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This was appendix C and is now appendix B.

Appendix C: The G Rule for a Tensor of Rank n

In this section we continue to use these shorthand operator notations,

$$\partial_S \equiv (d/dt)_S \qquad \partial_{S'} \equiv (d/dt)_{S'} \qquad \partial_t \equiv (d/dt)\ . \qquad (1.31) \qquad \text{(C.1)}$$

Recall from (1.37) that for a scalar function or for a component of any tensor one has,

$$(\partial_S T_{ijk\ldots}) = (\partial_{S'} T_{ijk\ldots}) = (\partial_t T_{ijk\ldots})\ , \qquad (1.37) \qquad \text{(C.2)}$$

and from (2.9) that

$$\partial_{S'} \mathbf{e}_i = -\boldsymbol{\omega} \times \mathbf{e}_i\ . \qquad (2.9) \qquad \text{(C.3)}$$

So far we know all about the G Rule for tensors of rank 0 and 1 (scalar and vector),

$$\partial_S A = \partial_{S'} A \qquad // = \partial_t A;\quad A \text{ is a scalar function of t} \qquad (C.2)$$

$$\partial_S \mathbf{A} = \partial_{S'} \mathbf{A} + \boldsymbol{\omega} \times \mathbf{A}\ . \qquad // \mathbf{A} \text{ is a vector} \qquad (2.8) \qquad \text{(C.4)}$$

What happens for tensors of rank 2 or more? To explore this question, we first collect a few facts.

A tensor of rank-n has the following expansion (where q represents the nth letter of the alphabet)

$$T = \Sigma_{abc..q} T_{abc..q}\ (\mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \ldots\ldots \otimes \mathbf{e}_q) \qquad \text{(C.5)}$$

Here T has n subscripts and there is a tensor product of n basis vectors. This expansion is a generalization of the expansion of a vector,

$$\mathbf{V} = \Sigma_i V_i \mathbf{e}_i\ . \qquad \text{(C.6)}$$

The basis vectors are axis-aligned unit vectors, so we have

$$(\mathbf{e}_n)_i = \delta_{n,i}\ . \qquad (1.3) \qquad \text{(C.7)}$$

The meaning of the $\otimes$ symbol is nothing more than the following,

$$[A \otimes B \otimes \ldots\ldots \otimes Q\ ]_{abc\ldots q} = A_a B_b C_c \ldots\ldots Q_q\ . \qquad \text{(C.8)}$$

This particular tensor $A \otimes B \otimes \ldots\ldots \otimes Q$ happens to be the direct product of the n vectors A,B,C...Q, but there are of course general tensors T which cannot be written as such a direct product.

As a simple tensor product example,

$$(\mathbf{e}_a \otimes \mathbf{e}_b)_{ij} = (\mathbf{e}_a)_i \, (\mathbf{e}_b)_j \; = \delta_{a,i} \, \delta_{b,j} \; . \qquad \text{(C.9)}$$

Here is the expansion of the direct product tensor T = **A**⊗**B**,

$$\mathbf{A}\otimes\mathbf{B} \; = \Sigma_{ab} \, [A\otimes B]_{ab} \; \mathbf{e}_a \otimes \mathbf{e}_b \; = \; \Sigma_{ab} \, A_a B_b \; \mathbf{e}_a \otimes \mathbf{e}_b \; . \qquad \text{(C.10)}$$

Before continuing, we develop two very similar Lemmas:

Lemma 1: $\quad \partial_s(\mathbf{A}\otimes\mathbf{B}) = \; (\partial_s \mathbf{A}) \otimes \mathbf{B} \; + \mathbf{A} \otimes (\partial_s \mathbf{B}) \qquad \text{(C.11)}$

where **A** and **B** are vectors whose components are functions of time.

Proof of Lemma 1:

$\partial_s(\mathbf{A}\otimes\mathbf{B}) \; = \partial_s(\, \Sigma_{ab} \, A_a B_b \; \mathbf{e}_a \otimes \mathbf{e}_b \,)$ // expansion (C.10)

$= \; \Sigma_{ab} \, \partial_s (A_a B_b) \; \mathbf{e}_a \otimes \mathbf{e}_b$ // since $\partial_s \mathbf{e}_n = 0$ by (1.27)

$= \; \Sigma_{ab} \, \partial_t (A_a B_b) \; \mathbf{e}_a \otimes \mathbf{e}_b$ // (C.2)

$= \; \Sigma_{ab} \, [(\partial_t A_a) B_b + A_a (\partial_t B_b)] \; \mathbf{e}_a \otimes \mathbf{e}_b$ // regular calculus product rule

$= \; \Sigma_{ab} \, [(\partial_s A_a) B_b + A_a (\partial_s B_b)] \; \mathbf{e}_a \otimes \mathbf{e}_b$ // restore ∂_s using (C.2)

$= \; \Sigma_{ab} \, [(\partial_s \mathbf{A})_a B_b + A_a (\partial_s \mathbf{B})_b] \; \mathbf{e}_a \otimes \mathbf{e}_b$ /// commutation rule (1.40b)

$= \Sigma_{ab} \, [(\partial_s \mathbf{A})_a B_b] \; \mathbf{e}_a \otimes \mathbf{e}_b + \Sigma_{ab} \; A_a (\partial_s \mathbf{B})_b \; \mathbf{e}_a \otimes \mathbf{e}_b$ // write as two terms

$= \; (\partial_s \mathbf{A}) \otimes \mathbf{B} \; + \mathbf{A} \otimes (\partial_s \mathbf{B}) \; .$ // expansion (C.10) QED

Recall from (1.5) that one can expand a vector **A** on either the $\mathbf{e}_i$ or the $\mathbf{e}'_i$ basis,

$$\mathbf{A} = \Sigma_i A_i \mathbf{e}_i \; = \Sigma_i A'_i \mathbf{e}'_i \; . \qquad \textit{(1.5)} \qquad \text{(C.12)}$$

Similarly, one can expand the tensor T of (C.5) on the $\mathbf{e}'_i$ basis to get

$$T = \Sigma_{abc..q} . T'_{abc..q} \; (\mathbf{e}'_a \otimes \mathbf{e}'_b \otimes \mathbf{e}'_c ..... \otimes \mathbf{e}'_q) \qquad \text{(C.13)}$$

with this special case

$$\mathbf{A}\otimes\mathbf{B} \; = \Sigma_{ab} \, [A\otimes B]'_{ab} \; \mathbf{e}'_a \otimes \mathbf{e}'_b \; = \; \Sigma_{ab} \, A'_a B'_b \; \mathbf{e}'_a \otimes \mathbf{e}'_b \; . \qquad \text{(C.14)}$$

We then have Lemma 2 which is the same as Lemma 1 but with S→S' ,

Lemma 2: $\partial_{\mathbf{s}'}(\mathbf{A}\otimes\mathbf{B}) = (\partial_{\mathbf{s}'}\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_{\mathbf{s}'}\mathbf{B})$ (C.15)

The proof exactly follows that of Lemma 1 but we show it anyway:

$\partial_{\mathbf{s}'}(\mathbf{A}\otimes\mathbf{B}) = \partial_{\mathbf{s}'}(\Sigma_{\mathbf{ab}} A'_{\mathbf{a}}B'_{\mathbf{b}}\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}})$ // expansion (C.14)

$= \Sigma_{\mathbf{ab}} \partial_{\mathbf{s}'}(A'_{\mathbf{a}}B'_{\mathbf{b}})\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // since $\partial_{\mathbf{s}'}\mathbf{e}'_{\mathbf{n}} = 0$ by (1.26)

$= \Sigma_{\mathbf{ab}} \partial_{\mathbf{t}}(A'_{\mathbf{a}}B'_{\mathbf{b}})\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // (C.2)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{t}}A'_{\mathbf{a}})B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{t}}B'_{\mathbf{b}})]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // regular calculus product rule

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}A'_{\mathbf{a}})B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{s}'}B'_{\mathbf{b}})]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // restore $\partial_{\mathbf{s}'}$ using (C.2)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}\mathbf{A})'_{\mathbf{a}}B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{s}}\mathbf{B})'_{\mathbf{b}}]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // commutation rule (1.40b)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}\mathbf{A})'_{\mathbf{a}}B'_{\mathbf{b}}]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}} + \Sigma_{\mathbf{ab}}\ A'_{\mathbf{a}}(\partial_{\mathbf{s}}\mathbf{B})'_{\mathbf{b}}\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // write as two terms

$= (\partial_{\mathbf{s}'}\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_{\mathbf{s}}\mathbf{B})$. // expansion (C.14) QED

G Rule for a Rank-2 Tensor

Recall from (C.5) that,

$\mathrm{T} = \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. *(C.5)* (C.16)

Then apply $\partial_{\mathbf{s}}$, using the facts $\partial_{\mathbf{s}}\mathbf{e}_{\mathbf{n}} = 0$ and (C.2),

$(\partial_{\mathbf{s}}\mathrm{T}) = \Sigma_{\mathbf{ab}}(\partial_{\mathbf{s}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) = \Sigma_{\mathbf{ab}}(\partial_{\mathbf{t}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. (C.17)

On the other hand,

$(\partial_{\mathbf{s}'}\mathrm{T}) = \Sigma_{\mathbf{ab}}\ (\partial_{\mathbf{s}'}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$

$= \Sigma_{\mathbf{ab}}\ (\partial_{\mathbf{t}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$ // (C.2)

$= (\partial_{\mathbf{s}}\mathrm{T}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. // (C.17) (C.18)

But,

$\partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) = (\partial_{\mathbf{s}'}\mathbf{e}_{\mathbf{a}})\otimes\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}}\otimes(\partial_{\mathbf{s}'}\mathbf{e}_{\mathbf{b}})$ // by Lemma 2 (C.15)

$$= \; - \; [\, (\boldsymbol{\omega} \times \mathbf{e}_a) \otimes \mathbf{e}_b \; + \; \mathbf{e}_a \otimes (\boldsymbol{\omega} \times \mathbf{e}_b) \,] \; . \qquad \text{// (C.3)} \tag{C.19}$$

Inserting this last result into (C.18) and swapping sides then gives the G rule for a rank-2 tensor,

$$(\partial_S T) \; = (\partial_{S'} T) \; + \; \Sigma_{ab} T_{ab} \, [\, (\boldsymbol{\omega} \times \mathbf{e}_a) \otimes \mathbf{e}_b \; + \; \mathbf{e}_a \otimes (\boldsymbol{\omega} \times \mathbf{e}_b) \,] \; . \tag{C.20}$$

We can write the G rule for a vector $\mathbf{T}$ (rank-1 tensor) in a similar form,

$$(\partial_S \mathbf{T}) = (\partial_{S'} \mathbf{T}) \; + \; \boldsymbol{\omega} \times \mathbf{T}$$

$$= (\partial_{S'} \mathbf{T}) \; + \; \boldsymbol{\omega} \times (\Sigma_a T_a \mathbf{e}_a)$$

$$= (\partial_{S'} \mathbf{T}) \; + \; \Sigma_a T_a \; \boldsymbol{\omega} \times \mathbf{e}_a \tag{C.21}$$

G Rule for a Rank-n Tensor

It is easy to show that the above general pattern applies to tensors of rank 3 and higher, and one ends up with the following lowest tensor G rules, where the tensor rank is shown on the left .

0 $\quad (\partial_S T) = (\partial_{S'} T)$

1 $\quad (\partial_S \mathbf{T}) = (\partial_{S'} \mathbf{T}) \; + \; \Sigma_a T_a \; \boldsymbol{\omega} \times \mathbf{e}_a$

2 $\quad (\partial_S T) = (\partial_{S'} T) \; + \; \Sigma_{ab} T_{ab} \, [\, (\boldsymbol{\omega} \times \mathbf{e}_a) \otimes \mathbf{e}_b \; + \; \mathbf{e}_a \otimes (\boldsymbol{\omega} \times \mathbf{e}_b) \,]$

3 $\quad (\partial_S T) = (\partial_{S'} T) \; + \; \Sigma_{abc} T_{abc} \, [\, (\boldsymbol{\omega} \times \mathbf{e}_a) \otimes \mathbf{e}_b \otimes \mathbf{e}_c \; + \; \mathbf{e}_a \otimes (\boldsymbol{\omega} \times \mathbf{e}_b) \otimes \mathbf{e}_c \; + \mathbf{e}_a \otimes \mathbf{e}_b \otimes (\boldsymbol{\omega} \times \mathbf{e}_c) \,]$ (C.22)

These G rule results may also be expressed in components. For example:

1 $\quad (\partial_S \mathbf{T})_i = (\partial_{S'} \mathbf{T})_i \; + \; \Sigma_a T_a \; (\boldsymbol{\omega} \times \mathbf{e}_a)_i$

$$= (\partial_{S'} \mathbf{T})_i \; + \; \Sigma_a T_a \; \varepsilon_{irs} \, \omega_r \, (\mathbf{e}_a)_s$$

$$= (\partial_{S'} \mathbf{T})_i \; + \; \Sigma_a T_a \; \varepsilon_{irs} \, \omega_r \, \delta_{a,s}$$

$$= (\partial_{S'} \mathbf{T})_i \; + \; \varepsilon_{irs} \, \omega_r T_s \quad . \tag{C.23}$$

2 $\quad (\partial_S T)_{ij} = (\partial_{S'} T)_{ij} \; + \; \Sigma_{ab} T_{ab} \, [\, (\boldsymbol{\omega} \times \mathbf{e}_a) \otimes \mathbf{e}_b \; + \; \mathbf{e}_a \otimes (\boldsymbol{\omega} \times \mathbf{e}_b) \,]_{ij}$

$$= (\partial_{S'} T)_{ij} \; + \; \Sigma_{ab} T_{ab} \, [\, (\boldsymbol{\omega} \times \mathbf{e}_a)_i \, (\mathbf{e}_b)_j \; + \; (\mathbf{e}_a)_i (\boldsymbol{\omega} \times \mathbf{e}_b)_j \,]$$

$$= (\partial_{S'} T)_{ij} \; + \; [\, \Sigma_{ab} \; T_{ab} \, (\boldsymbol{\omega} \times \mathbf{e}_a)_i \, \delta_{b,j} \; + \; \Sigma_{ab} \; T_{ab} \, \delta_{a,i} \, (\boldsymbol{\omega} \times \mathbf{e}_b)_j \,]$$

$$= (\partial_{s'}T)_{ij} + [\, \Sigma_a\, T_{aj}\, (\omega \times \mathbf{e}_a)_i + \Sigma_b\, T_{ib}\, (\omega \times \mathbf{e}_b)_j \,]$$

$$= (\partial_{s'}T)_{ij} + [\, \Sigma_a\, T_{aj}\, \varepsilon_{irs}\omega_r(\mathbf{e}_a)_s + \Sigma_b\, T_{ib}\, \varepsilon_{jrs}\omega_r(\mathbf{e}_b)_s \,]$$

$$= (\partial_{s'}T)_{ij} + [\, \Sigma_a\, T_{aj}\, \varepsilon_{irs}\omega_r\delta_{a,s} + \Sigma_b\, T_{ib}\, \varepsilon_{jrs}\omega_r\delta_{b,s} \,]$$

$$= (\partial_{s'}T)_{ij} + [\, T_{sj}\, \varepsilon_{irs}\omega_r + T_{is}\, \varepsilon_{jrs}\omega_r \,]$$

$$= (\partial_{s'}T)_{ij} + \varepsilon_{irs}\omega_r\, T_{sj} + \varepsilon_{jrs}\omega_r T_{is} \quad . \qquad \text{(C.24)}$$

We shall do one more case to establish the general pattern,

3 $(\partial_s T)_{ijk} - (\partial_{s'}T)_{ijk} = \Sigma_{abc}T_{abc}\, [\, (\omega x \mathbf{e}_a)\otimes\mathbf{e}_b\otimes\mathbf{e}_c + \mathbf{e}_a\otimes(\omega x \mathbf{e}_b)\otimes\mathbf{e}_c + \mathbf{e}_a\otimes\mathbf{e}_b\otimes(\omega x \mathbf{e}_c)\,]_{ijk}$

$= \Sigma_{abc}T_{abc}\, [\, (\omega x \mathbf{e}_a)_i(\mathbf{e}_b)_j(\mathbf{e}_c)_k + (\mathbf{e}_a)_i(\omega x \mathbf{e}_b)_j(\mathbf{e}_c)_k + (\mathbf{e}_a)_i(\mathbf{e}_b)_j(\omega x \mathbf{e}_c)_k \,]$

$= \Sigma_{abc}T_{abc}(\omega x \mathbf{e}_a)_i\delta_{b,j}\delta_{c,k} + \Sigma_{abc}T_{abc}\delta_{a,i}(\omega x \mathbf{e}_b)_j\delta_{c,k} + \Sigma_{abc}T_{abc}\delta_{a,i}\,\delta_{b,j}(\omega x \mathbf{e}_c)_k$

$= \Sigma_a T_{ajk}(\omega x \mathbf{e}_a)_i + \Sigma_b T_{ibk}(\omega x \mathbf{e}_b)_j + \Sigma_c T_{ijc}(\omega x \mathbf{e}_c)_k$

$= \Sigma_a T_{ajk}\varepsilon_{irs}\omega_r\,(\mathbf{e}_a)_s + \Sigma_b T_{ibk}\varepsilon_{jrs}\omega_r\,(\mathbf{e}_b)_s + \Sigma_c T_{ijc}\varepsilon_{krs}\omega_r\,(\mathbf{e}_c)_s$

$= \Sigma_a T_{ajk}\varepsilon_{irs}\omega_r\,\delta_{a,s} + \Sigma_b T_{ibk}\varepsilon_{jrs}\omega_r\,\delta_{b,s} + \Sigma_c T_{ijc}\varepsilon_{krs}\omega_r\delta_{c,s}$

$= T_{sjk}\varepsilon_{irs}\omega_r + T_{isk}\varepsilon_{jrs}\omega_r + T_{ijs}\varepsilon_{krs}\omega_r$

so we conclude that

$$(\partial_s T)_{ijk} = (\partial_{s'}T)_{ijk} + \varepsilon_{irs}\omega_r T_{sjk} + \varepsilon_{jrs}\omega_r T_{isk} + \varepsilon_{krs}\omega_r T_{ijs}\,. \qquad \text{(C.25)}$$

Looking at the last two results we can infer the rank-4 result

$$(\partial_s T)_{ijk\ell} = (\partial_{s'}T)_{ijk\ell} + \varepsilon_{irs}\omega_r T_{sjk\ell} + \varepsilon_{jrs}\omega_r T_{isk\ell} + \varepsilon_{krs}\omega_r T_{ijs\ell} + \varepsilon_{\ell rs}\omega_r T_{ijks} \qquad \text{(C.26)}$$

Summary of G rule for all ranks (tensor form) (C.27)

0 $(\partial_s T) = (\partial_{s'}T)$

1 $(\partial_s \mathbf{T}) = (\partial_{s'}\mathbf{T}) + \Sigma_a T_a\ \omega \times \mathbf{e}_a$

2 $(\partial_s T) = (\partial_{s'}T) + \Sigma_{ab}T_{ab}\, [\, (\omega \times \mathbf{e}_a)\otimes\mathbf{e}_b + \mathbf{e}_a\otimes(\omega \times \mathbf{e}_b)\,]$

3 $(\partial_s T) = (\partial_{s'}T) + \Sigma_{abc}T_{abc}\, [\, (\omega x \mathbf{e}_a)\otimes\mathbf{e}_b\otimes\mathbf{e}_c + \mathbf{e}_a\otimes(\omega x \mathbf{e}_b)\otimes\mathbf{e}_c + \mathbf{e}_a\otimes\mathbf{e}_b\otimes(\omega x \mathbf{e}_c)\,]$

$$4 \quad (\partial_S T) = (\partial_{S'} T) + \Sigma_{abcd} T_{abcd} [\, (\omega x \mathbf{e}_a) \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes \mathbf{e}_d + \mathbf{e}_a \otimes (\omega x \mathbf{e}_b) \otimes \mathbf{e}_c \otimes \mathbf{e}_d + \mathbf{e}_a \otimes \mathbf{e}_b \otimes (\omega x \mathbf{e}_c) \otimes \mathbf{e}_d + \mathbf{e}_a \otimes \mathbf{e}_b \otimes \mathbf{e}_c \otimes (\omega x \mathbf{e}_d) \,]$$

etc.

Summary of G rule for all ranks (component form) (C.28)

$$0 \quad (\partial_S T) = (\partial_{S'} T)$$

$$1 \quad (\partial_S \mathbf{T})_i = (\partial_{S'} \mathbf{T})_i + \varepsilon_{irs}\, \omega_r T_s = (\partial_{S'} \mathbf{T})_i + (\boldsymbol{\omega} \times \mathbf{T})_i$$

$$2 \quad (\partial_S T)_{ij} = (\partial_{S'} T)_{ij} + \varepsilon_{irs} \omega_r T_{sj} + \varepsilon_{jrs} \omega_r T_{is}$$

$$3 \quad (\partial_S T)_{ijk} = (\partial_{S'} T)_{ijk} + \varepsilon_{irs} \omega_r T_{sjk} + \varepsilon_{jrs} \omega_r T_{isk} + \varepsilon_{krs} \omega_r T_{ijs}$$

$$4 \quad (\partial_S T)_{ijk\ell} = (\partial_{S'} T)_{ijk\ell} + \varepsilon_{irs} \omega_r T_{sjk\ell} + \varepsilon_{jrs} \omega_r T_{isk\ell} + \varepsilon_{krs} \omega_r T_{ijs\ell} + \varepsilon_{\ell rs} \omega_r T_{ijks}$$

etc.

In the main body of our document we deal only with vector quantities **r**, **v** and **a** , but one can easily imagine "frames of reference problems" which deal also with tensors, such as the inertia tensor.