

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/259312490>

Study of the Apsidal Precession of the Physical Symmetrical Pendulum

Article in *Journal of Applied Mechanics* · December 2013

DOI: 10.1115/1.4029470 · Source: arXiv

CITATIONS

0

READS

43

3 authors, including:



[Rodolfo A. Diaz](#)

National University of Colombia

65 PUBLICATIONS 398 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Theoretical constraints and gauge theories [View project](#)

All content following this page was uploaded by [Rodolfo A. Diaz](#) on 20 October 2014.

The user has requested enhancement of the downloaded file. All in-text references [underlined in blue](#) are added to the original document and are linked to publications on ResearchGate, letting you access and read them immediately.

Study of the apsidal precession of the Physical Symmetrical Pendulum.

Héctor R. Maya^{1,2,*}; Rodolfo A. Diaz^{2,†}; William J. Herrera^{2,‡}

¹Universidad de Córdoba,

Departamento de Física. Montería, Colombia.

²Universidad Nacional de Colombia,

Departamento de Física. Bogotá, Colombia.

December 17, 2013

Abstract

We study the apsidal precession of a Physical Symmetrical Pendulum (Allais' precession) as a generalization of the precession corresponding to the Ideal Spherical Pendulum (Airy's Precession). Based on the Hamilton-Jacobi formalism and using the technics of variation of parameters along with the averaging method, we obtain approximate solutions, in terms of which the motion of both systems admits a simple geometrical description. The method developed in this paper is considerably simpler than the standard one in terms of elliptical functions and the numerical agreement with the exact solutions is excellent. In addition, the present procedure permits to show clearly the origin of the Airy's and Allais' precession, as well as the effect of the spin of the Physical Pendulum on the Allais' precession. Further, the method can be extended to the study of the asymmetrical pendulum in which an exact solution is not possible anymore.

PACS: 45.40.-f, 83.10.Ff, 45.20.Jj, 47.10.Df

Keywords: Ideal spherical pendulum, Physical symmetrical pendulum, Hamilton-Jacobi formalism, Averaging method, Apsidal precession.

1 Introduction

A physical symmetrical pendulum is a particular case of the symmetrical top in which the center of mass (CM) is located below the fixed point, and the precession and spin can be considered as small perturbations with respect to the nutation*. Lagrange [1]; Poisson [2]; Golubev [3] and Leimanis [4], found exact solutions (in terms of elliptical functions) for the dynamics and kinematics of the symmetrical top under the action of gravity. On the other hand, Johansen and Kane [5] obtained an approximate solution for the Ideal Spherical Pendulum (ISP) using the method of averaging by using canonical variables. In addition, Miles [6] analyzed the response of the ISP under a harmonic excitation, while Hemp and Sethana [7] studied the dynamics of the ISP when the support undergoes vertical motion.

More closely related with this paper are the articles of Airy [8], Olsson [9] and Synge [10]. By using different methods of approximation, these authors found the angular frequency of precession that undergoes the apsidal axis of the projection of the ISP trajectory (the so-called Airy's precession). In these studies it is assumed that the motion starts with small initial amplitudes. More recently, Gusev, Rudenko and Vinogradov [11], considered this precession when the pendulum is submitted to small perturbations coming from the anisotropy of the support and they found an analytical formula for the angular frequency of the plane of oscillation as a function of the initial conditions and the anisotropy of the support.

As for the Physical Symmetrical Pendulum (PSP), the most usual analysis of its dynamics is greatly simplified due to the assumption of spin dominance, in which the nutation and precession are considered small perturbations [12]. By contrast, we study the symmetrical top (i.e. the PSP) considering it as a pendulum with a fixed point, in a regime of nutation dominance. We shall assume that the pendulum is released near to the surface of the earth with small initial

*hrmaya@hotmail.com

†radiazs@unal.edu.co

‡jherreraw@unal.edu.co

*In the most usual scenario of the symmetrical top, the spin is the dominant part of its motion.

amplitudes and a small transverse initial velocity, but without initial spin (with respect to the earth). On the other hand, in an inertial reference frame the PSP has a small correction to the initial precession and spin (owing to the rotation of the earth), but they are very small (of the order of 10^{-4} rad/seg) and are kept small at all times, from which the nutation becomes the dominant motion. In this regime of initial conditions the PSP describes trajectories that are approximately elliptical and that precess very slowly. In the case of the ISP this effect is called “Airy’s precession” while in the case of a physical pendulum (which is our case) we shall call it “Allais’ precession” [13]. The results of this study represent a first approximation to the dynamics of the paraconical pendulum, originally designed by Allais, and currently used widely by many researchers in the characterization of gravitational anomalies during eclipses [14]. As we shall see, the spin introduces a significant correction to the precession of the PSP (with respect to the precession of the ISP).

Our paper is distributed as follows: In section 2, we study the Ideal Spherical Pendulum, using the Hamilton-Jacobi approach combined with the technics of variation of parameters and the averaging method. These approximate results are numerically compared with the exact solution showing an excellent agreement between them even for long times. In this approach, the Airy’s precession appears naturally. The methods developed in this section are applied to the symmetrical physical pendulum in section 3, and once again an excellent agreement with the exact solution even for large times is apparent. The corresponding Allais’ precession appears clearly from the formalism, as well as the correction to this precession coming from the spin. Section 4 shows our conclusions and appendix 6 shows some few technical details.

2 Apsidal precession in the ideal spherical pendulum

We now study the origin of the Apsidal precession of the ideal spherical pendulum, keeping in mind that our aim is just to establish the general framework to apply a similar procedure for the physical symmetrical pendulum. We assume that the (small) initial amplitudes are of the order of 0.1 rad (experimental conditions).

Let us consider a system of axes XYZ fixed on an inertial reference frame and the spherical pendulum of mass m and length l , as displayed in Fig. 1

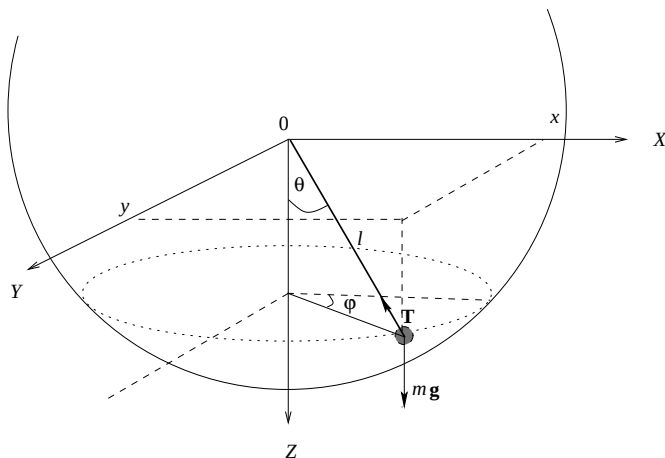


Figure 1: *Inertial system XYZ and the spherical pendulum. The angular coordinates θ and φ are shown, as well as the weight and tension.*

The natural coordinates that exhibit the symmetries of the IPS are the spherical coordinates θ and φ . The Lagrangian in such coordinates becomes

$$L = \frac{1}{2}ml^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + mgl \cos \theta. \quad (1)$$

The associated canonically conjugate momenta are given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}, \quad p_\varphi = ml^2 \dot{\varphi} \sin^2 \theta, \quad (2)$$

since φ is cyclic, its conjugate momentum p_φ is constant and can be identified as the z -component of the angular

momentum. The Hamiltonian of the system reads

$$h = \dot{\theta} p_\theta + \dot{\varphi} p_\varphi - L, \quad (3a)$$

$$h = \frac{1}{2ml^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right) - mgl \cos \theta, \quad (3b)$$

or in dimensionless units

$$H = \frac{1}{2} \left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2 \theta} \right) - \cos \theta, \quad (4)$$

where we have introduced the following definitions

$$H \equiv \frac{h}{mgl}, \quad \omega_0 \equiv \sqrt{\frac{g}{l}}, \quad \tau \equiv \omega_0 t, \quad P_\theta \equiv \frac{p_\theta}{ml^2 \omega_0}, \quad P_\varphi \equiv \frac{p_\varphi}{ml^2 \omega_0}. \quad (5)$$

Now we introduce a canonical transformation: $(\theta, \varphi, P_\theta, P_\varphi) \rightarrow (\bar{\theta}, \varphi, \bar{P}_\theta, P_\varphi)$ which keeps unaltered the variables associated with the precession (so that the constant of motion is still apparent), and permits to eliminate the circular functions from this Hamiltonian. An appropriate generating function of type II [12] for this Canonical Transformation (CT) reads

$$F_2(\theta, \varphi, \bar{P}_\theta, P_\varphi) = \bar{P}_\theta \sin \theta + \bar{P}_\varphi \varphi. \quad (6)$$

The formulas of transformation for the variables that describe the nutation are given by

$$\bar{\theta} = \frac{\partial F_2}{\partial \bar{P}_\theta} = \sin \theta, \quad P_\theta = \frac{\partial F_2}{\partial \theta} = \bar{P}_\theta \cos \theta, \quad (7)$$

while the new variables that describe the precession are identical to the old ones. Therefore, we continue using the same symbols φ and P_φ for them. Using this result in Eq. (4) we obtain the new Hamiltonian

$$\bar{H} = \frac{1}{2} \left[\bar{P}_\theta^2 (1 - \bar{\theta}^2) + \frac{P_\varphi^2}{\bar{\theta}^2} \right] - \sqrt{1 - \bar{\theta}^2}. \quad (8)$$

Now, since we are interested in a regime of small initial amplitudes (i.e. small values of θ_0) and since $\bar{\theta} \equiv \sin \theta$, we also have small values of the coordinate $\bar{\theta}$. Consequently, we can expand the radical in power series and keep terms up to fourth order in $\bar{\theta}$. We do this because this is the lowest order in which the apical precession appears. By doing such an expansion and omitting constant terms in the Hamiltonian we obtain

$$\bar{H} = \frac{\bar{P}_\theta^2}{2} + \frac{\bar{\theta}^2}{2} + \frac{P_\varphi^2}{2\bar{\theta}^2} - \frac{\bar{P}_\theta^2 \bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8}, \quad (9)$$

We identify the terms up to second order in the momenta and/or variables with the non-perturbed Hamiltonian (let us recall that all momenta and variables are dimensionless), while higher order terms are identified with the Hamiltonian of perturbation

$$H_0 = \frac{1}{2} \bar{P}_\theta^2 + \frac{\bar{\theta}^2}{2} + \frac{P_\varphi^2}{2\bar{\theta}^2}, \quad (10a)$$

$$H_1 = -\frac{\bar{P}_\theta^2 \bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8}. \quad (10b)$$

We shall use these Hamiltonians to determine the approximate solutions for the ISP. To do it, we shall apply the Hamilton-Jacobi formalism along with the technics of variation of parameters as well as the method of averaging.

2.1 Restricted Hamilton-Jacobi Method for the non-perturbed Hamiltonian

We start by finding the exact solution for the Hamiltonian H_0 by means of the implementation of the restricted Hamilton-Jacobi (RHJ) method, that is for the Hamilton's Characteristic Function [12]. Equation (10a) shows that φ is cyclic in H_0 , hence we can write the Hamilton's Characteristic Function W in the form

$$W = W_{\bar{\theta}} + \alpha_2 \varphi, \quad (11)$$

where $P_\varphi \equiv \alpha_2$ is the canonical conjugate momentum associated with φ , and the RHJ equation for H_0 becomes

$$\frac{1}{2} \left(\frac{\partial W}{\partial \bar{\theta}} \right)^2 + \frac{\bar{\theta}^2}{2} + \frac{\alpha_2^2}{2\bar{\theta}^2} = \alpha_1, \quad (12)$$

where α_1 corresponds to the numerical value of H_0 i.e. the mechanical energy of the system described by this Hamiltonian

$$\alpha_1 = \frac{1}{2} \bar{P}_{\theta 0}^2 + \frac{\bar{\theta}_0^2}{2} + \frac{\alpha_2^2}{2\bar{\theta}_0^2} \quad (13)$$

Equation (12) is an ordinary differential equation for $W_{\bar{\theta}}$ that can be solved directly

$$\frac{dW_{\bar{\theta}}}{d\bar{\theta}} = \pm \frac{\sqrt{-\alpha_2^2 + 2\alpha_1\bar{\theta}^2 - \bar{\theta}^4}}{\bar{\theta}}, \quad (14)$$

and the function W given by (11) becomes

$$W = \alpha_2 \varphi \pm \int \frac{\sqrt{-\alpha_2^2 + 2\alpha_1\bar{\theta}^2 - \bar{\theta}^4}}{\bar{\theta}} d\bar{\theta}. \quad (15)$$

The new angular variables are obtained from

$$\tau + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \pm \int \frac{\bar{\theta}}{\sqrt{-\alpha_2^2 + 2\alpha_1\bar{\theta}^2 - \bar{\theta}^4}} d\bar{\theta} = \mp \frac{1}{2} \arcsin \left(\frac{\alpha_1 - \bar{\theta}^2}{k} \right), \quad (16)$$

$$k^2 \equiv \alpha_1^2 - \alpha_2^2 \quad (17)$$

the initial conditions that we shall consider [Equations (32a) with $\theta_0 \ll 1$ Rad], combined with Eq. (13) lead to $\alpha_1^2 > \alpha_2^2$, and $0 < \theta < \pi/2$. The first condition says that k^2 is a positive constant while the second condition leads us to preserve only the upper sign in Eqs. (16). Further, the sign of the integral associated with the equation for $\partial W / \partial \alpha_2$ is chosen accordingly

$$\beta_2 = \frac{\partial W}{\partial \alpha_2} = \varphi - \int \frac{\alpha_2}{\bar{\theta} \sqrt{-\alpha_2^2 + 2\alpha_1\bar{\theta}^2 - \bar{\theta}^4}} d\bar{\theta}, \quad (18)$$

in order to evaluate this integral we use Eq. (16) to define the variable

$$u = \tau + \beta_1 = -\frac{1}{2} \arcsin \left(\frac{\alpha_1 - \bar{\theta}^2}{k} \right) \quad (19)$$

substituting (19) in (18) we can write φ as

$$\varphi = \beta_2 + \int \frac{\alpha_2}{\alpha_1 + k \sin(2u)} du = \beta_2 + \arctan \left[\frac{1}{\alpha_2} (k + \alpha_1 \tan u) \right] \quad (20a)$$

$$\varphi = \beta_2 + \arctan \left\{ \frac{1}{\alpha_2} [k + \alpha_1 \tan(\tau + \beta_1)] \right\} \quad (20b)$$

From Eqs. (16), (14) and (20) we have

$$\bar{\theta} = \sqrt{\alpha_1 + k \sin[2(\beta_1 + \tau)]}, \quad (21a)$$

$$\bar{P}_{\theta} = \frac{dW_{\bar{\theta}}}{d\bar{\theta}} = \frac{k \cos[2(\beta_1 + \tau)]}{\bar{\theta}}, \quad (21b)$$

$$\varphi = \beta_2 + \arctan \left\{ \frac{1}{\alpha_2} [k + \alpha_1 \tan(\beta_1 + \tau)] \right\}, \quad (21c)$$

these equations along with $P_\varphi = \alpha_2$, form the solution for the non-perturbed Hamiltonian H_0 .

2.2 Method of averaging using canonical variables for the complete Hamiltonian

The next step is to obtain an approximate solution for the complete Hamiltonian $\bar{H}$ of Eq. (9), based on the exact solution (21) for the non-perturbed Hamiltonian H_0 . To do this, we shall use the method of averaging using canonical variables. Our approach is a variation of the approximation proposed by K. F. Johansen and T. R. Kane [5]. The goal of such an approach is to obtain a set of approximate equations of easy solution for the canonical variables, by applying the averaging method in the version proposed by Krylov-Bogoliubov-Mitropolsky [17]. As we shall see, this technics is based on the variation of parameters and “the fast integration”.

In the framework of the Hamilton-Jacobi equation for Hamilton’s Principal Function, it is clear that the transformations (21) are associated with a generating function of type II given by

$$S_0 = S_0(\bar{\theta}, \varphi, \alpha_1, \alpha_2, \tau) = W(\bar{\theta}, \varphi, \alpha_1, \alpha_2) - \alpha_1 \tau \quad (22)$$

such that

$$P_j = \frac{\partial S_0}{\partial q_j}, \quad \beta_j = \frac{\partial S_0}{\partial \alpha_j}. \quad (23)$$

Note that at this step we are using the Hamilton-Jacobi formalism for the Hamilton’s principal function S . This is a natural choice since at this moment what we pretend is to see the way in which H_1 modifies the exact solution (21) of H_0 . We can do this by demanding that the associated CT reduces the total Hamiltonian $H_0 + H_1$ to the Hamiltonian H_1 . Of course, it means that the new Hamiltonian must be numerically different from the old one, and it is possible only if the generating function depends on time explicitly as is the case of S_0 . We can then propose a generating function S for a canonical transformation to a new set of canonical variables Q_1, Q_2, P_1 and P_2 if we replace in S_0 the constants α_1 and α_2 by P_1 and P_2 respectively, that is

$$S = S_0(\bar{\theta}, \varphi, P_1, P_2, \tau), \quad (24)$$

where now P_i and Q_i have become variables (this is the method of variation of parameters). The transformations induced by (24) are obviously (21) but with the replacements $\alpha_i \rightarrow P_i$, $\beta_i \rightarrow Q_i$

$$\bar{\theta} = \sqrt{P_1 + k \sin[2(Q_1 + \tau)]}, \quad (25a)$$

$$\bar{P}_\theta = \frac{k \cos[2(Q_1 + \tau)]}{\sqrt{P_1 + k \sin[2(Q_1 + \tau)]}}, \quad P_\varphi = P_2 \quad (25b)$$

$$\varphi = Q_2 + \arctan \left[\frac{1}{P_2} (k + P_1 \tan(Q_1 + \tau)) \right], \quad (25c)$$

where

$$k = \sqrt{P_1^2 - P_2^2}. \quad (26)$$

The new Hamiltonian K as a function of Q_1 , Q_2 , P_1 and P_2 is given by

$$K = H_0 + H_1 + \frac{\partial S}{\partial t} = H_1(Q, P), \quad (27)$$

where we have used $H_0 = P_1$ and $\partial S / \partial t = -P_1$. Expressing H_1 [Eq. (10b)] in terms of these new variables, yields

$$K = -\frac{\bar{P}_\theta^2 \bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8} \quad (28a)$$

$$K = \frac{1}{8} \{P_1 + k \sin[2(Q_1 + \tau)]\}^2 - \frac{1}{2} k^2 \cos^2[2(Q_1 + \tau)], \quad (28b)$$

Consequently, we can obtain an exact solution for the Hamiltonian (9), by solving the Hamilton equations for Q_1, Q_2, P_1 and P_2 as a function of τ and substituting them in Eq. (25). Nevertheless, we recall that it is not our goal. Rather, we shall obtain a set of approximate equations of motion with easy solution in terms of elementary functions. For this, we first consider in Eq. (25), that Q_i and P_i are functions that vary very slowly within a period π of the θ coordinate[†] (as it is customary in the quasi-harmonic approximation). Hence, an approximate solution for the canonical variables of the

[†]This is a reasonable ansatz since those variables are constant when we use the non-perturbed Hamiltonian H_0 . The time evolution of these variables arises from the introduction of the (much smaller) Hamiltonian H_1 .

complete Hamiltonian (9) is still given by (25) but with Q_i and P_i representing solutions of the Hamilton equations with K replaced by its average over a period T , this approximation yields

$$\langle K \rangle = \frac{1}{T} \int_0^T K(Q, P, t) dt = \frac{1}{16} (-3k^2 + 2P_1^2), \quad (29a)$$

$$\langle K \rangle = \frac{1}{16} (3P_2^2 - P_1^2) \quad (29b)$$

where we have neglected the variations of all Q_i and P_i within a period, and k is given by (26). Note that the new coordinates are all cyclic in this averaged Hamiltonian. Consequently, under this approximation all new canonical momenta are kept constant. The equations of motion for these new coordinates become

$$\dot{P}_1 = -\frac{\partial \langle K \rangle}{\partial Q_1} = 0, \quad \dot{P}_2 = -\frac{\partial \langle K \rangle}{\partial Q_2} = 0 \quad (30a)$$

$$\dot{Q}_1 = \frac{\partial \langle K \rangle}{\partial P_1} = -\frac{P_1}{8}, \quad \dot{Q}_2 = \frac{\partial \langle K \rangle}{\partial P_2} = \frac{3P_2}{8}. \quad (30b)$$

The integration of these equations is straightforward

$$P_1 = \alpha_1 = H_0, \quad P_2 = \alpha_2 = P_\varphi, \quad (31a)$$

$$Q_1 = -\frac{P_1}{8}\tau + Q_{10}, \quad Q_2 = \frac{3P_2}{8}\tau + Q_{20}, \quad (31b)$$

the constants Q_{10} and Q_{20} are obtained by using the initial conditions and evaluating (25) and (31b) at $\tau = 0$.

2.3 Approximate solution for the ideal spherical pendulum

Let us find an approximate solution for the ideal spherical pendulum associated with the elliptical mode characterized by initial conditions in which the pendulum is released with a small initial amplitude θ_0 and a small initial precession but without initial nutation (initial conditions with respect to the earth). As we have discussed, owing to the rotation of the earth the initial precession with respect to an inertial reference frame is slightly different. Therefore, in an inertial reference frame the initial conditions become

$$\theta(0) = \theta_0, \quad \varphi(0) = 0, \quad \dot{\theta}(0) = 0, \quad \dot{\varphi}(0) = \dot{\varphi}_0, \quad (32a)$$

we also take $l = 1m$. According with Eq. (7), the initial conditions (32a) are transformed into

$$\bar{\theta}(0) = \sin \theta_0, \quad \bar{P}_\theta(0) = P_\theta(0) \sec \theta_0 \quad (33)$$

By applying the initial conditions (32a) in Eqs. (2, 5) we see that $P_\theta(0) = 0$. Moreover, by combining Eqs. (13, 17, 31a, 31b), we obtain the following expressions for the constants

$$\alpha_2 = \frac{\dot{\varphi}_0}{\omega_0} \sin^2 \theta_0 = \frac{\dot{\varphi}_0}{\omega_0} \bar{\theta}_0^2, \quad \alpha_1 = \frac{\bar{\theta}_0^2}{2} + \frac{\alpha_2^2}{2 \bar{\theta}_0^2} \quad (34a)$$

$$k = \frac{\bar{\theta}_0^2}{2} - \frac{\alpha_2^2}{2 \bar{\theta}_0^2} \quad (34b)$$

$$Q_{10} = \frac{\pi}{4}, \quad Q_{20} = -\arctan\left(\frac{\bar{\theta}_0^2}{\alpha_2}\right), \quad (34c)$$

The new coordinates (31b) are

$$Q_1 = -\frac{\alpha_1}{8}\tau + \frac{\pi}{4}, \quad Q_2 = \frac{3P_2}{8}\tau - \arctan\left(\frac{\bar{\theta}_0^2}{\alpha_2}\right), \quad (35)$$

and the approximate solutions (25a) and (25c) yield

$$\bar{\theta} = \sqrt{\alpha_1 + k \cos\left[2\left(1 - \frac{\alpha_1}{8}\right)\tau\right]}, \quad (36a)$$

$$\varphi = Q_2 + \arctan\left[\frac{1}{\alpha_2}(k + \alpha_1 \tan(Q_1 + \tau))\right], \quad (36b)$$

In Figure 2 we show the solutions obtained with the approximation (36) and the exact solutions for the initial conditions $\theta_0 = 0.1\text{rad}$, $\dot{\varphi}_0 = 1.00167\text{ rad/s}$. In part *A* of this figure we superpose the graphics of the polar angle $\bar{\theta}$ and $\sin\theta$. Note that they cannot be distinguished. In part *B* it is shown the exact (increasing) azimuthal angle and the approximate one (monotonic piecewise). It is observed that both graphics coincide for $0 \leq \tau \leq \tau_{\pi/2}$ where $\tau_{\pi/2}$ corresponds to the value of τ for which the phase of the function $\tan x$ in (36b) is equal to $\pi/2$. It is clear that the discontinuities in the derivative appear for values $(2n+1)\pi/2$ of the argument where $n = 1, 2, \dots$

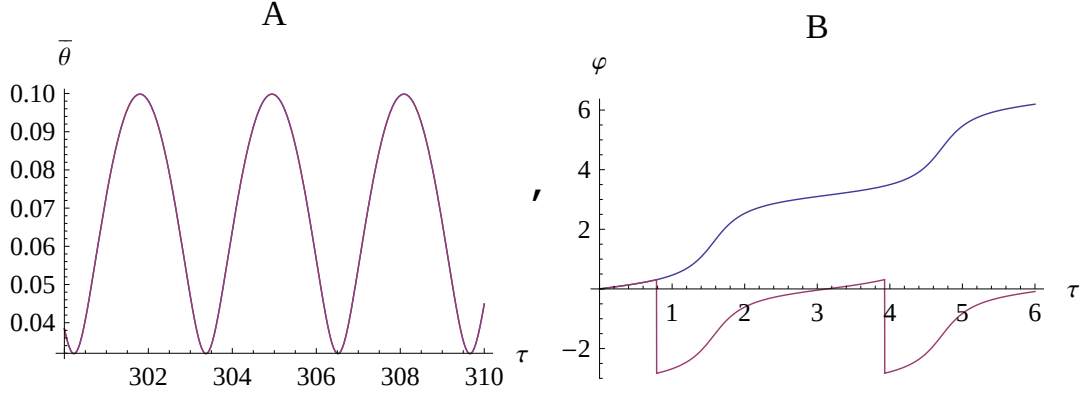


Figure 2: (A) Graphical form of the approximate solution for $\bar{\theta}$ with $300 \leq \tau \leq 310$. The graphics of the exact solution cannot be distinguished from the approximate one. (B) Graphical form of the (monotonic piecewise) approximate solution for the angle φ , and of the smooth approximate solution after the introduction of the γ angle, for $0 \leq \tau \leq 6$. The approximate (smooth) and exact solutions are superposed.

However, we can correct such a problem by introducing the angle γ such that

$$\arctan \left[\frac{k + \alpha_1 \tan(Q_1 + \tau)}{\alpha_2} \right] = (Q_1 + \tau) + \arctan \gamma, \quad (37)$$

solving for γ we obtain

$$\gamma = \frac{2k \cos^2(Q_1 + \tau) + (\alpha_1 - \alpha_2) \sin(2(Q_1 + \tau))}{(\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) \cos(2(Q_1 + \tau)) + k \sin(2(Q_1 + \tau))}. \quad (38)$$

Note that by means of the identity (37), the strictly increasing phase $(Q_1 + \tau)$ has been extracted from the argument of the function $\tan x$, and we have introduced a phase γ that oscillates and remains always finite as argument of the function $\arctan x$. Substituting in (36b) we finally get

$$\varphi = Q_2 + (Q_1 + \tau) + \arctan \gamma, \quad (39)$$

and the parameter Q_{20} in Eq. (34c) can be rewritten in terms of the initial conditions as

$$Q_{20} = -\frac{\pi}{4} + \arctan \left(\frac{\alpha_2 - \bar{\theta}_0^2}{\bar{\theta}_0^2 + \alpha_2} \right) \quad (40)$$

Figure 3 shows the superposition of the graphics for the azimuthal angle φ (curve) obtained from the exact solution and the approximation (39), for $500 \leq \tau \leq 510$. Like in the case of the polar angle we observe that the exact and approximate solutions cannot be distinguished from each other. The interval to plot was chosen in order to exhibit the asymptotic behavior of the approximate solution.

The period of motion is by definition twice the period of the coordinate $\bar{\theta}$ (36a), such that in this approximation the period (in dimensionless units) is given by

$$T = \frac{4\pi}{2 \left(1 - \frac{\alpha_1}{8}\right)} = \frac{16\pi}{8 - \alpha_1}, \quad (41)$$

and for our current example it yields $T = 2.00847\text{s}$, whose deviation with respect to the exact period is one part of 10^6 [16]. From Eq. (36b) it is followed that the approximate solution for $\dot{\varphi}$ has the same period T of θ . The straight line shown in Fig. 3 corresponds to the linear approximation for the function $\varphi(\tau)$ given by

$$\tilde{\varphi}_a(\tau) \equiv \varphi_0 + \frac{\varphi(T) - \varphi(0)}{T} \tau = \frac{1}{8} (8 - \alpha_1 + 3\alpha_2) \tau. \quad (42)$$

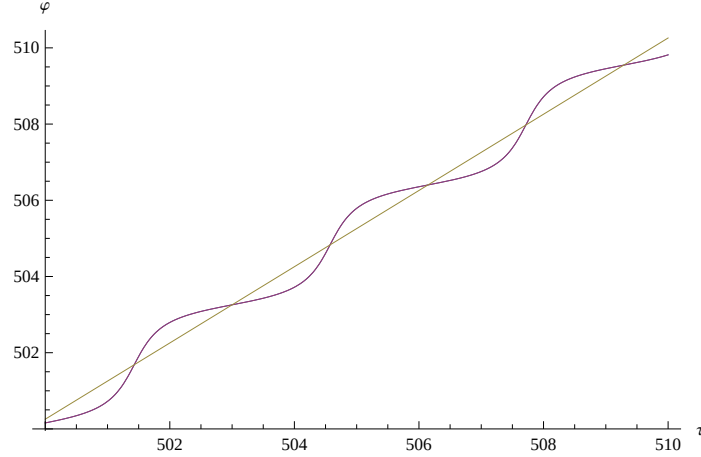


Figure 3: Graphical form of the approximate values of the angle φ for $500 \leq \tau \leq 510$. The straight line corresponds to the linear approximation given by (42). The approximate and exact solutions are superposed.

Finally, we obtain the average angular velocity of the apsidal precession. That is, the quotient between the angular excess over 2π and the period of the motion

$$\omega_a = \frac{\varphi(T) - \varphi(0) - 2\pi}{T} = \frac{3}{8}\alpha_2, \quad (43)$$

and using Eq. (34a) we see that it coincides with the angular velocity in the Airy's apsidal precession [8]

$$\omega_a = \frac{3}{8}\alpha_2 = \frac{3}{8} \frac{\dot{\varphi}_0}{\omega_0} \sin^2 \theta_0. \quad (44)$$

2.3.1 Geometrical interpretation of the solution

We conclude the study of the dynamics of the ISP by expressing the solutions obtained so far in terms of the cartesian coordinates. As we shall see, it permits us to give a simple geometrical interpretation to the projection of the motion on the XY plane, when we use the approximate solutions (36). In terms of the spherical coordinates, the dimensionless cartesian coordinates yield

$$q_1 \equiv \frac{x}{l} = \sin \theta \cos \varphi, \quad q_2 \equiv \frac{y}{l} = \sin \theta \sin \varphi, \quad (45)$$

and replacing the approximate solutions for θ and φ , we obtain an approximate solution for the dimensionless cartesian coordinates (see appendix)

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos(\omega_a \tau) & -\sin(\omega_a \tau) \\ \sin(\omega_a \tau) & \cos(\omega_a \tau) \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \cos(\omega \tau) \\ (\alpha_2 / \bar{\theta}_0) \sin(\omega \tau) \end{bmatrix} \quad (46)$$

Figure 4 shows the projection of the trajectory on the XY plane for the initial conditions $\theta_0 = 0.1$ rad, $\dot{\varphi}_0 = 1.00167$ Rad/s and $30nT \leq t \leq (30n+1)T$ with $n = 0, 1, 2$; using the approximation (46) and the exact solution. Once again, the comparison shows that the superposition of both solutions makes them indistinguishable.

It is easy to see that the parametric equations (46) describe an ellipse of semiaxes a and b with angular frequency ω (in standard units), given by

$$a = l\bar{\theta}_0 = l \sin \theta_0, \quad b = \frac{l\alpha_2}{\bar{\theta}_0}, \quad (47)$$

$$\omega = \omega_0 \left(1 - \frac{\alpha_1}{8}\right), \quad \alpha_1 = \frac{\bar{\theta}_0^2}{2} + \frac{\alpha_2^2}{2\bar{\theta}_0^2}, \quad (48)$$

the matrix of rotation in Eq. (46) shows that such an ellipse precesses counterclockwise with angular frequency

$$\omega_a = \left(\frac{3}{8}\alpha_2\right) \omega_0 = \frac{3}{8} \frac{ab}{l^2} \omega_0, \quad (49)$$

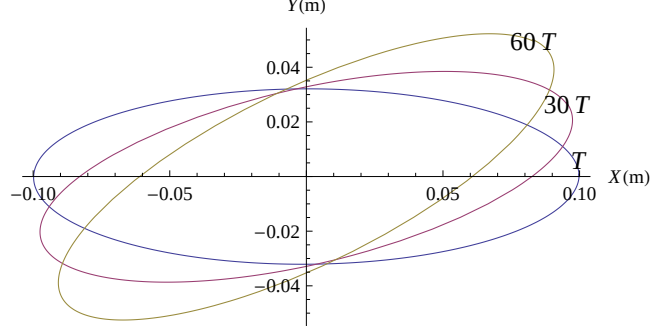


Figure 4: *Projection of the trajectory in the XY plane for $30nT \leq t \leq (30n+1)T$ with $n = 0, 1, 2$, using the approximate solution (46). The approximate and exact solutions cannot be distinguished.*

the last expression is the so-called apsidal Airy's precession [8]. As a matter of consistency, when $\dot{\varphi}_0 = 0$, and $\omega_a = 0$, the difference of phase $\alpha_1/8$ given by Eq. (48) coincides with the relative variation of first order of the frequency of a plane pendulum with finite amplitude θ_0 [12]. When $\dot{\varphi}_0 \neq 0$, equation (34a) shows that $\alpha_2 \neq 0$, so that this variation is corrected by $\alpha_2^2/(16\bar{\theta}_0^2)$ as an effect of the precession. Finally, we should emphasize that the expression obtained by us for the apsidal Airy's precession does not depend on the a priori assumption that Airy makes with respect to the elliptical trajectories of the projection of the pendulum motion [8]. Such a feature is deduced in our framework in a natural manner as a consequence of the slow variation of the new coordinates. We point out that the method also provides the correction to the period up to second order in the coordinates.

3 Physical Symmetrical Pendulum

We define a Physical Symmetrical Pendulum (PSP) as a rigid solid with an axis of symmetry that can rotate freely around a fixed point (support) located at one edge of the symmetry axis and in which the center of mass (CM) is located below the fixed point. An example is a disc supported by a cylindrical rod hung on an edge, as shown in Fig. 5. A comparison with the symmetrical top shows us that indeed we are dealing with the same system, and the only difference is the range in which nutation motion occurs: $0 \leq \theta \leq \pi/2$ for the PSP, and $\pi/2 \leq \theta \leq \pi$ for the top[‡].

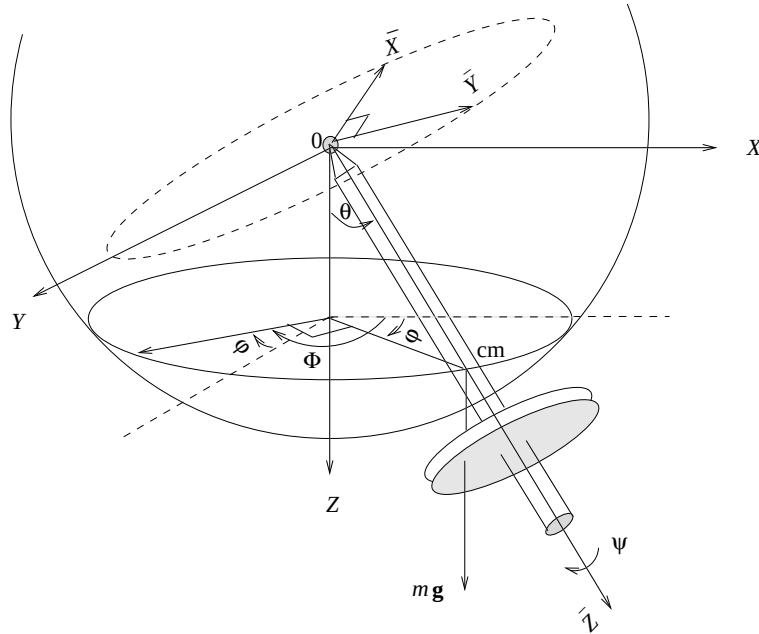


Figure 5: *Physical Symmetrical Pendulum hung on an edge. θ , Φ and ψ are the Euler angles used in the description of the motion. The spherical coordinate φ is shown as well as its relation with the Euler angle Φ .*

[‡]We are using the convention of positive direction of the Z -axis in the direction of the gravitational field (i.e. downwards).

We use the Euler angles θ , Φ , ψ as generalized coordinates. As shown in Fig. 5, these angles provide the orientation of the system of axes $\bar{X}\bar{Y}\bar{Z}$ fixed to the pendulum, with respect to the inertial system of axes XYZ . We can note that there is a difference of phase of $\pi/2$ between the Euler angle Φ and the azimuthal angle φ of the spherical coordinates, that is

$$\Phi = \varphi + \pi/2. \quad (50)$$

The components of the angular velocity $\boldsymbol{\omega}$ in the basis of axes $\bar{X}\bar{Y}\bar{Z}$ fixed to the pendulum, yield [12]

$$\begin{aligned} \omega_{\bar{x}} &= \dot{\Phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi, \\ \omega_{\bar{y}} &= \dot{\Phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi, \\ \omega_{\bar{z}} &= \dot{\Phi} \cos \theta + \dot{\psi} \end{aligned} \quad (51)$$

it is clear that in the system of axis $\bar{X}\bar{Y}\bar{Z}$, the position $\mathbf{r}_{\text{cm}}$ of the CM, gives

$$\mathbf{r}_{\text{cm}} = l \bar{\mathbf{k}} \quad (52)$$

where l is the distance from the fixed point to the CM, while in the inertial system of axis XYZ , it becomes

$$\mathbf{r}_{\text{cm}} = l \sin \theta \sin \Phi \mathbf{i} - l \sin \theta \cos \Phi \mathbf{j} + l \cos \theta \mathbf{k}. \quad (53)$$

Finally, the Lagrangian of the system is given by

$$L = \frac{1}{2} \left[I_{\bar{x}} \left(\dot{\theta}^2 + \dot{\Phi}^2 \sin^2 \theta \right) + I_{\bar{z}} \left(\dot{\Phi} \cos \theta + \dot{\psi} \right)^2 \right] + mgl \cos \theta, \quad (54)$$

where $I_{\bar{x}}$ and $I_{\bar{z}}$ are the moments of inertia of the pendulum with respect to the axes $\bar{X}$ and $\bar{Z}$ fixed to the body. The constant canonical momenta associated with the cyclic coordinates Φ and ψ are given by

$$p_{\Phi} = \frac{\partial L}{\partial \dot{\Phi}} = I_{\bar{x}} \dot{\Phi} \sin^2 \theta + I_{\bar{z}} \cos \theta \left(\dot{\psi} + \dot{\Phi} \cos \theta \right) \quad (55)$$

$$p_{\psi} = \frac{\partial L}{\partial \dot{\psi}} = I_{\bar{z}} \left(\dot{\Phi} \cos \theta + \dot{\psi} \right) \quad (56)$$

3.1 Hamiltonian of the PSP

The Lagrangian (54) is a homogenous function of second degree, the transformation from cartesian to generalized coordinates does not depend on time, and the potential does not depend on the generalized velocities. Thus, the Hamiltonian h becomes the total energy of the system and can be expressed in canonical variables as follows

$$h = \frac{1}{2} \left(\frac{p_{\theta}^2}{I_{\bar{x}}} + \frac{(p_{\psi} \cos \theta - p_{\Phi})^2}{I_{\bar{x}} \sin^2 \theta} + \frac{p_{\psi}^2}{I_{\bar{z}}} \right) - mgl \cos \theta, \quad (57)$$

or in dimensionless units

$$H = \frac{1}{2} \left(P_{\theta}^2 + \frac{(P_{\Phi} - P_{\psi} \cos \theta)^2}{\sin^2 \theta} + \alpha P_{\psi}^2 \right) - \cos \theta, \quad (58)$$

where we have used the definitions

$$\Omega_0 \equiv \sqrt{\frac{mgl}{I_{\bar{x}}}}, \quad \tau \equiv \Omega_0 t, \quad H \equiv \frac{h}{mgl}, \quad P_{\psi, \Phi} \equiv \frac{p_{\psi, \Phi}}{\sqrt{mgl I_{\bar{x}}}}, \quad \alpha \equiv \frac{I_{\bar{x}}}{I_{\bar{z}}} \quad (59)$$

the parameter α accounts on the shape of the pendulum. For a PSP like the one represented in Figure 5, $\alpha > 1$ and its ellipsoid of inertia is prolate. In this study we are interested in this case. When $\alpha < 1$ we would have an oblate ellipsoid of inertia while $\alpha = 1$ corresponds to an spherical ellipsoid.

Now we proceed in a way similar to the case of the ISP. That is, we introduce a generating function similar to (6), that leaves the variables of precession and spin unaltered

$$F_2(\theta, \Phi, \psi) = \bar{P}_{\theta} \sin \theta + \bar{P}_{\Phi} \Phi + \bar{P}_{\psi} \psi, \quad (60)$$

which leads to the following transformation between canonical coordinates

$$\bar{\theta} = \sin \theta, \quad P_{\theta} = \bar{P}_{\theta} \cos \theta, \quad (61)$$

and the Hamiltonian becomes

$$\bar{H} = \frac{1}{2} \left(\bar{P}_{\theta}^2 (1 - \bar{\theta}^2) + \frac{(P_{\Phi} - P_{\psi} \sqrt{1 - \bar{\theta}^2})^2}{\bar{\theta}^2} + \alpha P_{\psi}^2 \right) - \sqrt{1 - \bar{\theta}^2}, \quad (62)$$

where we have preserved the symbols of the variables that describe precession and spin. Expanding up to fourth order in $\bar{\theta}$ and neglecting constant terms, the Hamiltonian becomes

$$\bar{H} = \frac{1}{2} \left(\bar{P}_{\theta}^2 (1 - \bar{\theta}^2) + \frac{[P_{\Phi} - P_{\psi} (1 - \frac{\bar{\theta}^2}{2})]^2}{2\bar{\theta}^2} + \alpha P_{\psi}^2 \right) + \frac{\bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8}, \quad (63)$$

for future purposes, it is convenient the following separation

$$\bar{H} = \bar{H}_0 + \bar{H}_1 \quad (64)$$

$$\bar{H}_0 \equiv \frac{\bar{P}_{\theta}^2}{2} + \frac{\bar{\theta}^2}{2} + \frac{(P_{\Phi} - P_{\psi})^2}{2\bar{\theta}^2} - \frac{\bar{P}_{\theta}^2 \bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8}, \quad (65)$$

$$\bar{H}_1 \equiv \frac{P_{\psi}}{2} [P_{\Phi} + P_{\psi}(\alpha - 1)] + \frac{P_{\psi}^2}{8} \bar{\theta}^2, \quad (66)$$

It is easy to realize that this separation is not consequent with the perturbation theory, since in both $\bar{H}_0$ and $\bar{H}_1$ there are terms of order two and four. Indeed this separation has the aim of matching $\bar{H}_0$ with the Hamiltonian of the ISP (9), instead of implementing a standard perturbation theory.

To do that, we introduce a new generating function

$$G_2(\Phi, \psi, \bar{P}_{\Phi}, \bar{P}_{\psi}) = \Phi(\bar{P}_{\Phi} + \bar{P}_{\psi}) + \psi \bar{P}_{\psi}, \quad (67)$$

which leaves invariant the variables that describe the nutation and transforms the precession and spin variables as follows

$$\bar{\Phi} = \frac{\partial G_2}{\partial \bar{P}_{\Phi}} = \Phi, \quad P_{\Phi} = \frac{\partial G_2}{\partial \Phi} = \bar{P}_{\Phi} + \bar{P}_{\psi} \quad (68a)$$

$$\bar{\psi} = \frac{\partial G_2}{\partial \bar{P}_{\psi}} = \psi + \Phi, \quad P_{\psi} = \frac{\partial G_2}{\partial \psi} = \bar{P}_{\psi}. \quad (68b)$$

and the new Hamiltonian becomes

$$\bar{H}_0 = \frac{\bar{P}_{\theta}^2}{2} + \frac{\bar{\theta}^2}{2} + \frac{\bar{P}_{\Phi}^2}{2\bar{\theta}^2} - \frac{\bar{P}_{\theta}^2 \bar{\theta}^2}{2} + \frac{\bar{\theta}^4}{8}, \quad (69a)$$

$$\bar{H}_1 = \frac{\bar{P}_{\psi}}{2} (\bar{P}_{\Phi} + \alpha \bar{P}_{\psi}) + \frac{\bar{P}_{\psi}^2}{8} \bar{\theta}^2. \quad (69b)$$

Since (69a) coincides with (9), we can consider equations (25) as the solution of the Hamiltonian $\bar{H}_0$ and then we construct an additional CT that permits to reduce the new Hamiltonian to $\bar{H}_1$ alone. This is the task of the next section.

3.2 Variation of Parameters

According with the results of the previous section, we consider the approximate solution (25) as the solution of (69a), that is

$$\bar{\theta} = \sqrt{P_1 + k \sin[2(Q_1 + \tau)]}, \quad (70a)$$

$$\bar{P}_{\theta} = \frac{k \cos[2(Q_1 + \tau)]}{\bar{\theta}}, \quad \bar{P}_{\Phi} = P_2, \quad (70b)$$

$$\bar{\Phi} = Q_2 + \arctan \left[\frac{1}{P_2} (k + P_1 \tan(Q_1 + \tau)) \right], \quad (70c)$$

with

$$Q_1 = -\frac{P_1}{8}\tau + Q_{10}, \quad Q_2 = \frac{3P_2}{8}\tau + Q_{20}, \quad (71a)$$

$$P_1 = \frac{\bar{P}_{\theta 0}^2}{2} + \frac{\bar{\theta}_0^2}{2} + \frac{\bar{P}_{\Phi 0}^2}{2\bar{\theta}_0^2}, \quad (71b)$$

$$P_2 = \bar{P}_\Phi = P_\Phi - P_\psi, \quad (71c)$$

$$k^2 = P_1^2 - P_2^2. \quad (71d)$$

This solution suggests to construct a last CT to the new variables Γ_i , Π_i , with $i = 1, 2$, replacing the constants Q_{i0} by Γ_i , and P_i by Π_i (variation of parameters), while leaving unaltered the variables that describe the spin degrees of freedom

$$Q_1 = -\frac{\Pi_1}{8}\tau + \Gamma_1, \quad Q_2 = \frac{3\Pi_2}{8}\tau + \Gamma_2, \quad (72a)$$

$$Q_3 = \Gamma_3 = \bar{\psi}, \quad P_1 = \Pi_1, \quad P_2 = \Pi_2, \quad \Pi_3 = \bar{P}_\psi \quad (72b)$$

Indeed, the CT given by (72) is a particular case of a more general class of canonical transformations $(Q, P) \rightarrow (\Gamma, \Pi)$ that we call Linear Transformations (see appendix). There is a generating function of type II associated with these transformation $S(Q_1, Q_2, Q_3, \Pi_1, \Pi_2, \Pi_3, \tau)$, and the new Hamiltonian K as a function of Γ_i and Π_i is obtained from

$$K = \bar{H}_0 + \bar{H}_1 + \frac{\partial S}{\partial \tau} = \bar{H}_1(\Gamma, \Pi), \quad (73)$$

where we have used our a priori assumption that (70) is the exact solution for $\bar{H}_0$, that is

$$\bar{H}_0 + \frac{\partial S}{\partial \tau} = 0. \quad (74)$$

such that $\bar{H}_1$ described by (69b) in terms of these new variables, is the new Hamiltonian

$$K = \bar{H}_1(\Gamma, \Pi) = \frac{1}{2}\Pi_3(\Pi_2 + \alpha\Pi_3) + \frac{\Pi_3^2}{8}\{\Pi_1 + k \sin[2(\omega_1\tau + \Gamma_1)]\}, \quad (75)$$

where $\omega_1 = 1 - \Pi_1/8$. It is clear that Γ_1 is still a slowly varying coordinate, such that we can neglect its variation within a period of time $T = 2\pi/\omega_1$. Carrying out a “fast integration” with respect to τ we see that the oscillating term vanishes and the averaged Hamiltonian becomes

$$\langle K \rangle = \frac{1}{2}\Pi_3(\Pi_2 + \alpha\Pi_3) + \frac{1}{8}\Pi_3^2\Pi_1, \quad (76)$$

it is observed that the coordinates Γ are cyclic within this averaged Hamiltonian so that the new momenta are constant (within our approximation) and equal to their initial values

$$\Pi_1 = P_1, \quad \Pi_2 = P_2, \quad \Pi_3 = P_\psi \quad (77)$$

while the equations of motion for the new coordinates are

$$\dot{\Gamma}_1 = \frac{\partial \langle K \rangle}{\partial \Pi_1} = \frac{1}{8}\Pi_3^2, \quad \dot{\Gamma}_2 = \frac{\partial \langle K \rangle}{\partial \Pi_2} = \frac{\Pi_3}{2}, \quad (78a)$$

$$\dot{\Gamma}_3 = \frac{\partial \langle K \rangle}{\partial \Pi_3} = \frac{\Pi_2}{2} + \alpha\Pi_3 + \frac{1}{4}\Pi_3\Pi_1, \quad (78b)$$

integrating out we have

$$\Gamma_1 = \frac{1}{8}P_\psi^2\tau + Q_{10}, \quad (79a)$$

$$\Gamma_2 = \frac{P_\psi}{2}\tau + Q_{20}, \quad (79b)$$

$$\Gamma_3 = \left(\frac{P_2}{2} + \alpha P_\psi + \frac{1}{4}P_\psi P_1 \right) \tau + \bar{\psi}_0. \quad (79c)$$

By virtue of these results, the relations (70) are still approximate solutions for the variables $\bar{\theta}$, $\bar{P}_\theta$, $\bar{\Phi}$ of the system described by (63), where P_1 , P_2 , and k are the constants defined by (71b)-(71d), while the time evolution of the functions Q_i in (72) yields

$$Q_1 = \frac{1}{8} (P_\psi^2 - P_1) \tau + Q_{10}, \quad (80a)$$

$$Q_2 = \left(\frac{3P_2}{8} + \frac{P_\psi}{2} \right) \tau + Q_{20}. \quad (80b)$$

As for the spin angle ψ , it can be obtained from (68b)

$$\psi = \bar{\psi} - \Phi = \Gamma_3 - \Phi. \quad (81)$$

It is important to point out that despite the multiple canonical transformations carried out to reduce the exact Hamiltonian (58) to the approximate Hamiltonian (76), the new coordinates $(\Gamma_1, \Gamma_2, \Gamma_3)$ has an apparent physical description with respect to the original coordinates (θ, Φ, ψ) , that is: the coordinate Γ_1 still describes the nutation, the coordinate Γ_2 still describes the precession and Γ_3 describes the spin. As we shall see later, this fact simplifies considerably the geometrical analysis of the solutions. Now, we shall obtain the approximate solutions associated with the elliptic mode with this method, and then we compare them with the exact solutions. We are interested in the geometrical interpretation of these solutions and then verify the globality of them (that is we intend to check how close are the approximate solutions with respect to the exact solutions for long intervals of time).

3.3 Approximate solution for the PSP

The initial conditions that characterize the elliptical mode in an inertial reference frame are given by

$$\begin{aligned} \theta(0) &= \theta_0, \quad \Phi(0) = \pi/2, \quad \psi(0) = 0, \\ \dot{\theta}(0) &= 0, \quad \dot{\Phi}(0) = \dot{\Phi}_0, \quad \dot{\psi}(0) = 0, \end{aligned} \quad (82a)$$

for these conditions, we have again that $P_\theta(0) = 0$, and in terms of the new coordinates they transform according with (61) into

$$\bar{\theta}_0 = \sin \theta_0, \quad \bar{P}_{\theta 0} = P_\theta(0) \sec \theta_0. \quad (83)$$

The dimensionless constants of motion P_Φ and P_ψ are determined from these conditions and equations (55, 56, 59)

$$P_\Phi = \frac{p_\Phi}{\sqrt{mglI_{\bar{x}}}} = \frac{\dot{\Phi}_0}{\sqrt{mglI_{\bar{x}}}} [I_{\bar{x}} \bar{\theta}_0^2 + I_{\bar{z}} (1 - \bar{\theta}_0^2)], \quad (84a)$$

$$P_\psi = \frac{p_\psi}{\sqrt{mglI_{\bar{x}}}} = \frac{I_{\bar{z}}}{\sqrt{mglI_{\bar{x}}}} \left(\dot{\Phi}_0 \sqrt{1 - \bar{\theta}_0^2} \right), \quad (84b)$$

and the constants P_1 , P_2 , Q_{10} , Q_{20} are obtained from (71)

$$P_1 = \frac{P_2^2}{2\bar{\theta}_0^2} + \frac{\bar{\theta}_0^2}{2}, \quad P_2 = P_\Phi - P_\psi, \quad (85a)$$

$$k = \frac{\bar{\theta}_0^2}{2} - \frac{P_2^2}{2\bar{\theta}_0^2}. \quad (85b)$$

$$Q_{10} = \frac{\pi}{4}, \quad Q_{20} = \frac{\pi}{2} - \arctan \left(\frac{P_1 + k}{P_2} \right), \quad (86)$$

The coordinate $\bar{\theta}$, is given by (70a)

$$\bar{\theta} = \sqrt{P_1 + k \cos \left[\frac{1}{4} \left(8 - P_1 + P_\psi^2 \right) \tau \right]}, \quad (87)$$

while the azimuthal angle Φ given by (70c) presents the same discontinuities observed in (36b), it is worked out in a similar way of the previous case by introducing the auxiliary function

$$\gamma = \frac{2k \cos^2(Q_1 + \tau) + (P_1 - P_2) \sin[2(Q_1 + \tau)]}{(P_1 + P_2) - (P_1 - P_2) \cos[2(Q_1 + \tau)] + k \sin[2(Q_1 + \tau)]}, \quad (88)$$

and the parameter Q_{20} in (86) can be rewritten in terms of the initial conditions as

$$Q_{20} = \frac{\pi}{4} - \arctan\left(\frac{\bar{\theta}_0^2 - P_2}{\bar{\theta}_0^2 + P_2}\right), \quad (89)$$

such that the approximate solution for the azimuthal angle becomes

$$\Phi = Q_2 + (Q_1 + \tau) + \arctan \gamma. \quad (90)$$

And the spin angle ψ can be obtained from (81).

The period of the system, that by definition is twice the period of the angle $\bar{\theta}$, is given by

$$T = \frac{16\pi}{8 - P_1 + P_\psi^2}, \quad (91)$$

comparing with the corresponding period associated with the ISP Eq. (41), we note that the angular momentum associated with the spin P_ψ of the PSP, introduces two corrections: **(1)** In P_1 by means of the definition (85a) of P_2 [compare with Eqs. (31a, 34a)], and **(2)** In the quadratic correction of the denominator of (91).

From (70c) and (81) we infer that $\dot{\Phi}$ and $\dot{\psi}$ are periodic functions with the same period T given by (91). Therefore, we can obtain linear approximations for the angles Φ and ψ by using the following definitions

$$\begin{aligned} \tilde{\Phi}(\tau) &\equiv \Phi_0 + \left[\frac{1}{T} \int_0^T \dot{\Phi}(s) ds \right] \tau = \Phi_0 + \left[\frac{\Phi(T) - \Phi(0)}{T} \right] \tau, \\ \tilde{\Phi}(\tau) &= \frac{\pi}{2} + \frac{1}{8} (8 - P_1 + 3P_2 + 4P_\psi + P_\psi^2) \tau, \end{aligned} \quad (92)$$

$$\begin{aligned} \tilde{\psi}(\tau) &\equiv \psi_0 + \left[\frac{1}{T} \int_0^T \dot{\psi}(s) ds \right] \tau = \psi_0 + \left[\frac{\psi(T) - \psi(0)}{T} \right] \tau, \\ \tilde{\psi}(\tau) &= \frac{1}{8} (-8 + P_1 + P_2 - 4P_\psi - P_\psi^2 + 2P_1P_\psi + 8\alpha P_\psi^2) \tau. \end{aligned} \quad (93)$$

The mean angular velocity of apsidal precession (that we call Allais' precession), can be obtained analogously to the procedure in (43)

$$\Omega_{\text{Allais}} \equiv \frac{\Phi(T) - \Phi(0) - 2\pi}{T} \quad (94)$$

$$\Omega_{\text{Allais}} = \frac{3P_2}{8} + \frac{P_\psi}{2} = \frac{1}{8} (3P_\Phi + P_\psi), \quad (95)$$

where we have used Eqs. (90, 91) and Eq. (85a). By comparing Eq. (95) with Eq. (43) we see clearly the contribution of the spin to the Allais' precession, through the canonical momentum P_ψ [compare also the expressions (31a) and (85a) of P_2 for the ISP and PSP respectively]. We also note that the correction introduced by the spin is of the same order of the Z -component of the angular momentum P_2 .

Now, introducing the numerical values

$$\begin{aligned} I_{\bar{x}} &= I_{\bar{y}} = 1Kg \cdot m^2, \quad I_{\bar{z}} = 10^{-2}Kg \cdot m^2, \quad l = 1m, \quad m = 1Kg \\ \theta_0 &= 0.1, \quad \Phi_0 = \pi/2, \quad \psi_0 = 0, \quad (\text{rad}) \\ \dot{\theta}_0 &= 0, \quad \dot{\Phi}_0 = 1, \quad \dot{\psi}_0 = 0, \quad (\text{rad/seg}), \end{aligned} \quad (96)$$

we obtain

$$\begin{aligned} P_1 &= 0.0054868, \quad P_2 = 0.00316787, \quad P_\psi = 0.00317842 \\ k &= 0.00447991, \quad Q_{10} = \frac{\pi}{4}, \quad Q_{20} = 0.306786, \quad \bar{\psi}_0 = \pi/2. \end{aligned} \quad (97a)$$

the period of motion reads

$$T = \frac{16\pi}{8 - P_1 + P_\psi^2} \frac{1}{\Omega_0} = 2.00846 \text{ s}, \quad (98)$$

which is in excellent agreement with the exact solution [16]. The apical Allais' precession in this approximation gives

$$\Omega_{\text{Allais}} = 0.00871254 \text{ rad/s}. \quad (99)$$

In Fig. 6 we compare the exact solution for the nutation angle θ with the approximate one Eq. (87), for $500 \leq t \leq 504$ s. It is observed that both solutions are superposed and cannot be distinguished, showing the excellent agreement between them

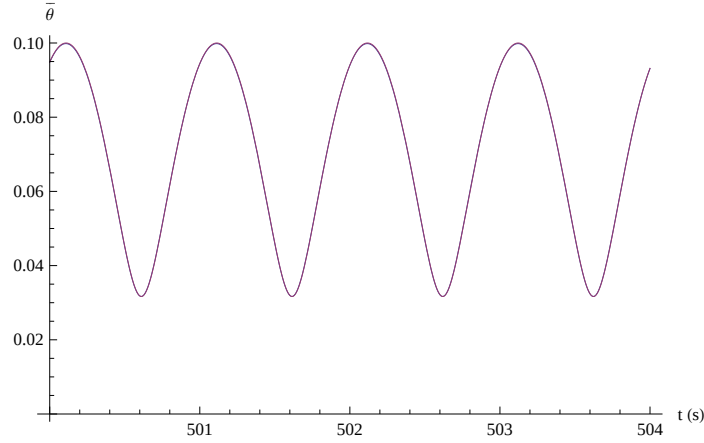


Figure 6: Plot of the approximate solution (87) for the nutation angle $\bar{\theta}$ ($\sin \theta$) as a function of time, for $500 \leq t \leq 504$ s. The exact solution is superposed to the approximate one.

In Fig. 7 we compare the exact and approximate solutions for the azimuthal angle Φ (90) and the spin angle ψ (81), for $500 \leq t \leq 504$ s. Once again, the solutions are indistinguishable showing the excellent agreement between them. The straight lines shown in these figures are the linear approximations (92) and (93). On the other hand, the interval of time chosen shows the global character of the approximate solutions.

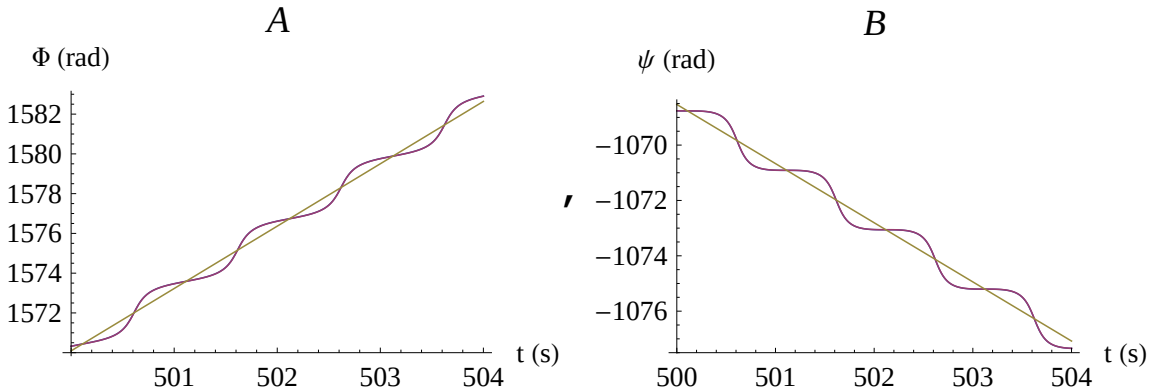


Figure 7: (A) Approximate solutions for the azimuthal angle Φ , for $500 \leq t \leq 504$ s, (B) Approximate solutions of the spin angle ψ , for $500 \leq t \leq 504$ s. The straight lines correspond to the linear approximations for each one of these angles. The exact solutions cannot be distinguished from the approximate ones.

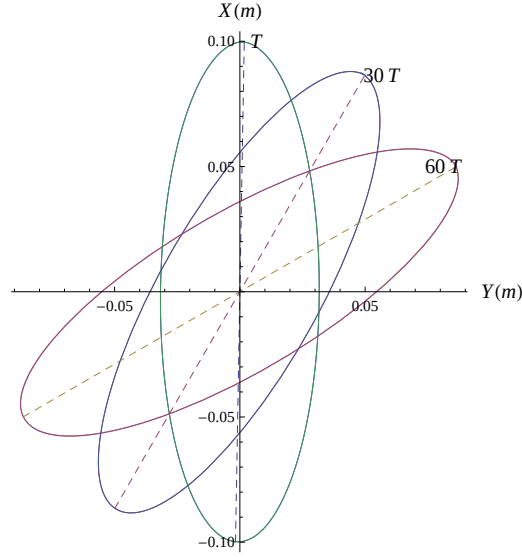


Figure 8: *Projections of the trajectory of the CM in the XY –plane, obtained with the approximate solutions, for time intervals of a period T , (A) for $0 \leq t \leq T$: (T). (B) For $29T \leq t \leq 30T$: ($30T$). (C) for $59T \leq t \leq 60T$: ($60T$). In all cases the exact solutions are superposed to the approximate ones.*

Finally, we compare in Fig. 8 the projections of the trajectory in the XY –plane obtained from the exact solution and from the approximate solution given by (see appendix):

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos(\Omega_{\text{Allais}}\tau) & -\sin(\Omega_{\text{Allais}}\tau) \\ \sin(\Omega_{\text{Allais}}\tau) & \cos(\Omega_{\text{Allais}}\tau) \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \cos(\omega\tau) \\ (P_2/\bar{\theta}_0) \sin(\omega\tau) \end{bmatrix} \quad (100)$$

where

$$q_1 = \frac{x}{l} = \bar{\theta} \sin \Phi, \quad q_2 = \frac{y}{l} = -\bar{\theta} \cos \Phi, \quad \omega = \frac{1}{8} (8 - P_1 + P_\psi^2). \quad (101)$$

once again the solutions are practically identical. The dotted straight lines show the evolution of the apsidal axis for the intervals of time plotted, which obey the linear equation

$$y = x \tan(\Omega_{\text{Allais}} nT), \quad \text{with } n = 1, 30, 60 \quad (102)$$

4 Concluding remarks

We have obtained approximate solutions for the equations of motion of the Ideal Spherical Pendulum (ISP) and the Physical Symmetrical Pendulum (PSP), by using a Hamilton-Jacobi approach along with the technics of variation of parameters and the averaging method. The success of the averaging method is due to the presence of variables that vary very slowly within a period of the polar angular coordinate.

In particular, we found approximate expressions for the apsidal precession that undergoes an ISP and a PSP in their elliptical modes, when the pendula are initiated with small initial amplitudes and small angular momenta of precession with respect to an inertial reference frame. In addition, we obtained approximate analytical expressions that describe the projections of the trajectories on the horizontal plane for both types of pendula.

The method implemented in this paper to obtain these expressions is operatively and conceptually simpler than the standard procedure used in obtaining the exact expressions in terms of elliptical functions of Jacobi and Weierstrass. In particular, our method permits to show with clarity the origin of the precession of the ISP and of the PSP. Moreover, in the case of the PSP, our approximate expressions introduce in a natural way the correction due to the spin for its apsidal precession. Up to our knowledge, this is a new approximation for an effect already observed in the paraconical pendulum with symmetrical support.

On the other hand, this method also outlines the way to obtain approximate solutions for the Physical Asymmetrical Pendulum (paraconical pendulum), which is still an open problem. It is because canonical transformations to slowly varying coordinates are also possible for the Asymmetrical Pendulum, and in terms of such kind of coordinates the new averaged Hamiltonian leads to trivial (though approximate) equations of motion.

The examples presented here show an excellent numerical agreement of our approximations with respect to the exact solutions. Further, such an agreement is preserved even for long periods of time, so that they are globally valid. Finally, as a consequence of the geometrical interpretation of the approximate trajectories, we have associated the period of elliptical motion T with the period of oscillations of the coordinate $\bar{\theta}$, this association permits to propose T as the period of motion, which coincides with the exact period up to the order of microseconds.

5 Acknowledgments

We acknowledge to División de Investigación de Bogotá (DIB) of Universidad Nacional de Colombia, for its financial support.

6 Appendix

We present in this appendix the deduction of Eqs. (46) and (100), which expresses the geometrical interpretation that we have done for the approximate solutions obtained in this paper. Further, we prove the canonical character of the Linear Transformation introduced in (72).

1. In order to deduce Eq. (46)

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos(\omega_a \tau) & -\sin(\omega_a \tau) \\ \sin(\omega_a \tau) & \cos(\omega_a \tau) \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \cos(\omega \tau) \\ (\alpha_2/\bar{\theta}_0) \sin(\omega \tau) \end{bmatrix}. \quad (103)$$

we start with the following definitions

$$q_1 = \frac{x}{l} = \sin \theta \cos \varphi = \bar{\theta} \cos \varphi, \quad (104a)$$

$$q_2 = \frac{y}{l} = \sin \theta \sin \varphi = \bar{\theta} \sin \varphi, \quad (104b)$$

and from the approximate solution for the angle of precession φ Eqs. (34)-(36b) we get

$$Q_1 = -\frac{\alpha_1}{8}\tau + Q_{10}, \quad Q_2 = \frac{3\alpha_2}{8}\tau + Q_{20}, \quad (105a)$$

$$\alpha_1 = \frac{\bar{\theta}_0^2}{2} + \frac{\alpha_2^2}{2\bar{\theta}_0^2}, \quad k = \frac{\bar{\theta}_0^2}{2} - \frac{\alpha_2^2}{2\bar{\theta}_0^2}, \quad (105b)$$

$$k^2 = \alpha_1^2 - \alpha_2^2 \quad (105c)$$

$$Q_{10} = \frac{\pi}{4}, \quad Q_{20} = -\arctan\left(\frac{k + \alpha_1}{\alpha_2}\right) \quad (105d)$$

$$\varphi = Q_2 + \arctan\left\{\frac{1}{\alpha_2}\left[k + \alpha_1 \tan\left(\omega\tau + \frac{\pi}{4}\right)\right]\right\} \quad (106)$$

$$\varphi = \omega_a \tau + Q_{20} + \eta \quad (107)$$

where

$$\eta \equiv \arctan\left\{\frac{1}{\alpha_2}\left[k + \alpha_1 \tan\left(\omega\tau + \frac{\pi}{4}\right)\right]\right\}, \quad (108)$$

$$\omega_a = \frac{3\alpha_2}{8}, \quad \omega = 1 - \frac{\alpha_1}{8}. \quad (109)$$

From the definitions (108) and (105c) it can be proved that

$$\cos \eta = \frac{\alpha_2 \cos \left(\omega \tau + \frac{\pi}{4} \right)}{\sqrt{\alpha_1} \bar{\theta}}, \quad (110a)$$

$$\sin \eta = \frac{k \cos \left(\omega \tau + \frac{\pi}{4} \right) + \alpha_1 \sin \left(\omega \tau + \frac{\pi}{4} \right)}{\sqrt{\alpha_1} \bar{\theta}} \quad (110b)$$

and without any kind of approximation we find

$$\cos (Q_{20} + \eta) = \frac{\sqrt{\alpha_1 + k} \cos (\omega \tau)}{\bar{\theta}}, \quad (111)$$

$$\sin (Q_{20} + \eta) = \frac{\sqrt{\alpha_1 - k} \sin (\omega \tau)}{\bar{\theta}}, \quad (112)$$

and from the initial conditions associated with the elliptical mode (105b) it follows that

$$\sqrt{\alpha_1 + k} = \bar{\theta}_0, \quad \sqrt{\alpha_1 - k} = \frac{\alpha_2}{\bar{\theta}_0}$$

Picking up all these results in Eq. (104a) we see that

$$q_1 = \bar{\theta} \cos \varphi = \bar{\theta}_0 \cos (\omega \tau) \cos (\omega_a \tau) - \frac{\alpha_2}{\bar{\theta}_0} \sin (\omega \tau) \sin (\omega_a \tau), \quad (113)$$

with a similar result for q_2 .

2. In order to deduce Eq. (100)

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos (\Omega_A \tau) & -\sin (\Omega_A \tau) \\ \sin (\Omega_A \tau) & \cos (\Omega_A \tau) \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \cos (\omega \tau) \\ (P_2 / \bar{\theta}_0) \sin (\omega \tau) \end{bmatrix} \quad (114)$$

We take into account that $\Phi = \varphi + \pi/2$, where φ is the spherical coordinate and Φ is the Euler angle. The definitions (104) are converted into

$$q_1 = \frac{x}{l} = \sin \theta \sin \Phi = \bar{\theta} \sin \Phi, \quad (115a)$$

$$q_2 = \frac{y}{l} = -\sin \theta \cos \Phi = -\bar{\theta} \cos \Phi. \quad (115b)$$

On the other hand, the approximate solution for the azimuthal angle Eqs. (70c), (80b) and (86) are given by

$$\Phi = \left(\frac{3P_2}{8} + \frac{P_\psi}{2} \right) \tau + Q_{20} + \arctan \left\{ \frac{1}{P_2} [k + P_1 \tan (Q_1 + \tau)] \right\}, \quad (116a)$$

$$Q_1 = \frac{1}{8} (P_\psi^2 - P_1) \tau + Q_{10}, \quad (116b)$$

$$Q_{10} = \frac{\pi}{4}, \quad Q_{20} = \frac{\pi}{2} - \arctan \left(\frac{P_1 + k}{P_2} \right), \quad (116c)$$

$$P_1 = \frac{P_2^2}{2\bar{\theta}_0^2} + \frac{\bar{\theta}_0^2}{2}, \quad P_2 = P_\Phi - P_\psi, \quad (116d)$$

$$k = P_1^2 - P_2^2 = \frac{\bar{\theta}_0^2}{2} - \frac{P_2^2}{2\bar{\theta}_0^2}. \quad (116e)$$

which can be expressed as

$$\Phi = \Omega_A \tau + \frac{\pi}{2} - \lambda + \eta, \quad (117)$$

where we have used the auxiliary functions

$$\Omega_A \equiv \frac{3P_2}{8} + \frac{P_\psi}{2}, \quad \omega \equiv 1 - \frac{P_1}{8} + \frac{P_\psi^2}{8} \quad (118)$$

$$\lambda \equiv \arctan \left(\frac{P_1 + k}{P_2} \right), \quad \eta \equiv \arctan \left\{ \frac{1}{P_2} \left[k + P_1 \tan \left(\omega \tau + \frac{\pi}{4} \right) \right] \right\} \quad (119)$$

from these definitions, it is easy to prove that

$$\cos \lambda = \frac{P_2}{\sqrt{2P_1}\sqrt{P_1+k}}, \quad \sin \lambda = \frac{\sqrt{P_1+k}}{\sqrt{2P_1}} \quad (120)$$

$$\cos \eta = \frac{P_2 (\cos \omega \tau - \sin \omega \tau)}{\bar{\theta} \sqrt{2P_1}}, \quad (121)$$

$$\sin \eta = \frac{(P_1+k) \cos \omega \tau + (P_1-k) \sin \omega \tau}{\bar{\theta} \sqrt{2P_1}}. \quad (122)$$

using these results and Eq. (117) in Eqs. (115a, 115b), we have

$$\begin{aligned} q_1 &= \bar{\theta} \sin \Phi = \sqrt{P_1+k} \cos(\Omega_A \tau) \cos \omega \tau - \sqrt{P_1-k} \sin(\Omega_A \tau) \sin \omega \tau, \\ q_2 &= -\bar{\theta} \cos \Phi = \sqrt{P_1+k} \sin(\Omega_A \tau) \cos \omega \tau + \sqrt{P_1-k} \cos(\Omega_A \tau) \sin \omega \tau, \end{aligned}$$

and considering finally the conditions (116d) and (116e) it is easy to see that

$$\sqrt{P_1+k} = \bar{\theta}_0, \quad \sqrt{P_1-k} = \frac{P_2}{\bar{\theta}_0}, \quad (123)$$

which proves Eq. (114).

- Now, to prove the canonical character of the linear transformation in (72), we introduce a more general linear transformation $(Q, P) \rightarrow (\Gamma, \Pi)$ whose definition is

$$Q_i = f_i(\Pi) \tau + \Gamma_i, \quad P_i = \Pi_i, \quad (124)$$

where f_i is any polynomic function of the momenta Π_k . The canonical character of this family of transformations follows from the Poisson brackets of the variables Q_i, P_i with respect to the new variables Γ_k, Π_k

$$[Q_i, P_j]_{\Gamma_k, \Pi_k} = \frac{\partial Q_i}{\partial \Gamma_k} \frac{\partial P_j}{\partial \Pi_k} - \frac{\partial Q_i}{\partial \Pi_k} \frac{\partial P_j}{\partial \Gamma_k} = \delta_{ij} \quad (125)$$

References

- [1] Lagrange, J. L.: *Mécanique Analytique*. Veuve Desaint, Paris (1788)
- [2] Poisson, S. D.: *Sur un cas particulier du mouvement de rotation des corps pesants*. Journal de l'École Polytechnique. 16, 247-267 (1813)
- [3] Golubev, V. V.: *Lectures on Integration of Equations of Motion of a Rigid Body about a Fixed Point*. Israeli Program for Scientific Translations, Israel (1960)
- [4] Leimanis, E.: *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*. Springer Tracts in Natural Philosophy Vol. 7. Springer Verlag, Berlin (1965)
- [5] K. F. Johansen and T. R. Kane. *A simple description of the motion of a spherical Pendulum*. J. Appl. Mech., 36, 76-82. (1969)
- [6] J. W. Miles. *Stability of forced oscillations of a spherical pendulum*. Q. Appl. Mat., 20, 21-32. (1962)
- [7] G. W. Hemp., and P. R. Sethna. *The effect of high-frequency support oscillation on the motion of a spherical pendulum*. J. Appl. Mech., 31, 351-354. (1964)
- [8] G. B. Airy, *On the vibration of a free pendulum in an oval differing little from a straight line*. R. Astron. Soc. XX, 121-130 (1851). (<http://home.t01.itscom.net/allais/whiteprior/airy/airyprecession.pdf>)
- [9] M. G. Olsson, *The precessing spherical pendulum*, Am. J. Phys. 46 (1978), 1118.

- [10] J. Synge and B. Griffith, *Principles of Mechanics*, McGraw-Hill, New York, 1959
- [11] [A. V. Gusev and M. P. Vinogradov, *Angular velocity of rotation of the swing plane of a spherical pendulum with anisotropic suspension*, *Measurement Techniques*, Vol **36**, N°10, \(1993\).](#)
- [12] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, 3rd Ed. Addison-Wesley (2002).
- [13] Allais, M.: *The Allais effect and my experiment with the paraconical pendulum 1954-1960*. Memories prepared for the NASA, Paris (1999), 167 pp
- [14] T. J. Goodey et al., *Correlated anomalous effects observed during the August 1st 2008 solar eclipse*, Journal of Advanced Research in Physics 1(2), 021007 (2010).
- [15] D. Olenici, V. A. Popescu, and B. Olenici, *A confirmation of the Allais and Javerdan-Rusu-Antonescu effects during the solar eclipse from 22 September 2006, and the quantization behavior of pendulum*, Proceedings of the 7th European meeting of the Society for Scientific Exploration, (2007).
- [16] [A. J. Brizard. *A primer on elliptic functions with applications in classical mechanics*. arXiv:0711.4064; \(2007\).](#)
- [17] Bogolyubov, N. and Mitropolsky, Y.: *Asymptotic methods in theory on nonlinear oscillations*, Ed. Gordon Breachs, New York (1961).
- [18] [E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Body*, 4 ed. Dover, New York \(1944\).](#)
- [19] [N. Bogolyubov and Y. Mitropolsky, *Asymptotic methods in theory on nonlinear oscillations*, Ed. Gordon Breachs, New York \(1961\).](#)