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C.4 The Simple Pendulum

In this section we temporarily turn off the rotation of the Earth to study the behavior of pendulum without that complication. Setting $\omega = 0$ in (C.3.9) the pendulum equations of motion become

$$\begin{aligned} &\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/\ell)\sin\theta && \hat{\boldsymbol{\theta}} \\ &2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \text{(C.4.1)}$$

We now seek a solution of (C.4.1) for which $\dot{\varphi} = 0$. In this case the last equation requires $\ddot{\varphi} = 0$ and we are left with just two equations,

$$\begin{aligned} &\dot{\theta}^2 = -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ &\ddot{\theta} = -(g/\ell)\sin\theta \quad . && \hat{\boldsymbol{\theta}} \end{aligned} \qquad \text{(C.4.2)}$$

These equations describe a pendulum that seems to swing in the plane φ = constant and so this is an example of a plane pendulum, also known as a simple pendulum. One can solve the second equation for $\theta(t)$, and then the first equation gives the tension $T(t)$.

If the pendulum is started at some polar angle θ_0 and released perfectly so $\varphi = \varphi_0$ and $\dot{\varphi} = 0$, the pendulum swings in a plane, but at it swings through $\theta = 0$ the coordinate $\varphi = \varphi_0$ discontinuously jumps to $\varphi = \varphi_0 + \pi$, so there is a continuity issue for φ at $\theta = 0$. We could treat this technically using Heaviside and delta functions, but shall not go down that road. It is a simple fact that, for spherical coordinates, points on the z-axis where $\theta = 0$ have an undefined value of φ.

For small angles the θ one finds for the above initial condition that

$$\ddot{\theta} + (g/\ell)\theta = 0 \;\Rightarrow\; \theta(t) = \theta_0\cos(\Omega t) \quad \Omega \equiv \sqrt{g/\ell} \qquad \{\theta(0) = \theta_0\,.\dot{\theta}(0) = 0\} \qquad \text{(C.4.3)}$$

On the other hand, if the pendulum is initialized with $\theta = 0$ and some velocity sufficient to take it up to a maximum angle θ_0 we get

$$\ddot{\theta} + (g/\ell)\theta = 0 \;\Rightarrow\; \theta(t) = \theta_0\sin(\Omega t) \quad \Omega \equiv \sqrt{g/\ell} \qquad \{\theta(0) = 0,\ \theta_{\mathbf{max}} = \theta_0\} \qquad \text{(C.4.4)}$$

Exact Solution for the Simple Pendulum

We provide this detailed solution because it does not generally appear in textbooks.

For larger θ_0 the simple pendulum equation $\ddot{\theta} + (g/\ell)\sin\theta = 0$ falls into a class of second order non-linear ODE's which have the form $\ddot{x} = f(x)$ where $\ddot{x}$ means $\partial_t^2 x$ and we seek x(t). The solution is not hard to obtain, as we now outline. The first step is to define $v \equiv \dot{x}$:

$$v \equiv \dot{x} \quad \Rightarrow \quad \ddot{x} = (dv/dt) = (dv/dx)(dx/dt) = (dv/dx)v \quad \Rightarrow \quad (dv/dx)v = f(x)$$

$$\Rightarrow \quad vdv = f(x)dx \quad \Rightarrow \quad (v^2/2) = \int^x f(x')dx' + C$$

$$\Rightarrow \quad (dx/dt) = \sqrt{2}\ \sqrt{\int^x f(x')dx' + C} \quad \Rightarrow \quad dt = \frac{1}{\sqrt{2}}\ \frac{dx}{\sqrt{\int^x f(x')dx' + C}}$$

$$\Rightarrow \quad t(x) = \left(\frac{1}{\sqrt{2}}\ \int^x \frac{dx''}{\sqrt{\int^{x''} f(x')dx' + C}}\right) + C' \qquad \text{(C.4.5)}$$

where integration constants C and C' are determined by initial conditions. This solution gives t = t(x) which one must then "invert" to obtain x = x(t).

If we take $f(x) = -\Omega^2\sin x$ and then $x \rightarrow \theta$ the $(v^2/2)$ result in (C.4.5) becomes

$$(\dot{\theta}^2/2) = \int^\theta f(\theta')d\theta' + C = -\Omega^2 \int^\theta \sin(\theta')d\theta' + C = \Omega^2\cos\theta + C\ . \qquad \text{(C.4.6)}$$

We set the zero of potential energy for the pendulum at the bottom, $\theta = 0$. At angle θ the pendulum has risen a height $h = \ell - \ell\cos\theta$ so the potential energy at θ is then $V = mgh = mg\ell(1-\cos\theta)$.

We shall now apply the boundary conditions of (C.4.4). At t = 0 the pendulum is at $\theta = 0$ and we give it a kick in the $+\varphi$ direction with some initial velocity $\ell\dot{\theta}(0)$. We assume that this causes the pendulum to rise up to some max angle $\theta_0 < \pi$. A too-large kick results in over-the-top behavior which we exclude.

At t = 0 the total pendulum energy is $(1/2)m[\ \ell\dot{\theta}(0)]^2$.

At the top of the swing the total energy is $mg\ell(1-\cos\theta)$. Therefore from energy conservation,

$$(1/2)m[\ \ell\dot{\theta}(0)]^2 = mg\ell(1-\cos\theta)$$

or

$$(1/2)[\ \dot{\theta}(0)]^2 = (g/\ell)(1-\cos\theta_0) = \Omega^2(1-\cos\theta_0)\ . \qquad \text{(C.4.7)}$$

Evaluating (C.4.6) at t = 0 gives

$$(1/2)[\ \dot{\theta}(0)]^2 = \Omega^2\cos(0) + C = \Omega^2 + C$$

Comparing these last two equations one concludes that.

$$C = -\Omega^2\cos\theta_0 \ . \qquad \text{(C.4.8)}$$

From (C.4.5) the solution for t(θ) is then

$$\Rightarrow\ t(\theta) = \left(\frac{1}{\sqrt{2}} \int^{\theta} \frac{d\theta''}{\sqrt{\int^{\theta''} f(\theta')d\theta' + C}}\right) + C' = \left(\frac{1}{\sqrt{2}} \int^{\theta} \frac{d\theta''}{\sqrt{\Omega^2\cos\theta'' + C}}\right) + C'$$

$$= \left(\frac{1}{\sqrt{2}\,\Omega} \int^{\theta} \frac{d\theta''}{\sqrt{\cos\theta'' - \cos\theta_0}}\right) + C' \qquad \text{(C.4.9)}$$

The integral of interest appears on page 179 of GR7 (where we shall use the second form),

4. $$\int \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2}, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$

$$= \sqrt{\frac{2}{b}} F\left(\gamma, \frac{1}{r}\right) \qquad \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$

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$$\gamma = \arcsin\sqrt{\frac{b\,(1-\cos x)}{a+b}}, \quad r = \sqrt{\frac{2b}{a+b}} \qquad \text{(C.4.10)}$$

and where F(φ,k) is "the elliptic integral of the first kind". In our case b = 1 and a = $-\cos\theta_0$ so

$$r = \sqrt{\frac{2}{a+1}} = \sqrt{\frac{2}{1-\cos\theta_0}} = \frac{1}{\sin(\theta_0/2)}$$

$$\gamma = \sin^{-1}\left[\sqrt{\frac{1-\cos\theta}{1-\cos\theta_0}}\ \right] = \sin^{-1}\left[\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right] \ . \qquad \text{(C.4.11)}$$

Therefore,

$$t(\theta) = \frac{1}{\Omega} F\left(\sin^{-1}\left[\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right], \sin(\theta_0/2)\right) + C'$$

$$= \frac{1}{\Omega} F(\varphi, k) + C' \qquad \text{where} \quad \varphi = \sin^{-1}\left[\frac{\sin(\theta/2)}{k}\right], \qquad k = \sin(\theta_0/2) \qquad \text{(C.4.12)}$$

The function F is defined in GR7 on page 860,

2. The elliptic integral of the first kind:

$$F(\varphi,k)=\int_0^{\varphi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}=\int_0^{\sin\varphi}\frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}} \qquad \text{(C.4.13)}$$

from which we see that F(0,k) = 0. Recall our boundary condition that θ(0) = 0 with some $\dot{\theta}(0) > 0$. From (C.4.12) we find at t = 0 and θ = 0 that

$$0 = \frac{1}{\Omega}\,F(0,\,k) + C' = 0 + C' \qquad \Rightarrow \qquad C' = 0\,. \qquad \text{(C.4.14)}$$

Our final solution before inversion is then

$$t(\theta) = \frac{1}{\Omega}\,F(\varphi,\,k) \qquad \text{where} \quad \varphi = \sin^{-1}\left[\frac{\sin(\theta/2)}{k}\right],\ k = \sin(\theta_0/2), \qquad \Omega \equiv \sqrt{g/\ell} \qquad \text{(C.4.15)}$$

The Jacobi elliptic function sn(u) is usually defined in this rather obscure manner,

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1. $$u=\int_0^{\mathrm{sn}\,u}\frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}} \qquad \text{(C.4.16)}$$

If one writes sn(u) = sin(φ), then the right sides of (C.4.13) and (C.4.16) are the same, so

$$F(\varphi,k) = u = \mathrm{sn}^{-1}(\sin\varphi) \qquad \text{or} \qquad \mathrm{sn}(F(\varphi,k)) = \sin\varphi\,. \qquad \text{(C.4.17)}$$

We now apply this last equation to (C.4.15) as follows:

$$\mathrm{sn}(F(\varphi,k)) = \sin\varphi$$

or

$$\mathrm{sn}(\Omega t) = \sin\left\{\sin^{-1}\left[\frac{\sin(\theta/2)}{k}\right]\right\} = \frac{\sin(\theta/2)}{k}$$

so

$$\sin(\theta/2) = k\,\mathrm{sn}(\Omega t) \qquad k = \sin(\theta_0/2)$$

and

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\Omega t)) \qquad k = \sin(\theta_0/2) \qquad \text{(C.4.18)}$$

and the inversion of (C.4.15) is now complete. Following the convention of GR7, since sn(u) has an implicit parameter k, we make it explicit by writing sin(u) = sin(u,k) and then our solution is

$$\sin(\theta/2) = k\,\mathrm{sn}(\Omega t,k) \qquad k = \sin(\theta_0/2)$$

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\Omega t,k)) \qquad k = \sin(\theta_0/2) \qquad \text{(C.4.19)}$$

or

$$\sin(\theta/2) = \sin(\theta_0/2)\ \mathrm{sn}(\Omega t, \sin(\theta_0/2))$$
$$\theta(t) = 2\sin^{-1}[\ \sin(\theta_0/2)\ \mathrm{sn}(\Omega t, \sin(\theta_0/2))\ ] \qquad \text{(C.4.20)}$$

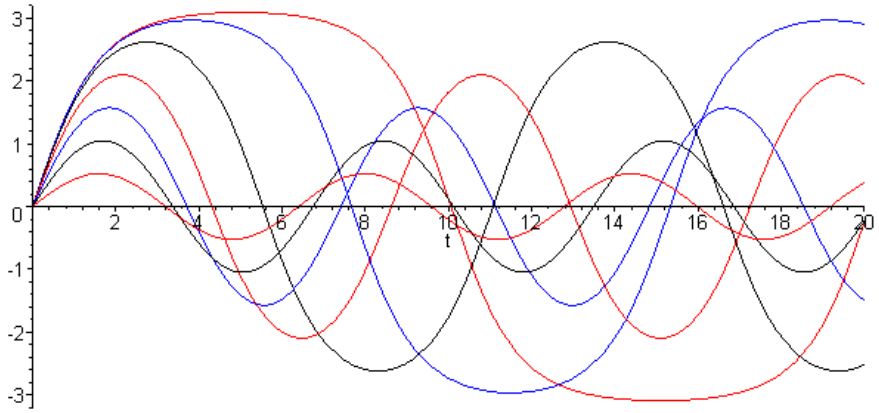
When k is not too large, one has $\mathrm{sn}(x,k) \approx \sin(x)$. If θ_0 is small, then so is θ and the first line of (C.4.20) reads,

$$\theta/2 \approx (\theta_0/2)\sin(\Omega t) \qquad \Rightarrow \qquad \theta(t) = \theta_0 \sin(\Omega t) \qquad \text{(C.4.21)}$$

which is the correct small-angle solution for $\theta(0) = 0$ and max angle θ_0 as was stated in (C.4.4).

Here is a Maple plot of $\theta(t)$ from (C.4.20) with a selection of peak angles θ_0 with $\Omega = 2\pi$so that the small-angle period is $T = 2\pi/\Omega = 1$:

```
restart; alias(sn=JacobiSN): with(plots):
theta := 2*arcsin(k*sn(Omega*t,k));
                                θ := 2 arcsin(k sn(Ω t, k))
k := sin(theta0/2):  Omega := 1:
theta0 := th0deg * (Pi/180):
deg := [30,60,90,120,150,170,177]:  n := 7:
col := [red,black,blue,red,black,blue,red]:
for i from 1 to n do
   th0deg := deg[i];
   s[i] := plot(theta,t=0..20, color=[col[i]]):
od:
display(seq(s[i],i=1..n));
```



(C.4.22)

For small θ_0 the function $\theta(t)$ is very sine-like with period $T = 1$, but for larger θ_0 the period increases and the top flattens out. For $\theta_0 = 177°$ the top is quite flat, indicating a long "hang time" when the pendulum (on its rigid massless stick) lingers in the near-vertical position.

The *complete* elliptic integral of the first kind is defined by (some sources capitalize the letter K),

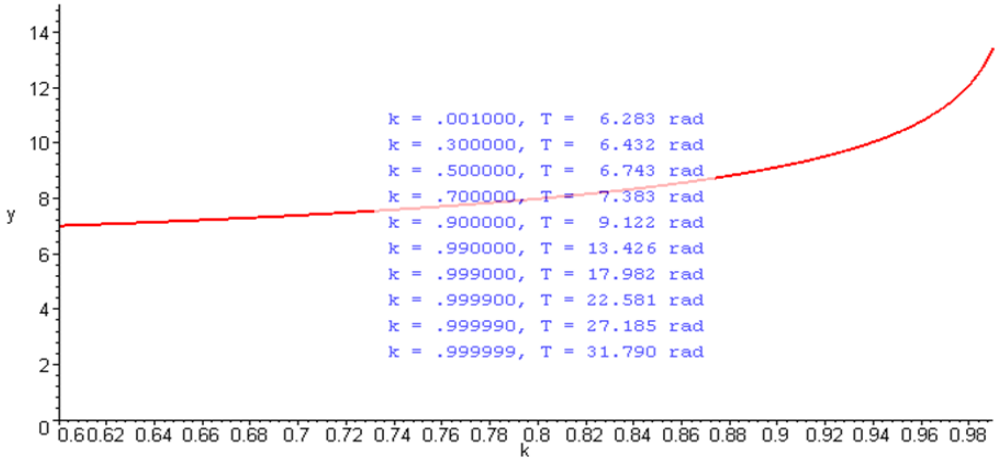
$$K(k) = F(\pi/2,k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \qquad \text{// see (C.4.13).} \qquad \text{(C.4.23)}$$

Since sn(u) = sin(φ) as shown above (C.4.17), we know that the peak value of sn(u) is 1 and this occurs one quarter of a wave into the sn(u) waveform. From (C.4.17) we know that

$$\text{sn}(F(\varphi,k)) = \sin\varphi \quad \Rightarrow \quad \text{sn}(F(\pi/2,k)) = 1 \quad \Rightarrow \quad \text{sn}(K(k)) = 1 \qquad \text{(C.4.23)}$$

Thus, a quarter period of the function sn(u,k) along the real u axis is just K(k) and so the full period is then T = 4K(k). Here is a plot of 4K(k) showing how it increases from 2π to larger values slowly approaching the limit 4K(∞) = ∞ :

```
plot(4*EllipticK(k),k = 0..0.99,y=0..10,thickness=2);
```



(C.4.24)

A quarter period of sn(Ωt,k) is then K(k)/Ω. For the more standard boundary conditions shown in (C.4.3) where θ(0) = θ_0 and $\dot{\theta}(0) = 0$, the solution is obtained by shifting (C.4.20) ahead by a quarter period, which means taking t → t + K(k)/Ω or Ωt → Ωt + K(k) . The result is then, with k = sin(θ_0/2),

$$\sin(\theta/2) = \sin(\theta_0/2)\ \text{sn}(\Omega t + K(\sin(\theta_0/2)), \sin(\theta_0/2))$$

$$\theta(t) = 2\sin^{-1}[\ \sin(\theta_0/2)\ \text{sn}(\Omega t + K(\sin(\theta_0/2)), \sin(\theta_0/2))\ ] \qquad \theta(0) = \theta_0 \qquad \dot{\theta}(0) = 0 \qquad \text{(C.4.25)}$$

Now for small angles the first line gives

$$\theta/2 \approx \theta_0/2\ \sin(\Omega t+\pi/2) = \theta_0/2\ \cos(\Omega t) \quad \Rightarrow \quad \theta(t) = \theta_0\ \cos(\Omega t) \qquad \text{(C.4.26)}$$

in agreement with (C.4.3)

Conventions: We have adopted the notation for K(k) and sn(u,k) which is used by the 2010 *NIST Handbook of Mathematical Functions*, by Gradshteyn and Ryzhik 7th edition (2007), and by the Bateman Manuscript Project (1953), though the last two sources write K as **K**. The reader is warned that there is

another common convention which appears in the precursor to the NIST Handbook, namely the heavily-used 1964 *Handbook of Mathematical Functions* of Abramowitz and Segun (AS). The connection is this

$$\begin{aligned} K(k) &= K_{AS}(k^2) && // \ k^2 = m \\ sn(u,k) &= sn_{AS}(u,k^2) = sn(u \mid k^2) && // \text{ sometimes} = sn(u; k^2) \end{aligned} \qquad (C.4.27)$$

In using these functions one must be very careful to learn the convention used by a given source. For example, ancient Maple V states that (notice the k^2 in the integral but k in the argument),

- JacobiAM is the inverse of the trigonometric form of the elliptic integral of the first kind, EllipticF:

```
EllipticF(phi,k) = int(1/sqrt(1-k^2*sin(theta)^2),theta=0..phi)
JacobiAM(EllipticF(sin(phi),k),k) = phi
```

```
JacobiSN(z,k) = sin(JacobiAM(z,k))
```

So this version of Maple is using our "modern" NIST 2010 notation for argument k.