(\theta)\,\hat{\mathbf{y}} &&= \mathcal{R}\hat{\mathbf{y}} && /\!/\ R_{\mathbf{y}}(\theta)\text{ does nothing here}
\end{aligned} \qquad (A.3)$$

which we rewrite as

$$\begin{aligned}
\hat{\boldsymbol{e}}_1 &= \mathcal{R}\,\mathbf{e}_3 && \text{where} && \hat{\boldsymbol{e}}_1 = \hat{\mathbf{r}} && \mathbf{e}_3=\hat{\mathbf{z}}\\
\hat{\boldsymbol{e}}_2 &= \mathcal{R}\,\mathbf{e}_1 && && \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{\theta}} && \mathbf{e}_1=\hat{\mathbf{x}}\\
\hat{\boldsymbol{e}}_3 &= \mathcal{R}\,\mathbf{e}_2 && && \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{\varphi}} && \mathbf{e}_2=\hat{\mathbf{y}}\ .
\end{aligned} \qquad (A.4)$$

We can repair the ordering of the basis vectors $\mathbf{e}_n$ on the right using $R_2 = (\hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}})$ as follows

$$\mathbf{e}_3 = R_2\,\mathbf{e}_1 \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad R_2=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=(\hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}})$$

$$\mathbf{e_1} = R_2\,\mathbf{e_2} \qquad \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathbf{e_2} = R_2\,\mathbf{e_3} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad . \tag{A.5}$$

Maple tells us that $R_2^{-1} = R_2^{T}$ and $\det(R_2) = 1$, confirming that R_2 is a rotation. Putting (A.5) into (A.4),

$$\begin{aligned}\hat{\boldsymbol{e}}_1 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_1}\\ \hat{\boldsymbol{e}}_2 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_2}\\ \hat{\boldsymbol{e}}_3 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_3}\end{aligned} \tag{A.6}$$

or

$$\hat{\boldsymbol{e}}_i = R(\xi)\,\mathbf{e}_i \quad . \tag{A.7}$$

Recalling (14.4),

$$\begin{aligned}&\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \quad \Rightarrow \quad \mathbf{e}_n = R(\xi)_{nm}\hat{\boldsymbol{e}}_m \quad \text{or} \quad \hat{\boldsymbol{e}}_n = [R(\xi)]^{-1}_{nm}\,\mathbf{e}_m\\ &\hat{\boldsymbol{e}}'_n = R'(\xi')\,\mathbf{e}'_n \quad \Rightarrow \quad \mathbf{e}'_n = R'(\xi')_{nm}\hat{\boldsymbol{e}}'_m \quad \text{or} \quad \hat{\boldsymbol{e}}'_n = [R'(\xi')]^{-1}_{nm}\,\mathbf{e}'_m \quad .\end{aligned} \tag{14.4}$$

we have therefore found the sought-after matrix $R(\xi)$ appearing in the top left of (14.4).

$$R(\xi) = \boldsymbol{\mathcal{R}}\,R_2 \quad . \tag{A.8}$$

Specific evaluation gives

$$\boldsymbol{\mathcal{R}} = R_z(\varphi)\,R_y(\theta) = \begin{pmatrix}\cos\varphi&-\sin\varphi&0\\\sin\varphi&\cos\varphi&0\\0&0&1\end{pmatrix}\begin{pmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi&\cos\varphi\sin\theta\\\sin\varphi\sin\theta&\cos\varphi&\sin\varphi\sin\theta\\-\sin\theta&0&\cos\theta\end{pmatrix} \tag{A.9}$$

and then

$$R(\xi) = \boldsymbol{\mathcal{R}}\,R_2 = \begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi&\cos\varphi\sin\theta\\\sin\varphi\sin\theta&\cos\varphi&\sin\varphi\sin\theta\\-\sin\theta&0&\cos\theta\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}\cos\varphi\sin\theta&\cos\varphi\cos\theta&-\sin\varphi\\\sin\varphi\sin\theta&\sin\varphi\cos\theta&\cos\varphi\\\cos\theta&-\sin\theta&0\end{pmatrix} \tag{A.10}$$

from which we find

$$[R(\xi)]^{-1} = [R(\xi)]^{T} = \begin{pmatrix}\cos\varphi\sin\theta&\sin\varphi\sin\theta&\cos\theta\\\cos\varphi\cos\theta&\sin\varphi\cos\theta&-\sin\theta\\-\sin\varphi&\cos\varphi&0\end{pmatrix} \quad . \tag{A.11}$$

From the top right equation of (14.4) (quoted above) we can write

$$\hat{\boldsymbol{e}}_i = [R(\xi)]^{-1}_{ij}\,\mathbf{e}_j = [R(\xi)]^{-1}_{i1}\,\mathbf{e_1} + [R(\xi)]^{-1}_{i2}\,\mathbf{e_2} + [R(\xi)]^{-1}_{i3}\,\mathbf{e_3} \quad . \tag{A.12}$$

This can be written in this matrix notation:

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \text{(A.13a)}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \qquad \text{(A.13b)}$$

or

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\varphi\sin\theta\, \hat{\mathbf{x}} + \sin\varphi\sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\varphi\cos\theta\, \hat{\mathbf{x}} + \sin\varphi\cos\theta\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \end{aligned} \qquad \text{(A.13c)}$$

which are the well known expressions for the spherical unit vectors in terms of the Cartesian ones. The inverse of the last three equations can be obtained in the following manner (matrix from (A.10))

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = [R(\xi)] \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} \qquad \text{(A.14a)}$$

or

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)] \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} \qquad \text{(A.14b)}$$

or

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\varphi\sin\theta\, \hat{\mathbf{r}} + \cos\varphi\cos\theta\, \hat{\boldsymbol{\theta}} - \sin\varphi\, \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin\varphi\sin\theta\, \hat{\mathbf{r}} + \sin\varphi\cos\theta\, \hat{\boldsymbol{\theta}} + \cos\varphi\, \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}} \ . \end{aligned} \qquad \text{(A.14c)}$$

Appendix E contains further information on spherical coordinates.

Some Properties of Rotation Matrices

A rotation matrix R is always real orthogonal which means $R^{-1} = R^T$. It follows that

$$\begin{aligned} RR^T = 1 \quad &\Rightarrow \quad \Sigma_k R_{ik}(R^T)_{kj} = \delta_{i'j} \quad \Rightarrow \quad \Sigma_k R_{ik}R_{jk} = \delta_{i,j} \\ R^TR = 1 \quad &\Rightarrow \quad \Sigma_k (R^T)_{ik}R_{kj} = \delta_{i'j} \quad \Rightarrow \quad \Sigma_k R_{ki}R_{kj} = \delta_{i,j} \ . \end{aligned} \qquad \text{(A.15)}$$

Also

$$RR^T = 1 \quad \Rightarrow \quad \det(RR^T) = \det(1) \quad \Rightarrow \quad [\det(R)]^2 = 1$$

For a rotation matrix, $\det(R) = +1$. (A.16)

The determinant of R may be written using a standard expansion for the determinant of a matrix,

$$1 = \det(R) = \Sigma_{ijk}\varepsilon_{ijk}R_{i1}R_{j2}R_{k3} \,, \qquad (A.17)$$

where the permutation tensor was described in (1.5.3). This can be generalized to read

$$\varepsilon_{abc} = \Sigma_{ijk}\varepsilon_{ijk}R_{ia}R_{jb}R_{kc} \qquad \text{for any a,b,c} \qquad (A.18)$$

where (A.17) is the particular case with abc = 123. We leave the proof of (A.18) as a Reader Exercise.

Now apply $\Sigma_a R_{na}$ to both sides and sum on a to get

$$\Sigma_a R_{na}\,\varepsilon_{abc} = \Sigma_a R_{na}\,\Sigma_{ijk}\varepsilon_{ijk}R_{ia}R_{jb}R_{kc}$$

$$= \Sigma_{ijk}\varepsilon_{ijk}\,[\,\Sigma_a R_{na}R_{ia}]\,R_{jb}R_{kc}$$

$$= \Sigma_{ijk}\varepsilon_{ijk}\,[\delta_{n,i}]\,R_{jb}R_{kc} \qquad \text{// (A.15)}$$

$$= \Sigma_{jk}\,\varepsilon_{njk}R_{jb}R_{kc} \,.$$

We have just derived the following rarely-stated property of any 3x3 rotation matrix R :

$$\Sigma_a R_{na}\varepsilon_{abc} = \Sigma_{jk}\varepsilon_{njk}R_{jb}R_{kc} \qquad \text{for any n,b,c .} \qquad (A.19)$$

Theorem: If **A'** = R**A**, **B'** =R**B** and **C'** = R**C**, then **C** = **A** x **B** $\Leftrightarrow$ **C'** = **A'** x **B'** . (A.20)

Proof $\Leftarrow$: **C'** = **A'** x **B'**

$$\Rightarrow\ C'_n = \Sigma_{jk}\varepsilon_{njk}\,A'_jB'_k \quad \Rightarrow\ (R\mathbf{C})_n = \Sigma_{jk}\varepsilon_{njk}\,(R\mathbf{A})_j(R\mathbf{B})_k$$

$$\Rightarrow\ (\Sigma_a R_{na}C_a) = \Sigma_{jk}\varepsilon_{njk}(\Sigma_b R_{jb}A_b)(\Sigma_c R_{kc}B_c) =$$

$$= \Sigma_{bc}\,[\,\Sigma_{jk}\varepsilon_{njk}R_{jb}R_{kc}]\,A_bB_c$$

$$= \Sigma_{bc}\,[\,\Sigma_a R_{na}\varepsilon_{abc}]\,A_bB_c \qquad \text{// (A.19)}$$

$$= \Sigma_a R_{na}\,[\Sigma_{bc}\varepsilon_{abc}A_bB_c\,]$$

$$= \Sigma_a\ R_{na}\,[\mathbf{A} \text{ x } \mathbf{B}\,]_a \,.$$

In vector notation we have just shown that R**C** = R(**A**x**B**). Apply R^{-1} from the left to conclude that **C** = **A** x **B**, QED. Run the steps in reverse to prove $\Rightarrow$.

This theorem confirms the intuitive fact that if **A**,**B**,**C** all transform as normal vectors under R, then if **C** = **A**x**B** in Frame S, then **C'** = **A'**x**B'** in rotated Frame S' . The fact that **A**•**B** = **A'**•**B'** is more obvious and requires only (A.15) : $\Sigma_j(A')_j(B')_j = \Sigma_j(\Sigma_b R_{jb}A_b)(\Sigma_c R_{jc}B_c) = \Sigma_{bc}\,\delta_{bc}A_bB_c = \Sigma_b A_bB_b$.