Archive of old C.4 and C.5 on Sun Jan 29 PhL 1.29.17

C.4 The Spherical Pendulum

We now digress for a subsection on the spherical pendulum, then in the Section C.5 we resume discussion of the use of such a pendulum in its "Foucault pendulum" mode.

(a) The equations of motion

If we momentarily *turn off* the rotation of the Earth by setting $\omega = 0$, equations (C.3.9) become

$$\begin{aligned} &\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/\ell)\sin\theta && \hat{\boldsymbol{\theta}} \\ &2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \tag{C.4.1}$$

These are the equations of motion for a spherical pendulum in the presence of a uniform gravitational field of strength g. It is useful to know something about the solution of these equations before we turn the Earth's rotation back on.

Operationally, in the sense of a numerical solution, one can regard the last two equations of (C.4.1) as a pair of coupled second-order non-linear ODE's for functions $\theta(t)$ and $\varphi(t)$ subject to initial conditions θ_0, φ_0, $\dot{\theta}_0$ and $\dot{\varphi}_0$. Once these equations are solved (physically we know a unique solution must exist), the first equation may be used to determine T(t), the string tension. In practice, there are some difficulties with numerical integration since $\theta = 0$ is a singular point in spherical coordinates (φ is undefined there). This is discussed a bit in Section F.7 for the dumbbell satellite which has similar equations of motion.

(b) L_z as constant of the motion

The last equation in (C.4.1) may be written in this form

$$d/dt\,(\sin^2\theta\,\dot{\varphi}) = 0 \tag{C.4.2}$$

which says that $\sin^2\theta\,\dot{\varphi}$ must be a "constant of the motion". To understand the meaning of this constant, we first compute the torque about the pivot point due to the gravitational force on the mass m,

$$\mathbf{N} = \mathbf{r} \times m\mathbf{g} = \mathbf{r} \times mg\hat{\mathbf{z}} = mg\ell\,\hat{\mathbf{r}} \times [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] = -mg\ell\sin\theta\,\hat{\boldsymbol{\varphi}} \quad . \tag{C.4.3}$$

Since this torque lies in a plane normal to the z axis, we may conclude that the Cartesian torque component $N_z = 0$. The angular version of Newton's Law says $\mathbf{N} = d\mathbf{L}/dt$, and therefore we expect that the quantity L_z will be a constant of the motion. Direct calculation shows that

$$\begin{aligned} L_z &= \mathbf{L} \bullet \hat{\mathbf{z}} = m\,(\mathbf{r} \times \mathbf{v}) \bullet \hat{\mathbf{z}} = m\,(\hat{\mathbf{z}} \times \mathbf{r}) \bullet \mathbf{v} = m\,\ell\sin\theta\,\hat{\boldsymbol{\varphi}} \bullet (\ell\,[\dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}]) \\ &= m\ell^2\sin^2\theta\,\dot{\varphi} \end{aligned} \tag{C.4.4}$$

where we have made use of (C.1.4) for $\hat{\mathbf{z}}$ and (C.3.7) for **v**. Thus, we see that our third equation of motion in (C.4.1), rewritten as in (C.4.2), is just the statement that $dL_z/dt = 0$. In the Lagrangian formulation one finds that L_z is one of the canonical momenta which is constant because φ is a cyclic coordinate, meaning it does not appear in the Lagrangian.

In terms of the spherical unit vectors, one finds (using (C.3.7) for **v)** that the vector angular momentum of the pendulum mass is given by

$$\mathbf{L} = m\,\mathbf{r} \times \mathbf{v} = m\ell^2\,[\dot{\theta}\,\hat{\boldsymbol{\varphi}} - \sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\theta}}] \quad . \tag{C.4.5}$$

Then application of $\mathbf{N} = d\mathbf{L}/dt$, with **N** as in (C.4.3) and unit-vector change rates as in (C.3.6), simply reproduces the last two equations of motion in (C.4.1).

So we conclude that

$$h \equiv L_z/(m\ell^2) = \sin^2\theta\,\dot{\varphi} \tag{C.4.6}$$

is a constant of the motion of the spherical pendulum.

(c) E as another constant of the motion

We have shown in (C.4.6) that $\dot{\varphi} = h/\sin^2\theta$. It this is substituted into the second equation of (C.4.1) one obtains

$$\ddot{\theta} - h^2\cos\theta\,/\sin^3\theta + (g/\ell)\sin\theta = 0 \tag{C.4.7}$$

and we just put this equation on hold for a moment, noticing that it is a second order ODE.

Another constant of the motion is the total energy E which may be regarded as E = T + V = kinetic energy + potential energy,

$$E = (1/2)mv^2 - mg\ell\cos\theta = (1/2)\,m\,\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mg\ell\cos\theta \tag{C.4.8}$$

where $v^2 = \ell^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)$ according to (C.3.7) for **v**. By rescaling the energy, we can take this constant of the motion to be

$$\mathcal{E} = \dot{\theta}^2/2 + \sin^2\theta\dot{\varphi}^2/2 - (g/\ell)\cos\theta \quad . \qquad E = m\,\ell^2\mathcal{E} \tag{C.4.9}$$

Using (C.4.6) to eliminate $\dot{\varphi}$ gives,

$$\mathcal{E} = \dot{\theta}^2/2 + (h^2/2\sin^2\theta) - (g/\ell)\cos\theta$$

or (C.4.10)

$$\mathcal{E} = \dot{\theta}^2/2 + V_e(\theta) \qquad \text{where} \qquad V_e(\theta) \equiv (h^2/2\sin^2\theta) - (g/\ell)\cos\theta \quad .$$

Unlike (C.4.7), equation (C.4.10) is a first-order ODE so we prefer it to (C.4.7). In fact, if one multiplies (C.4.7) by $\dot{\theta}$ and notes that $\dot{\theta}\,\ddot{\theta} = (1/2)\,\partial_t(\dot{\theta}^2)$ and does a few easy integrals, one obtains an equation of the form ∂_t[stuff] = 0 so then stuff = constant. That "stuff" is the right side of (C.4.10) and we have then provided an interpretation for the constant (energy) [$\dot{\theta}$ is an "integrating factor" for (C.4.7)].

(d) Analytic solution to the spherical pendulum problem (outline)

Equation (C.4.10) is a first order non-linear ODE for θ(t) which we know how to solve:

$$\dot{\theta} = (2\mathcal{E} - 2V_e(\theta))^{1/2} \qquad \Rightarrow \qquad d\theta = (2\mathcal{E} - 2V_e(\theta))^{1/2} dt$$

$$\Rightarrow dt = (2\mathcal{E} - 2V_e(\theta))^{-1/2}\, d\theta$$

$$\Rightarrow \quad t(\theta) = \int_{\theta_0}^{\theta} d\theta' \, \frac{1}{\sqrt{2\mathcal{E} - (h^2/\sin^2\theta') + (2g/\ell)\cos\theta'}} \quad . \qquad \text{(C.4.11)}$$

Letting $z' = \cos\theta'$, so $dz' = -\sqrt{1-z'^2}\, d\theta'$, this integral can be written as

$$t(\theta) = \int_{z}^{z_0} dz' \, \frac{1}{\sqrt{2\mathcal{E}\,[1-z'^2] - h^2 - (2g/\ell)z'[1-z'^2]}} \qquad // \; z = \cos\theta,\; z_0 = \cos\theta_0$$

$$= \int_{z}^{z_0} dz' \, \frac{1}{\sqrt{(2g/\ell)\,[\; z'^3 - (\mathcal{E}\ell/g)\, z'^2 - z' + (\ell/g)\,(\mathcal{E} - h^2/2\,)\,]}}$$

$$= \sqrt{\ell/(2g)} \int_{z}^{z_0} dx \, \frac{1}{\sqrt{(x-a)(x-b)(x-c)}}$$

where a,b,c are the roots of the cubic equation $x^3 - (\mathcal{E}\ell/g)\, x^2 - x + (\ell/g)\,(\mathcal{E} - h^2/2\,) = 0$. The dimensionless integral appearing on the last line can be evaluated in closed form using this indefinite integral,

```
Int(((x-a)*(x-b)*(x-c))^(-1/2),x);
```

$$\int \frac{1}{\sqrt{(x-a)(x-b)(x-c)}}\, dx$$

```
value(%);
```

$$2\,\frac{(-a+b)\sqrt{\frac{x-a}{-a+b}}\sqrt{\frac{x-c}{a-c}}\sqrt{\frac{x-b}{a-b}}\,\text{EllipticF}\left(\sqrt{\frac{x-a}{-a+b}},\sqrt{\frac{a-b}{a-c}}\right)}{\sqrt{x^3 - x^2 c - b x^2 + b x c - a x^2 + a x c + a b x - a b c}} \quad . \qquad \text{(C.4.12)}$$

Here EllipticF is the incomplete elliptic integral of the first kind,

$$\text{EllipticF}(\sin(\varphi),k) = \int_0^{\sin(\varphi)} dt \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} = F(\varphi,k) = F(\varphi|m) = F(\varphi\backslash\alpha) = sn^{-1}(\sin\varphi,k) \tag{C.4.13}$$

where $k = \sin\alpha$, $m = k^2$ and sn^{-1} is the inverse Jacobi sn function. Therefore, we have a closed form result for $t(\theta)$ which can then be "inverted" to obtain a solution for $\theta(t)$. Then from (C.4.6) we get

$$\dot{\varphi} = h/\sin^2(\theta(t)) \quad \Rightarrow \quad \varphi(t) = \varphi_0 + h \int_0^t dt' /\sin^2(\theta(t')) \tag{C.4.14}$$

and the problem of the spherical pendulum is completely solved more or less in closed form.

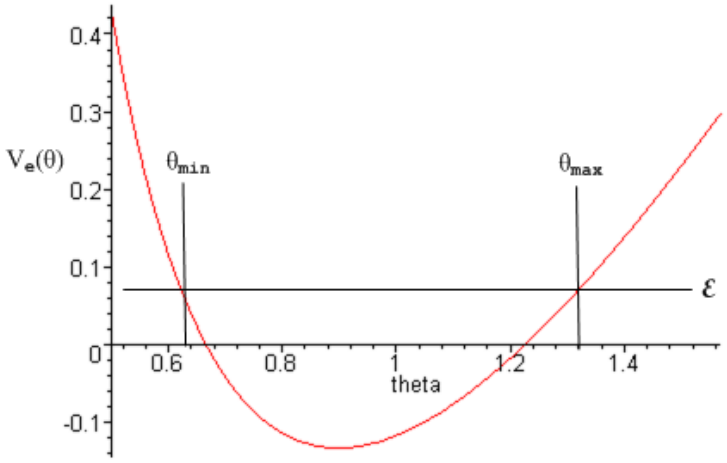
(e) The nature of the general solution

Now recall (C.4.10), which one regards as kinetic plus potential energy for the pendulum mass,

$$\mathcal{E} = \dot{\theta}^2/2 + V_e(\theta) \quad \text{where} \quad V_e(\theta) \equiv (h^2/2\sin^2\theta) - (g/\ell)\cos\theta \ . \tag{C.4.10}$$

The effective potential $V_e(\theta)$ has the following shape (plotted here for $h^2/2 = .3$ and $(g/\ell) = 1$)

```
plot(.3/sin(theta)^2 - cos(theta),theta=0.5..Pi/2);
```



(C.4.15)

Since $\dot{\theta}^2/2 = (\mathcal{E} - V_e(\theta))$ must be positive, one must have $V_e(\theta) \le \mathcal{E}$ which is valid only between θ_{min} and θ_{max} as shown. These are the "turning points" where $\dot{\theta} = 0$. Differentiation of (C.4.10) tells us that,

$$0 = \dot{\theta}\,\ddot{\theta} + V_e'(\theta)\,\dot{\theta} \qquad // \ V_e'(\theta) \equiv dV_e(\theta)/d\theta$$

or

$$\ddot{\theta} = -V_e'(\theta) \quad . \qquad \text{(C.4.16)}$$

At the right turning point the slope $V_e'(\theta)$ is positive so $\ddot{\theta} < 0$ which accelerates the particle to the left. At the left turning point the slope $V_e'(\theta)$ is negative so $\ddot{\theta} > 0$ which accelerates the particle to the right. The particle therefore bounces back and forth between these two turning points in some manner. Thus the motion of the pendulum is constrained on the spherical surface $r = \ell$ between two horizontal circles of angles θ_{min} and θ_{max} and hits both these angles once per "oscillation".

Meanwhile, from (C.4.14) the action in the φ dimension of the problem is controlled by

$$\dot{\varphi} = h/\sin^2(\theta(t)) \qquad => \qquad \varphi(t) = \varphi_0 + h \int_0^t dt' /\sin^2(\theta(t')) \qquad \textit{(C.4.14)}$$

so as $\theta(t)$ does the oscillation just discussed between θ_{min} and θ_{max}, $\varphi(t)$ increases as shown in its own complicated functional manner.

As we shall see in Section C.8, if the spherical pendulum is started in some narrow elliptical orbit, that orbit precesses clockwise roughly in proportion to the area of that ellipse. This "intrinsic apsidal precession" occurs even if the Earth is not rotating. In order to use a spherical pendulum as a "Foucault" pendulum (to be discussed below), one must launch the spherical pendulum with a sufficiently small elliptical area (ideally that area is zero) so the Foucault precession is not masked by the intrinsic precession. Zero area corresponds to $h = 0$ and $L_z = 0$ so $\dot{\varphi} = 0$ in (C.4.14).

The Foucault/Intrinsic signal ratio is improved by making the pendulum very long (67 m in Paris). In theory, perhaps by using the "burned string" launch method, one can have $h = 0$ so $\dot{\varphi} = 0$ exactly and then the spherical pendulum becomes a plane pendulum moving in the plane $\varphi = \varphi_0$. Schumacher and Tarbet (2009) discuss this subject and propose an active electronic device to neutralize the intrinsic precession so that one can have a much shorter length Foucault pendulum.

A quick tour of web animations of the spherical pendulum shows the amazing complexity of the possible motions. If the string is replaced with a massless stiff rod, over the top motions are included. Some examples (search youtube if these are dead links) :

http://www.youtube.com/watch?v=6hCLkTENfSA .
http://www.youtube.com/watch?v=VS1dU5HpfOM&feature=relmfu

Both animations demonstrate the $\theta_{min} \le \theta \le \theta_{max}$ idea, though it takes a while in the first animation.

(f) The spherical pendulum for small θ is still complicated unless h is very small

Setting $\sin\theta = \theta$ and $\cos\theta = 1$, the equations of motion (C.4.1) become

$$\begin{aligned} \ddot{\theta}^2 + \theta^2 \dot{\varphi}^2 &= -(g/\ell) + T/(m\ell) && \hat{\mathbf{r}} \\ \ddot{\theta} - \theta \dot{\varphi}^2 &= -(g/\ell)\,\theta && \hat{\boldsymbol{\theta}} \\ 2\dot{\theta}\dot{\varphi} + \theta\ddot{\varphi} &= 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \text{(C.4.17)}$$

Since in general $\dot{\varphi}^2$ is not small relative to (g/ℓ), one cannot trivially solve the second equation to find simple harmonic motion for θ. If one assumes that $\dot{\varphi}^2 << (g/\ell)$ (meaning very small h) and then neglects the $\dot{\varphi}^2$ term in the second equation, one obtains $\theta(t) \approx \theta_0 \cos(\Omega t)$ where $\Omega = \sqrt{g/\ell}$ which is the usual angular frequency for a small-angle plane pendulum.

(g) The Conical Motion solution

The spherical pendulum has an obvious simple solution where $\theta = \theta_0 =$ constant, so the string motion traces out a cone. In this case the equations of motion (C.4.1) become

$$\begin{aligned} \sin^2\theta_0\, \dot{\varphi}^2 &= -(g/\ell)\cos\theta_0 + T/(m\ell) && \hat{\mathbf{r}} \\ \cos\theta_0\, \dot{\varphi}^2 &= (g/\ell) && \hat{\boldsymbol{\theta}} \\ \ddot{\varphi} &= 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \text{(C.4.18)}$$

The second equation says $\omega_\varphi \equiv \dot{\varphi} = \sqrt{(g/\ell)\sec\theta_0}$. Since this is a constant, the third equation is satisfied as well, and then the first says $T = mg\sec\theta_0$. In the small angle limit for θ_0, ω_φ slows down to its smallest possible value $\omega_\varphi \equiv \sqrt{g/\ell} = \Omega$ which is the frequency of a small-angle plane pendulum. Conversely, as we try to achieve $\theta_0 \to \pi/2$, $\sec\theta_0 \to \infty$ and both ω_φ and tension T become infinite, which seems pretty reasonable. This solution can of course be obtained by elementary methods as well.

C.5 The Foucault Pendulum

The equations of motion for the spherical pendulum on the rotating Earth were stated in (C.3.9),

$$\begin{aligned} \ddot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2 &= -(g/\ell)\cos\theta + T/(m\ell) + 2\omega\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot{\varphi} + \sin\beta\sin\varphi\,\dot{\theta}\,] \\ \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 &= -(g/\ell)\sin\theta + 2\omega\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\sin\theta\,\dot{\varphi}] \\ 2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} &= -2\omega\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\dot{\theta}\,] \quad . \end{aligned} \qquad \textit{(C.3.9)}$$

Recall that the first equation serves only to determine string tension T when θ(t) and φ(t) are known, so we ignore this equation for the time being. We launch the pendulum as a "plane pendulum" with $\dot{\varphi}(t=0) = \dot{\varphi}_0 = 0$, but right away on the first half-swing the Coriolis deflection generates a small $\dot{\varphi} \neq 0$. It is then assumed that $\dot{\varphi}$ remains small enough that we may neglect it in the second equation. In the third equation, we neglect $\ddot{\varphi}$ which we expect to be smaller than $\dot{\varphi}$. We will check these assumptions after the fact. The equations of motion then become,

$$\begin{aligned} \ddot{\theta} &= -(g/\ell)\sin\theta \\ \dot{\varphi} &= \omega\,(\cos\beta - \sin\beta\cos\varphi\tan\theta) \quad . \end{aligned} \qquad \text{(C.5.1)}$$

At this point, we have two options, both of which involve approximations.

Option 1. (arm waving) We imagine that the pendulum is launched at t = 0 with θ = 0 and some velocity so it reaches a maximum angle $\theta_0 < \pi$ and then swings back and forth. The first equation in (C.5.1) is decoupled and is the usual equation which describes a large-swing plane (simple) pendulum. The exact solution θ(t) can be expressed in terms of the Jacobi sn function (see e.g. Lundmark's pendulum section),

$$\sin(\theta/2) = k\ \mathrm{sn}(\Omega t,k) \qquad k = \sin(\theta_0/2) \qquad \Omega = \sqrt{g/\ell} \qquad \theta(0) = 0 \qquad \text{(C.5.2a)}$$

or

$$\theta(t) = 2\ \sin^{-1}\left[\ k\ \mathrm{sn}(\Omega t, k)\right] \quad . \qquad \text{(C.5.2b)}$$

If θ_0 happens to be very small, then $k \approx \theta_0/2$ and (C.5.2a) says

$$(\theta/2) \approx (\theta_0/2)\ \mathrm{sn}(\Omega t, 0) = (\theta_0/2)\ \sin(\Omega t) \qquad \text{or} \qquad \theta(t) = \theta_0\ \sin(\Omega t)$$

which is the expected solution. The quarter period of sn(x;k) is not π/2 but rather **K**(k) where **K** is a complete elliptic integral. If we define $\Omega t' = \Omega t - \mathbf{K}(k)$, then at t' = 0 the pendulum is at θ_0 instead of 0. The solution for this initial condition is then

$$\sin(\theta/2) = k\ \mathrm{sn}(\Omega t + \mathbf{K}(k), k) \qquad k = \sin(\theta_0/2) \qquad \Omega = \sqrt{g/\ell} \qquad \theta(0) = \theta_0 \qquad \text{(C.5.3a)}$$

or

$$\theta(t) = 2\ \sin^{-1}\left[\ k\mathrm{sn}(\Omega t + \mathbf{K}(k),k)\right] \quad . \qquad \text{(C.5.3b)}$$

[Some sources use $\mathrm{sn}(z,k) = \mathrm{sn}(z \mid k^2)$]

The second equation in (C.5.1) is not decoupled, since both θ and φ appear, but we then make the following wobbly argument. The pendulum follows some path as suggested in Fig (C.2.1) which winds around the $\hat{\mathbf{z}}$ axis so that angle φ(t) monotonically increases in some non-linear but oscillatory manner. We conjecture that the long term average (many swings) of cosφ is therefore 0. Going even further out on this limb, we might conjecture that the long term time average of the product cosφ tanθ also vanishes as well. This would take some effort to study, so it is just an arm-waving conjecture which is the basis of Option 1. Then one finds from (C.5.1) that

$$\langle\dot{\varphi}\rangle = \omega\ (\cos\beta - \sin\beta\ \langle\cos\varphi\ \tan\theta\rangle) \quad = \omega\ \cos\beta \qquad \text{(C.5.4)}$$

so the effective version of (C.5.1) is this

$$\ddot{\theta} + (g/\ell)\ \sin\theta \ = 0$$
$$\langle\dot{\varphi}\rangle = \ \omega\ \cos\beta \qquad \text{(C.5.5)}$$

which says that the Foucault pendulum precesses on average at ωcosβ even if it has a large swing angle.

Why is sign different from that in (C.5) ??

Option 2. Here we are on firmer ground by assuming that θ_0 and therefore $\theta(t)$ are small angles. In this case, setting $\sin\theta = \tan\theta = \theta$ and $\cos\theta = 1$, (C.5.1) becomes

$$\ddot{\theta} = -(g/\ell)\theta$$
$$\dot{\varphi} = \omega(\cos\beta - \sin\beta\cos\varphi\,\theta) \quad . \qquad \text{(C.5.6)}$$

Unless we are extremely close to the equator of the Earth where $\cos\beta = 0$, the second term in parentheses is much smaller than the first, and we then have

$$\ddot{\theta} + (g/\ell)\theta = 0$$
$$\dot{\varphi} = \omega\cos\beta \quad . \qquad \text{(C.5.7)}$$

The solution of the first equation is $\theta(t) = \theta_0\cos(\Omega t)$ where $\Omega = \sqrt{g/\ell}$, the usual frequency of the small angle plane pendulum. The second equation then gives the precession rate of the Foucault pendulum. This is the main result of this Appendix.

If the pendulum is located at the North Pole, since the Earth rotates counterclockwise when viewed from above the North Pole, we expect the Foucault pendulum to rotate *clockwise* in non-inertial Frame S. Equation (C.5.6) in this case says $\dot{\varphi} = \omega$. Recall that $\omega > 0$ for the Earth in Fig (C.1.1) so $\dot{\varphi} > 0$. According to the left side of Fig (C.1.3), $\dot{\varphi} > 0$ does in fact imply clockwise rotation.

If the pendulum is located at the South Pole, since the Earth rotates clockwise when viewed from beneath the South Pole, we expect the Foucault pendulum to rotate counterclockwise in non-inertial Frame S, so we expect $\dot{\varphi} < 0$. This is the case since $\cos(\pi) = -1$.

In terms of the period of the precession, $\dot{\varphi} = \omega\cos\beta$ may be written as (ignoring direction)

$$(2\pi/T_{\mathbf{prec}}) = (2\pi/T_{\mathbf{sday}})\,|\cos\beta| = (2\pi/T_{\mathbf{sday}})\,|\sin(\theta_{\mathbf{LAT}})|$$

$$\Rightarrow \qquad T_{\mathbf{prec}} = T_{\mathbf{sday}}/\,|\sin(\theta_{\mathbf{LAT}})| \qquad \text{(C.5.8)}$$

and this is the fundamental Foucault precession result.

The sidereal day (rotation relative to the stars) is 23.93447 hours. The Foucault pendulum at the Pantheon in Paris should have this precession period :

$T_{\mathbf{prec}} = 23.9344696/\sin(48.846285°) = 31.787731$ hours $= 31$ hours 47.26 minutes $\qquad$ (C.5.9)

http://www.thegpscoordinates.com/france/paris/pantheon/ // latitude

Swing Period $= 2\pi/\Omega = 2\pi/\sqrt{g/\ell} \approx 16.4$ sec $\qquad \ell = 67$ m

```
Tprec := evalf(23.9344696 /sin(48.846285*Pi/180));
                    Tprec := 31.787731307617963437
(Tprec-31)*60;
                          47.2638784570778062
period := evalf(2*Pi/sqrt(9.81/67));
                          period := 16.42036347
```

(C.5.10)

Justification of assumptions

In the second of equations (C.2.9) we neglected both $\dot{\varphi}$ terms. Was this justified?

$$| \sin\theta\cos\theta\, \dot{\varphi}^2 | \approx \theta\, \dot{\varphi}^2 \approx \theta\omega^2 \cos\beta \sim \theta\, \omega^2$$

$$|- 2\omega\, [(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\sin\theta\, \dot{\varphi}] | \approx |2\omega \cos\beta\, \theta\, \omega\, (\cos\beta)| \sim \theta\, \omega^2 \ .$$

Both these terms are far smaller than the non-neglected term

$$| (g/\ell) \sin\theta | \sim \Omega^2\theta$$

since $\omega^2 << \Omega^2$. For a typical Foucault pendulum

$\Omega = 2\pi/T_{\Omega}$	$\omega = 2\pi/T_{day}$		
$T_{day}^2 >> T_{\Omega}^2$		?	
$(24*3600)^2 >> (16.4)^2$		?	// Pantheon
$7.46 \times 10^9 >> 269$	.	yes	