A Review of Appendix C (as in the 10.24.12 release) PhL 11.26.16

I am looking for a G rule for tensors. I start with a claimed Lemma

Lemma 1: $(d[\mathbf{A}\otimes\mathbf{B}]/dt)_S = (d\mathbf{A}/dt)_S \otimes \mathbf{B} + \mathbf{A} \otimes (d\mathbf{B}/dt)_S$ (C.1)

What does the reader make of the ⊗ symbol which has suddenly appeared? I could just say

$$[\mathbf{A}\otimes\mathbf{B}]_{ij} = A_i B_j$$

and this is then a rank-2 tensor and that would explain ⊗. I write

$$\mathbf{A}\otimes\mathbf{B} = [\mathbf{A}\otimes\mathbf{B}]_{ab}\, \mathbf{e}_a \otimes \mathbf{e}_b \quad (C.4)$$

where $\mathbf{e}_n$ are the frames doc Frame S basis elements and I point out that $\partial_S(\mathbf{e}_n) = 0$. I claim that

$$[(d[\mathbf{A}\otimes\mathbf{B}]/dt)_S]_{ab} = [(d[\mathbf{A}\otimes\mathbf{B}]_{ab}/dt)_S] \quad . \quad (C.3)$$

I don't prove this, but why not prove it directly in this manner

$$\partial_S(\mathbf{A}\otimes\mathbf{B}) = \partial_S \{ [\mathbf{A}\otimes\mathbf{B}]_{ab}\, \mathbf{e}_a \otimes \mathbf{e}_b \}$$

$$= \partial_S \{ (\mathbf{A}\otimes\mathbf{B})_{ab} \}\ \mathbf{e}_a \otimes \mathbf{e}_b$$

Then

$$[\partial_S(\mathbf{A}\otimes\mathbf{B})]_{ij} = \partial_S \{ (\mathbf{A}\otimes\mathbf{B})_{ab} \}\ [\mathbf{e}_a \otimes \mathbf{e}_b]_{ij}$$

$$= \partial_S \{ (\mathbf{A}\otimes\mathbf{B})_{ab} \} (\mathbf{e}_a)_i (\mathbf{e}_b)_j$$

$$= \partial_S \{ (\mathbf{A}\otimes\mathbf{B})_{ab} \} \delta_{a,i}\, \delta_{b,j}$$

$$= \partial_S [(\mathbf{A}\otimes\mathbf{B})_{ij}]$$

This is really what (C.3) says. You can differentiate and then take component, or you can take component and then differentiate. I have presented this poorly in frames doc I think. Now I could go on with the above

$$= \partial_S [A_i B_j]$$

$$= (\partial_S A_i) B_j + A_i (\partial_S B_j) \quad \text{// just d/dt and product rule}$$

$$= (\partial_S A)_i\, B_j + A_i (\partial_S B)_j \quad \text{// using a previous result (d/dt on components)}$$

$$=[(\partial_s A) \otimes B]_{ij} + [A \otimes (\partial_s B)]_{ij}$$

$$= [\ (\partial_s A) \otimes B\ + A \otimes (\partial_s B)]_{ij}$$

and this I have shown that

$$[\partial_s(\mathbf{A} \otimes \mathbf{B})]_{ij} = [\ (\partial_s A) \otimes B\ + A \otimes (\partial_s B)]_{ij}$$

and therefore for the tensor itself

$$\partial_s(\mathbf{A} \otimes \mathbf{B}) = (\partial_s A) \otimes B\ + A \otimes (\partial_s B)$$

and this is the claim of Lemma 1, so I am happy with it.

Just fiddling now:

$$\mathbf{V} = V_i \mathbf{e}_i$$

$$\partial_s \mathbf{V}\ = (\partial_s V_i)\ \mathbf{e}_i + \ V_i\ (\partial_s \mathbf{e}_i)\ = \ (\partial_s V_i)\ \mathbf{e}_i\ = (\partial_t V_i)\ \mathbf{e}_i$$

On the other hand

$$\partial_{s'} \mathbf{V}\ = \ (\partial_{s'} V_i)\ \mathbf{e}_i + \ V_i\ (\partial_{s'} \mathbf{e}_i)\ = (\partial_t V_i) \mathbf{e}_i + V_i[\ - \boldsymbol{\omega} \times \mathbf{e}_i]$$

$$= \ (\partial_t V_i) \mathbf{e}_i - \omega \times (V_i \mathbf{e}_i)$$

$$= \ \partial_s \mathbf{V} - \omega \times \mathbf{V}$$

This seems to be a much simpler proof of the G Rule.

Now try this for a tensor

$$T = T_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j$$

$$\partial_s T = (\partial_t T_{ij})\ \mathbf{e}_i \otimes \mathbf{e}_j$$

On the other hand

$$\partial_{s'} T\ = (\partial_t T_{ij}) \mathbf{e}_i \otimes \mathbf{e}_j + T_{ij}\ \partial_{s'} (\mathbf{e}_i \otimes \mathbf{e}_j)$$

Now I need a Lemma which shows this

$$\partial_t(\mathbf{A} \otimes \mathbf{B})\ = (\partial_t \mathbf{A}) \otimes \mathbf{B}\ + \mathbf{A} \otimes\ (\partial_t \mathbf{B})$$

which I would show by looking at components. Then I can apply this to get

$$\partial_{s'}(\mathbf{e}_i \otimes \mathbf{e}_j) = (\partial_{s'}\mathbf{e}_i) \otimes \mathbf{e}_j + \mathbf{e}_i \otimes (\partial_{s'}\mathbf{e}_j)$$

$$= (-\boldsymbol{\omega} \times \mathbf{e}_i) \otimes \mathbf{e}_j + \mathbf{e}_i \otimes (-\boldsymbol{\omega} \times \mathbf{e}_j)$$

I don't think I can take this further, so insert into the above to get

$$\partial_{s'}T = (\partial_t T_{ij})\mathbf{e}_i \otimes \mathbf{e}_j + T_{ij}\,\partial_{s'}(\mathbf{e}_i \otimes \mathbf{e}_j)$$

$$= (\partial_t T_{ij})\mathbf{e}_i \otimes \mathbf{e}_j + T_{ij}\,[(-\boldsymbol{\omega} \times \mathbf{e}_i) \otimes \mathbf{e}_j + \mathbf{e}_i \otimes (-\boldsymbol{\omega} \times \mathbf{e}_j)]$$

$$= \partial_s T - T_{ij}\,[(\boldsymbol{\omega} \times \mathbf{e}_i) \otimes \mathbf{e}_j + \mathbf{e}_i \otimes (\boldsymbol{\omega} \times \mathbf{e}_j)]$$

Lets do ε notation to write

$$(\partial_{s'}T)_{ab} = (\partial_s T)_{ab} - T_{ij}\,[(\boldsymbol{\omega} \times \mathbf{e}_i) \otimes \mathbf{e}_j + \mathbf{e}_i \otimes (\boldsymbol{\omega} \times \mathbf{e}_j)]_{ab}$$

$$= (\partial_s T)_{ab} - T_{ij}\,[(\boldsymbol{\omega} \times \mathbf{e}_i)_a (\mathbf{e}_j)_b + (\mathbf{e}_i)_a (\boldsymbol{\omega} \times \mathbf{e}_j)_b]$$

$$= (\partial_s T)_{ab} - T_{ij}\,(\boldsymbol{\omega} \times \mathbf{e}_i)_a \delta_{j,b} - T_{ij}\delta_{i,a}(\boldsymbol{\omega} \times \mathbf{e}_j)_b]$$

$$= (\partial_s T)_{ab} - T_{ib}\,(\boldsymbol{\omega} \times \mathbf{e}_i)_a - T_{aj}(\boldsymbol{\omega} \times \mathbf{e}_j)_b]$$

$$= (\partial_s T)_{ab} - T_{ib}\,\varepsilon_{ars}\omega_r(\mathbf{e}_i)_s - T_{aj}\,\varepsilon_{brs}\omega_r(\mathbf{e}_j)_s]$$

$$= (\partial_s T)_{ab} - T_{ib}\,\varepsilon_{ars}\omega_r\delta_{i,s} - T_{aj}\,\varepsilon_{brs}\omega_r\delta_{j,s}]$$

$$= (\partial_s T)_{ab} - T_{sb}\,\varepsilon_{ars}\omega_r - T_{as}\,\varepsilon_{brs}\omega_r$$

$$= (\partial_s T)_{ab} - \varepsilon_{ars}\omega_r T_{sb} - \varepsilon_{brs}\omega_r T_{as}$$

So then

$$(\partial_s T)_{ab} = (\partial_{s'}T)_{ab} + \varepsilon_{ars}\omega_r T_{sb} + \varepsilon_{brs}\omega_r T_{as}$$

This agrees with what I already quote

$$(\partial_s A)_{ab} = (\partial_{s'}A)_{ab} + \varepsilon_{ars}\omega_r A_{sb} + \varepsilon_{brs}\omega_r A_{as} \qquad 2$$