This is the Title PhL 1.11.15

Here is my Foucault in Cartesian coordinates.

$$\ddot{x} = -(T/mr)x - 2\omega\cos\beta\dot{y}$$
$$\ddot{y} = -(T/mr)y + 2\omega\,(\cos\beta\dot{x} + \sin\beta\dot{z})$$
$$\ddot{z} = g - (T/mr)z - 2\omega\sin\beta\dot{y} \quad . \qquad \text{(C.6.6)}$$

If I turn off the earth, seems that here is the spherical pendulum,

$$\ddot{x} = -(T/mr)x$$
$$\ddot{y} = -(T/mr)y$$
$$\ddot{z} = g - (T/mr)z \quad . \qquad (C.6.6)_{\omega=0}$$

Eliminating T the equations are

1 $y\ddot{x} - x\ddot{y} = -2\omega y\cos\beta\dot{y} - 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z})$

2 $z\ddot{x} - x\ddot{z} = -xg - 2\omega\dot{y}(z\cos\beta - x\sin\beta)$

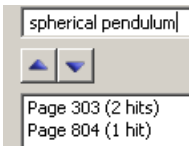
and

1 $y\ddot{x} - x\ddot{y} = 0$

2 $z\ddot{x} - x\ddot{z} = -xg$

This is too simple! What happened here? Have I omitted some fictional force? I think I have messed up the centrifugal force somehow! I said it was taken care of by g, but maybe that is wrong. The above says you have independent simple harmonic motion even for large amplitudes. [wrong! it only says this for small oscillations in which case $T \approx mg$.]

Who does a good basic treatment of "the spherical pendulum"? Time for a serious search, I need some external verification of things.

Taylor: 

Page 303 is a problem 7.40 setup but quotes no results. Wants us to use Lagrangian. Has three stars on the problem. Page 804 is this hit in the index. So nada.

Goldstein original. Only simple pendulum appears in the index.

Goldstein et al. This is djvu non-searchable I have no PDF. Index shows spherical, 83, 428 for spherical pendulum. P 83 comments on Lissajouz figures for small amplitude spherical pend. Page 428 says to set up the problem in Hamiltonian, just a problem. 9.37. So nothing here.

Osgood. Page 183 mentions small oscillations and ellipse. Full treatment topic on page 306. Writes T and U and then Lagrange equations of motion, his z is opposite mine. He gets these equations,

$$(1) \qquad \frac{d^2\theta}{dt^2} - \left(\frac{d\varphi}{dt}\right)^2 \sin\theta\cos\theta = \frac{g}{a}\sin\theta.$$

$$\frac{d}{dt}\left(\frac{d\varphi}{dt}\sin^2\theta\right) = 0.$$

These are the same as mine in (C.4.1). Osgood implies these apply to NOT just small angles. He combines the two equations to get

$$(3) \qquad \frac{d^2\theta}{dt^2} = \frac{h^2\cos\theta}{\sin^3\theta} + \frac{g}{a}\sin\theta.$$

Which is an equation involving only θ. I have a similar equation which is

$$2\mathcal{E} = \dot{\theta}^2 + (h^2/\sin^2\theta) - 2(g/\ell)\cos\theta \qquad E = m\,\ell^2\mathcal{E} \qquad (C.4.9)$$

But if I now replace $\theta \rightarrow \pi-\theta$ to convert to his system, then $\cos(\theta) \rightarrow \cos(\pi-\theta) = -\cos\theta$. $\sin\theta \rightarrow -\sin\theta$, so if I translate his equation I would get

$$2\mathcal{E} = \dot{\theta}^2 + (h^2/\sin^2\theta) + (g/2\ell)\cos\theta$$

But this is very different from his equation! Start over. Here are my equations:

$$\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) \qquad \hat{\mathbf{r}}$$

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$$

$$2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . \qquad \hat{\boldsymbol{\varphi}} \qquad (C.4.1)$$

where the last equation says

$$d/dt\,(\sin^2\theta\,\dot{\varphi}) = 0 \qquad \text{or} \qquad \sin^2\theta\,\dot{\varphi} = h \qquad \text{or} \qquad \dot{\varphi} = h/\sin^2\theta$$

If I install this into my second equation, I get

$$\ddot{\theta} - \sin\theta\cos\theta\,[h/\sin^2\theta]^2 = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$$

or

$\ddot{\theta} - \sin\theta\cos\theta\ h^2/\sin^4\theta = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$

or

$\ddot{\theta} - \cos\theta\ h^2/\sin^3\theta = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$

or

$\ddot{\theta} = \cos\theta\ h^2/\sin^3\theta - (g/\ell)\sin\theta$

which compare to Osgood

$$(3) \qquad \frac{d^2\theta}{dt^2} = \frac{h^2\cos\theta}{\sin^3\theta} + \frac{g}{a}\sin\theta.$$

Now the change $\theta \to \pi - \theta$ makes the two equations the same. So this is a simple way to get an ODE for θ for the spherical pendulum. My energy approach is a different path. Not clear what HIS (3) looks like for small θ.

OK, let's continue with Osgood and yes, he is NOT doing only small angles. He then switches to my θ angle calling it θ' and he gets

$$(3') \qquad \frac{d^2\theta'}{dt^2} = \frac{h^2\cos\theta'}{\sin^3\theta'} - \frac{g}{a}\sin\theta'.$$

which then matches my equation. He then finishes off by saying

be interpreted in terms of spherical pendulum motion.

A first integral of (3) can be obtained in the usual manner:

$$2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = \frac{2h^2\cos\theta}{\sin^3\theta}\frac{d\theta}{dt} + \frac{2g}{a}\sin\theta\frac{d\theta}{dt}.$$

Integrating each side with respect to θ, we obtain:

$$(5) \qquad \frac{d\theta^2}{dt^2} = -\frac{h^2}{\sin^2\theta} - \frac{2g}{a}\cos\theta + k.$$

For a further discussion of the problem, cf. Appell, *Mécanique rationelle*, vol. i, § 277.

So he bails out at this point.

So there are *two different* θ equations one can work with. In my notation

$2\mathcal{E} = \dot{\theta}^2 + (h^2/\sin^2\theta) - 2(g/\ell)\cos\theta \qquad E = m\ell^2\mathcal{E} \qquad (C.4.9)$

$\ddot{\theta} = \cos\theta\ h^2/\sin^3\theta - (g/\ell)\sin\theta \qquad h \equiv L_z/(m\ell^2) \qquad$ // above

Recall that

$$h \equiv L_z/(m\ell^2) = \sin^2\theta\, \dot{\varphi}$$

Therefore

$$0 = 2\sin\theta\cos\theta\, \dot{\theta}\, \dot{\varphi} + \sin^2\theta\, \ddot{\varphi}$$

Now my angular equations are

$$\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) \qquad \hat{\mathbf{r}}$$

$$\ddot{\theta} - \sin\theta\cos\theta\, \dot{\varphi}^2 = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$$

$$2\cos\theta\, \dot{\theta}\, \dot{\varphi} + \sin\theta\, \ddot{\varphi} = 0 \quad . \qquad \hat{\boldsymbol{\varphi}} \qquad \text{(C.4.1)}$$

where the third I have just derived again.

Comment: Osgood has **confirmed** my θ,φ equations for non-small motions! At least the last two of (C.4.1). I obtained these equations from $m\mathbf{a} = mg\hat{\mathbf{z}} + \mathbf{T} - 2m\,\boldsymbol{\omega} \times \mathbf{v}$ where $\boldsymbol{\omega}$ was Earth rotation, and then I set ω = 0. This says

$$m\mathbf{a} = mg\hat{\mathbf{z}} + \mathbf{T}$$

or

$$m\mathbf{a} = mg\hat{\mathbf{z}} - T\,\hat{\mathbf{r}}$$

There are NO OTHER FORCES ??? With no earth rotation, Frame S of mine is an inertial frame, and these are the correct equations! These correct equations give

$$\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) \qquad \hat{\mathbf{r}}$$

$$\ddot{\theta} - \sin\theta\cos\theta\, \dot{\varphi}^2 = -(g/\ell)\sin\theta \qquad \hat{\boldsymbol{\theta}}$$

$$2\cos\theta\, \dot{\theta}\, \dot{\varphi} + \sin\theta\, \ddot{\varphi} = 0 \quad . \qquad \hat{\boldsymbol{\varphi}} \qquad \text{(C.4.1)}$$

and Osgood has verified the last two equations.

Now go to Cartesian coordinates: I start from the same inertial-frame equation (if I set ω = 0)

$$m\mathbf{a} = mg\hat{\mathbf{z}} - T\,\hat{\mathbf{r}}$$

and this results in

$$\ddot{x} = -(T/mr)x$$

$$\ddot{y} = -(T/mr)y$$

$$\ddot{z} = g - (T/mr)z \qquad \text{(C.6.6)}$$

But beware that T is not a constant except for small oscillations!! My later equations are then

1 $\quad y\ddot{x} - x\ddot{y} = 0$

2 $\quad z\ddot{x} - x\ddot{z} = -xg$

3 $\quad x^2+y^2+z^2 = r^2$

where T is eliminated. Do these have some simple solution? Try this on the first two equations

$$zy\ddot{x} - zx\ddot{y} = 0$$

$$yz\ddot{x} - yx\ddot{z} = -xyg$$

so subtract to get

$$-zx\ddot{y} + yx\ddot{z} = xyg$$

or

$$-z\ddot{y} + y\ddot{z} = yg$$

or

$$y\ddot{z} - z\ddot{y} = yg$$

This does not really help. So the three variables x,y,z are entangled and it is unclear how you would get a solution! I have these equations then

$$y\ddot{x} - x\ddot{y} = 0$$

$$z\ddot{x} - x\ddot{z} = -xg$$

$$y\ddot{z} - z\ddot{y} = yg$$

So in fact spherical coordinates is much better in terms of trying to state an ODE!!

Conclusions at this point

1. For small oscillations, if we can say T ≈ mg, then the Cartesian equations for the spherical pendulum are these:

$$\ddot{x} = -(g/\ell)x$$
$$\ddot{y} = -(g/\ell)y$$

$$\ddot{z} = g - (g/\ell)z = g\,[\,1 - z/\ell\,]\ . \qquad (C.6.6)_{\omega=0}$$

The last equation can be interpreted as $\ddot{z} = g$ which is reasonable.

So in this limit, you get:

$$x(t) = A\cos(\Omega t+\varphi_1) \qquad \Omega = \sqrt{g/\ell}$$
$$y(t) = B\cos(\Omega t+\varphi_2)$$

We now have

$$(x/A)^2 = \cos^2(\Omega t+\varphi_1)$$
$$(y/B)^2 = \cos^2(\Omega t+\varphi_2)$$

I think we have a rotated ellipse. What is the equation of same? Start with this

$$x^2/C^2 + y^2/D^2 = 1$$

Now rotate by an angle α which then means

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{r}' = R\mathbf{r}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \mathbf{r}' = R\mathbf{r}$$

$$x = \cos\alpha\, x' + \sin\alpha\, y'$$

$$y = -\sin\alpha\, x' + \cos\alpha\, y'$$

Then the equation of the rotated ellipse centered at the origin is this

$$[\,\cos\alpha\, x' + \sin\alpha\, y']^2/C^2 + [-\sin\alpha\, x' + \cos\alpha\, y']^2/D^2 = 1$$

Now switch back to x y

$$[\,\cos\alpha\, x + \sin\alpha\, y]^2/C^2 + [-\sin\alpha\, x + \cos\alpha\, y]^2/D^2 = 1$$

and this is then a rotated ellipse. Compare to what I have which is

$$x(t) = A\cos(\Omega t+\varphi_1) \qquad \Omega = \sqrt{g/\ell}$$
$$y(t) = B\cos(\Omega t+\varphi_2)$$

Can you find α, C and D as functions of A,B,φ_1.φ_2 which cause this to be an ellipse? It is pretty messy, but Maple makes it look like an ellipse. The equation to solve is then this

$$[\cos\alpha\ A\cos(\Omega t+\varphi_1) + \sin\alpha\ B\cos(\Omega t+\varphi_2)]^2/C^2 + [-\sin\alpha\ A\cos(\Omega t+\varphi_1) + \cos\alpha\ B\cos(\Omega t+\varphi_2)]^2/D^2 = 1$$

and you want to solve for α,C,D. I tried this in Maple, and I got nowhere.

Go back to

$$x(t) = A\cos(\Omega t+\varphi_1) \qquad \Omega = \sqrt{g/\ell}$$
$$y(t) = B\cos(\Omega t+\varphi_2)$$

Why might this be an ellipse?

I have done this somewhere before!!! Maybe ocean wave particle motion? How can I find my document on this. Ransack on ellipse AND ocean gives 33 hits in physics. It is in lines doc and here is my claim there:

Reader Exercise: Show using **F** = m**a** and **F** = q**E** (ignore magnetic fields) that, with a time-harmonic **E** field, a classical electron inside a transmission line conductor traverses a tiny elliptical path and thus never really goes anywhere. That path is traversed once per period T= 2π/ω. Mathematically, show that this amounts to proving that the three equations

$$x = A\cos(\omega t-a)$$
$$y = B\cos(\omega t-b)$$
$$z = C\cos(\omega t-c)$$

are parametric equations for an ellipse with some orientation in 3D space. As just noted above, this goes-

So did I actually *do* this somewhere?? Search Acos AND Bcos in physics get 18 hits. But not here. But in math I finally found it:

D:\Work\My Interests\Math\Geometry\Rotated Ellipse Stuff of Dec 2013

and I have a whole doc on exactly this subject!!! Specifically in Section 6! I do in fact prove there that the two equations shown above define an ellipse. Here is what I show:

$$x = A'\cos(t-a)$$
$$y = B'\cos(t-b)$$

The ellipse has major and minor axes A,B and is at fixed angle θ where

$$A^{-2}, B^{-2} = (1/2)\csc^2(b-a)\frac{1}{A'^2B'^2}\left[A'^2+B'^2 \pm \sqrt{A'^4+B'^4-2A'^2B'^2[1+\cos(b-a)]}\right]$$

$\theta = (1/2)\tan^{-1}[2\cos(b-a)(A'B'), A'^2-B'^2]$

And Maple has proven this as well, so let's now move on:

Conclusions at this point (restarted)

1. For small oscillations, if we can say T ≈ mg, then the Cartesian equations for the spherical pendulum are these:

$$\ddot{x} = -(g/\ell)x$$
$$\ddot{y} = -(g/\ell)y$$
$$\ddot{z} = g - (g/\ell)z = g[1-z/\ell] \quad . \qquad (C.6.6)_{\omega=0}$$

The last equation can be interpreted as $\ddot{z} = g$ which is reasonable. So in this limit, you get:

$$x(t) = A\cos(\Omega t+\varphi_1) \qquad \Omega = \sqrt{g/\ell}$$
$$y(t) = B\cos(\Omega t+\varphi_2)$$

The orbit here in the x,,y plane is a rotated ellipse at a fixed angle, so there is no precession for small oscillations for the spherical pendulum.

Fact 1: There is no precession for small oscillations for the spherical pendulum.

2. For large oscillations the Cartesian equations of motion look very difficult to solve. One gets these equations either of which can be studied to obtain θ(t) :

$$\ddot{\theta} - h^2\cos\theta/\sin^3\theta + (g/\ell)\sin\theta = 0 \qquad h = L_z/(m\ell^2)$$

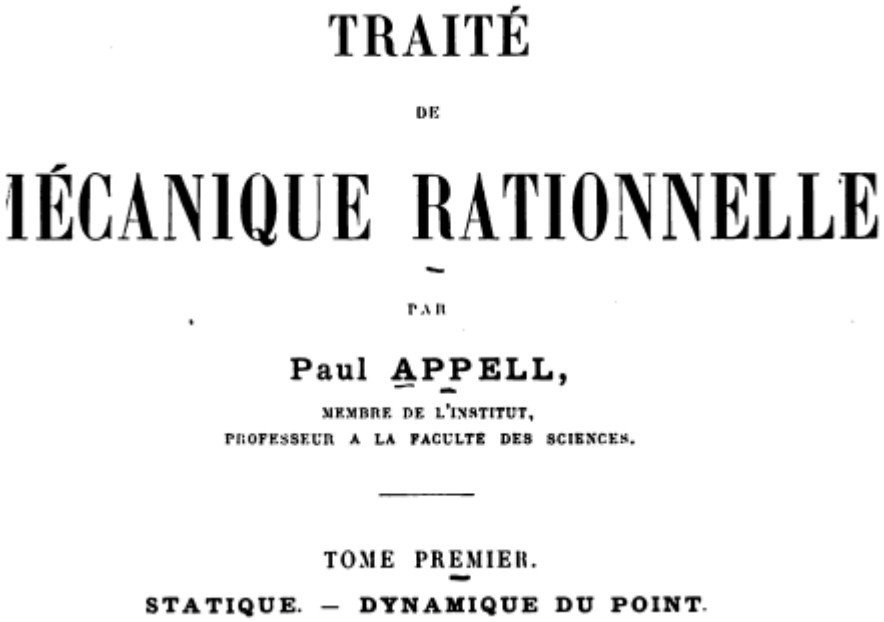
$$\mathcal{E} = \dot{\theta}^2/2 + (h^2/2\sin^2\theta) - (g/\ell)\cos\theta \qquad E = m\ell^2\mathcal{E}$$

Osgood directs us to

> For a further discussion of the problem, cf. Appell, *Mécanique rationelle*, vol. i, § 277.

for a discussion of the first equation. I can access this book in google books! But I need "volume i ? ?? I was able to download this item,

I will take a look at these pages: The cycloid is some weird thing and the simple has friction.



TRAITÉ

DE

MÉCANIQUE RATIONNELLE

PAR

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S

Wiki lists his pubs

Publications [edit]

- *Traité de mécanique rationnelle*, 4 Vols. (Gauthier-Villars, 1893–1896)
- Traité de mécanique rationnelle Tome I
- Traité de mécanique rationnelle Tome II
- Traité de mécanique rationnelle Tome III
- Traité de mécanique rationnelle Tome IV Fasc. 1
- Traité de mécanique rationnelle Tome IV Fasc. 2
- Traité de mécanique rationnelle Tome V[3]
- Les mouvements de roulement en dynamique with Jacques Hadamard (C. Hérissey, Évreux, 1899)
- Éléments de la théorie des vecteurs et de la géométrie analytique (Payot, 1926)
- Éléments d'analyse mathématique à l'usage des ingénieurs et des physiciens : cours professé à l'École centra Villars, 1921)
- Principes de la théorie des fonctions elliptiques et applications with E. Lacour (Gauthier-Villars, 1897)
- Le problème géométrique des déblais et remblais (Gauthier-Villars, 1928)
- Souvenirs d'un alsacien , autobiographic (Payot, 1923)
- Théorie des fonctions algébriques et de leurs intégrales with Édouard Goursat.[4]
- Fonctions hypergéométriques et hypersphériques with Joseph-Marie Kampé de Fériet (Gauthier-Villars, 1926)
- See catalogue of the French National Library for a more detailed list

Osgood earlier has

sentation and lacunae in the theory; Appell, *Mécanique rationelle*, vols. i and ii — a charming book, which the student may open at any chapter for supplementary reading and examples. Jeans,

So I really have vol 1 and it really does have a spherical pendulum section 277 on page 499 of PDF.

le bas. En coordonnées semipolaires, l'équation de la sphère sera

$$r^2 + z^2 = l^2,$$

en désignant par l la longueur du pendule.

so ℓ is the string and I guess $r^2 = x^2+y^2$. It is a very long piece, but I don't see the Osgood equation mentioned. It is on topic yes, but not helpful, too bad, a canoe path that did not pan out.

Let's continue the search.

Marion. On page 355 he gets equations of motion but he is doing Foucault. He gets my (C.6.14) as his (9) on page 355. He does the same complex solution I do. Nothing new here.

Landau & Lip. page 60 sees some action. he does my energy equation approach I think

$$ml^2\dot{\phi}\sin^2\theta = M_z = \text{constant}.$$

$$\begin{aligned} E &= \tfrac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - mgl\cos\theta \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + \tfrac{1}{2}M_z^2/ml^2\sin^2\theta - mgl\cos\theta. \end{aligned}$$

$$t = \int \frac{d\theta}{\sqrt{\{2[E-U_{\text{eff}}(\theta)]/ml^2\}}},$$

Nothing on the 2nd order equation.

Web hit #1 PDF.

[PDF] THE SPHERICAL PENDULUM
www.maths.surrey.ac.uk/explore/michaelspages/documentation/Spherical ▾
The **spherical pendulum** is similar to the simple pendulum, but moves in 3-dimensional space. This means we need to introduce a new variable ϕ in order to ...

Nothing useful.

Web hit #2 PDF.

Spherical Pendulum - Richard Fitzpatrick
farside.ph.utexas.edu/teaching/336k/Newtonhtml/node82.html
Spherical Pendulum. Consider a pendulum consisting of a compact mass m on the end of light inextensible string of length l . Suppose that the mass is free ...
You visited this page on 1/25/17.

z is neg of mine. He writes the 2nd order equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta - h^2\frac{\cos\theta}{\sin^3\theta} = 0, \qquad (667)$$

Suppose, finally, that the motion is almost conical: *i.e.*, the value of θ remains close to the value θ_0. Let

So here is treating a slightly tipped circular orbit, tipped by δφ and he gets precession in this case with a formula. But nothing more.

Web hit #3 PDF.

Spherical pendulum
physique.unice.fr/sem6/2011-2012/PagesWeb/PT/Pendule/.../study2_**spherical**.html
We will now study the case of the **spherical pendulum**. We add another degree of freedom to the case of the simple pendulum : the mass can now rotate around ...

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \Rightarrow \ddot{\theta} = \dot{\varphi}^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta$$
$$\frac{\partial\mathcal{L}}{\partial\varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \Rightarrow \ddot{\varphi} = -2\dot{\theta}\dot{\varphi}\cot\theta$$

Has nice orbit plot numerique but nothing helpful

[PDF] Notes on spherical pendulum - Imperial College London
wwwf.imperial.ac.uk/~dholm/classnotes/M3-4A16-NotesNov11.pdf
What will we study about the **spherical pendulum**? 1. Newtonian, Lagrangian and Hamilto- nian formulations of motion equation. 2. S1 symmetry and Noether's ...

No good

[PDF] Applications of the Lagrange equations
ph.qmul.ac.uk/sites/default/files/Examples1.pdf
r from the Sun, it is pretty clear that spherical polar coordinates r, θ, φ provide a .... As a second example, we consider the **spherical pendulum** which consists of a.

No good

[PDF] Advanced mechanics PHYS*3400 - Department of Physics - Universit...
https://www.physics.uoguelph.ca/poisson/research/mech.pdf
2.4.2 Equations of motion in spherical coordinates. 65. 2.4.3 Motion on the surface of a cone. 66. 2.4.4 **Spherical pendulum**. 70. 2.4.5 Rotating pendulum. 72.

$$\ddot{\theta} - \frac{h^2 \cos\theta}{\sin^3\theta} + \omega^2 \sin\theta = 0.$$

Multiplying each term by $\dot{\theta}$ allows us to integrate this equation. The result is the conservation statement

$$\frac{1}{2}\dot{\theta}^2 + \nu(\theta) = \varepsilon = \text{constant}, \qquad (2.4.22)$$

where ε is the pendulum's reduced total mechanical energy, and

$$\nu(\theta) = \frac{h^2}{2\sin^2\theta} - \omega^2 \cos\theta \qquad (2.4.23)$$

So how does that integration work exactly ? Start with

$\ddot{\theta}$ - $h^2\cos\theta/\sin^3\theta$ + $(g/\ell)\sin\theta$ = 0 $\qquad$ $h = L_z/(m\ell^2)$

$\dot{\theta}\ddot{\theta}$ - $h^2\dot{\theta}\cos\theta/\sin^3\theta$ + $\dot{\theta}(g/\ell)\sin\theta$ = 0 $\qquad$ $h = L_z/(m\ell^2)$

$(1/2)\partial_t(\dot{\theta})^2$ - $h^2(\partial_t\theta)\cos\theta/\sin^3\theta$ + $(\partial_t\theta)(g/\ell)\sin\theta$ = 0

In the second term I want

$\partial_t(f) = \cos\theta/\sin\theta^3\ (\partial_t\theta)$

$df/dt = \cos\theta/\sin\theta^3\ d\theta/dt$

$df = \cos\theta/\sin\theta^3\ d\theta$

$f = -(1/2)\ 1/\sin^2\theta + C$ // says Maple

In the third term I want

$\partial_t(g) = \sin\theta(\partial_t\theta)$

$dg/dt = \sin\theta\, d\theta/dt$

$dg = \sin\theta d\theta$

$g = -\cos\theta + C$

Then my equation says

$$(1/2)\partial_t(\dot{\theta})^2 - h^2\partial_t[-(1/2)\ 1/\sin^2\theta + C] + (g/\ell)\partial_t[\ -\cos\theta + C'] = 0$$

which then says

$$(1/2)\partial_t(\dot{\theta})^2 - h^2\partial_t[-(1/2)\ 1/\sin^2\theta\ ] + (g/\ell)\partial_t[\ -\cos\theta] = 0$$

or

$$\partial_t\ \{\ (1/2)(\dot{\theta})^2 - h^2[-(1/2)\ 1/\sin^2\theta\ ] + (g/\ell)[\ -\cos\theta]\ \} = 0$$

or

$$\{\ (1/2)(\dot{\theta})^2 - h^2[-(1/2)\ 1/\sin^2\theta\ ] + (g/\ell)[\ -\cos\theta]\ \} = \text{constant}$$

or

$$(1/2)(\dot{\theta})^2 + (1/2)h^2/\sin^2\theta - (g/\ell)\cos\theta = \text{constant}$$

or

$$\dot{\theta}^2/2 + h^2/(2\sin^2\theta) - (g/\ell)\cos\theta = \text{constant} = \mathcal{E}$$

So he is exactly correct in his observation! So nobody is every going to mess with the 2nd order equation. He does add this perhaps useful result

We start the integration at $\theta = \theta_-$, setting $v = 0$ and $\phi = 0$. The constant h can be determined in terms of θ_- and θ_+ by using the relation $\nu(\theta_-) = \nu(\theta_+)$, which follows from Eq. (2.4.22). The result, after some algebra, is

$$h^2 = 2\omega^2 \frac{(\sin\theta_+ \sin\theta_-)^2(\cos\theta_- - \cos\theta_+)}{(\sin\theta_+ - \sin\theta_-)(\sin\theta_+ + \sin\theta_-)}.$$

But no mention of precession!

So my web search is turning up very little. If I add precession to "spherical pendulum".

[PDF] Study of the apsidal precession of the Physical Symmetrical Pendulum
https://arxiv.org/pdf/1312.4019
by HR Maya - 2013 - Related articles
Dec 14, 2013 - **precession** corresponding to the Ideal **Spherical Pendulum** (Airy's ... spin introduces a significant correction to the **precession** of the PSP (with ...
You visited this page on 1/25/17.

This is a very complicated paper, but at least uses my variables θ and φ.

ISP = Ideal Spherical Pendulum

apsidal precession = Airy's precession

Aha! Airy wrote a paper on this in 1851 and here is the title

VIII. *On the Vibration of a Free Pendulum in an Oval differing little from a Straight Line.* By G. B. AIRY, Esq., Astronomer Royal.

Read May 9, 1851.

That is exactly what I am looking for. He starts with these equations

$$x\frac{dy}{dt} - y\frac{dx}{dt} - A = 0$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - 2gz + B = 0,$$

where the first says L_z is constant while the second says energy = constant. Very good. He uses ψ(t) as the precessing angle of interest.

16. The second line of the equation gives $m^2 = \frac{g}{a}\left(1 - \frac{b^2}{8a^2}\right)$; and the time of describing the oval, or $\frac{2\pi}{m}$, is therefore $2\pi\sqrt{\frac{a}{g}} \cdot \left(1 + \frac{b^2}{16a^2}\right)$ nearly.

His big conclusion is this

$$q = -\frac{3}{16} \cdot \frac{b^2 c}{a^2} \cos^2 mt \cdot \sin mt.$$

But I don't know what q is because it is defined in Article 5 which he write earlier. m is the base frequency I think.

a = string length.

b and c are defined by the assumed form of the basic ellipse.

$$X = b \cdot \cos m t + p$$
$$Y = c \cdot \sin m t + q,$$

Now here is what Article 5 is -- it is in his Published Memoirs, but where are they. I have this,

AUTOBIOGRAPHY

OF

SIR GEORGE BIDDELL AIRY, K.C.B.,

M.A., LL.D., D.C.L., F.R.S., F.R.A.S.,

HONORARY FELLOW OF TRINITY COLLEGE, CAMBRIDGE,

ASTRONOMER ROYAL FROM 1836 TO 1881.

He was an astro guy.

Some idea of his activity as a writer on mathematical and physical subjects during these early years may be gathered from the fact that previous to this appointment he had contributed no less than three important memoirs to the *Philosophical Transactions* of the Royal Society, and eight to the Cambridge Philosophical Society. At the Cambridge

Pendulum Investigations and Experiments 66, 67, 71, 79, 83, 84, 220, 221, 223, 359

https://books.google.com/books?id=-mUSAAAAIAAJ&pg=PA342&dq=edward+maunder+astronomer&hl=en#v=onepage&q&f=false

This has a list of all his papers and books near the end page 369.

(Lecture.)

1851 May 9 On the Vibration of a Free Pendulum in an Oval differing little from a Straight Line. R. Astr. Soc. (Memoirs.)

Peel, Sir Robert 105—108, 151, 179
Pendulum Investigations and Experiments 66, 67, 71, 79, 83, 84, 220, 221, 223, 359

The item 1851 is all I see in this index related to the topic, and that is the article I have. He says in this paper that is earlier paper was in "the 11th volume of the memoirs". So how do I find that article. I am sure it is the Royal Astronomical Society.

https://catalog.hathitrust.org/Record/000519124 has full view vol 11 two items

ROYAL ASTRONOMICAL SOCIETY.

VOL. XI.

WITH SIX COPPER-PLATES.

XIII. *On the Determination of the Orbits of Comets, from Observations.* *By G. B. Airy, Esq. Astronomer Royal* 181

But the claimed paper is not there, only the above paper.

In a paper communicated to this Society several years since, and printed in the eleventh volume of their *Memoirs*, I investigated the motion of a pendulum in the case in which it describes an oval differing little from a circle; and I showed that, if the investigation is limited to the first power

But it is just not there. Not in vol 12 either.

In a paper communicated to this Society several years since, and printed in the eleventh volume of their *Memoirs*, I investigated the motion of a pendulum in the case in which it describes an oval differing little from a circle; and I showed that, if the investigation is limited to the first power of ellipticity, and if α is the mean value of the angle made by the pendulum-rod with the vertical, then the proportion of the time occupied in passing from one distant apse to the next distant apse, to the mean time of a revolution, is the proportion of 1 to the square root of $4 - 3 \sin^2\alpha$. When α is small, this proportion is nearly the same as the proportion of $\frac{1}{2}$ to $1 - \frac{3}{8} \sin^2\alpha$; or the time of moving from one distant apse to another distant apse is equal to the time of half a revolution divided by $1 - \frac{3}{8} \sin^2\alpha$. This shows that the major axis of the oval is not stationary, but that its line of apses progresses; and that, while the ellipticity is small, the velocity of progress of the apses is sensibly independent of the ellipticity, and may be assigned in finite terms for any value of the mean inclination of the pendulum-rod.

So α = the mean value of θ and he is claiming

$T_{precession} / T_{swing} = 4 - 3\sin^2\alpha$ // maybe factor of 2, what is apse?

but he says this is not very good if thin ellipse.

Maybe this is his 1851 rsult

19. The result which it was the special object of this investigation to obtain is that given by the equation $\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2}$. From this we find that the time in which the line of apses would perform a complete revolution is $\frac{8}{3} \cdot \frac{a^2}{bc}$ multiplied by the time of a complete double vibration.

OK, here he is saying

$T_{precession} = (8/3)(a^2/bc)\, T_{osc}$

where a = length of string, b and c are what appear in the ellipse equations

$$X = b \cdot \cos m t + p$$
$$Y = c \cdot \sin m t + q,$$

Here p and q are small correction functions, they are not phases.

16. The second line of the equation gives $m^2 = \frac{g}{a}\left(1 - \frac{b^2}{8a^2}\right)$; and the time of describing the oval, or $\frac{2\pi}{m}$, is therefore $2\pi\sqrt{\frac{a}{g}} \cdot \left(1 + \frac{b^2}{16a^2}\right)$ nearly.

So notice that m is not quite what you think if b is large.

The assumption was therefore made

$$x = X \cdot \cos nt - Y \cdot \sin nt$$
$$y = X \cdot \sin nt + Y \cdot \cos nt,$$

I think n is his precession frequency.

sidered that $\frac{b}{a}$ and $\frac{c}{b}$ are small quantities of the first order;

$$\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2},$$

Thus if a pendulum 52 feet long (which performs its double vibration in 8 seconds), vibrated in an ellipse whose major axis is 52 inches and minor axis 6 inches, the line of apses would perform a complete revolution *from this cause* in 30 hours nearly.

If a common seconds' pendulum (which performs its double vibration in 2 seconds) vibrated in an ellipse whose major axis is 4 inches and minor axis $\frac{1}{13}$th-inch, the line of apses would perform a complete revolution *from this cause* in 30 hours nearly.

He is giving the full major axes, while the formula wants the semi-major axes , so

```
restart;
Tprec := evalf((1/3600)*Tosc / ((3/8)*b*c/a^2));
```

$$Tprec := .0007407407407\,\frac{Tosc\,a^2}{b\,c}$$

```
a := 52*12: b := 52/2: c := 6/2:  Tosc := 8;
```

$$Tosc := 8$$

```
Tprec;
```

$$29.58222222$$

```
a := 39.1: b := 4/2: c := (1/13)/2:  Tosc := 2;
```

$$Tosc := 2$$

```
Tprec;
```

$$29.44374815$$

Now that I know about his pioneering study of "Airy precession", there MUST be some contemporary discussion of this in the open world.

I try for the Olsson reference but I am blocked by paywalls,

olsson "the precessing spherical pendulum"

But then here is a good hit, look at the date!!

A Short Foucault Pendulum Free of Ellipsoidal Precession

Reinhard A. Schumacher and Brandon Tarbet
Department of Physics, Carnegie Mellon University, Pittsburgh, PA 15213
(Dated: February 12, 2009)

Any ellipsoidal motion that develops in a pendulum will result in an intrinsic precession rate Ω that is wholly unrelated to the Foucault precession Ω_F of Eq (1). This has been worked out in detail by Olsson [2, 3] and Pippard [4], among others, and it appears in some textbooks, for example in Ref [5]. The main result is

$$\Omega = \frac{3}{8}\omega_0 \frac{ab}{L^2}, \tag{2}$$

They also quote this adjusting factor for the swing frequency

$$\omega = \omega_0 \left(1 - \frac{1}{16}\frac{a^2 + b^2}{L^2}\right).$$

Textbook ref [5] is

[5] J. Synge and B. Griffith, *Principles of Mechanics* (McGraw Hill, 1959), 3rd ed., pp 335-342.

I have a 1949 copy of this thing, and it has a LOT of mention of the word pendulum! The analysis of the simple case begins on page 370. The spherical starts on page 373! They do small osc, then conical, then general motion. They use R,z,φ as cylindrical coordinates! They then have the following u coordinate,

$$u = \sqrt{z - z_1},$$

The ODE of interest is then

$$\dot{u}^2 = \frac{g}{2a^2}(z_2 - z_1 - u^2)(z_3 - z_1 - u^2). \tag{13.312}$$

function sn. Let us define

$$\begin{cases} v = \dfrac{u}{\sqrt{z_2 - z_1}} = \sqrt{\dfrac{z - z_1}{z_2 - z_1}}, \\ k = \sqrt{\dfrac{z_2 - z_1}{z_3 - z_1}}, \qquad p = \sqrt{\dfrac{g(z_3 - z_1)}{2a^2}}. \end{cases} \tag{13.313}$$

Then (13.312) may be written

$$\dot{v}^2 = p^2(1 - v^2)(1 - k^2v^2), \tag{13.314}$$

and so

$$v = \operatorname{sn}[p(t - t_0)], \tag{13.315}$$

where t_0 is a constant of integration. Hence we have

$$\begin{cases} z - z_1 = (z_2 - z_1)\operatorname{sn}^2[p(t - t_0)], \\ z_2 - z = (z_2 - z_1)\operatorname{cn}^2[p(t - t_0)], \\ z_3 - z = (z_3 - z_1)\operatorname{dn}^2[p(t - t_0)]. \end{cases} \tag{13.316}$$

On it goes. Finally comes precession

Now if R_1, R_2 are the distances from the central vertical to the apsides, we have accurately

$$R_1^2 = a^2 - z_1^2, \qquad R_2^2 = a^2 - z_2^2, \qquad SD = R_1R_2,$$

and so the approximate formula (13.331) becomes

$$\alpha = \tfrac{1}{2}\pi\left(1 + \frac{3R_1R_2}{8a^2}\right). \tag{13.332}$$

For that curve the apsidal angle is $\frac{1}{2}\pi$. Now we see, from (13.332), that the apsidal angle is a little greater than $\frac{1}{2}\pi$. This means that the apse advances; the path is approximately a central ellipse, but *this ellipse turns slowly forward* (i.e., in the same sense as that in which the particle describes the path). In one rotation of the particle, the apse advances through an angle

$$4\alpha - 2\pi = \frac{3\pi R_1R_2}{4a^2} = \frac{3A}{4a^2}, \tag{13.333}$$

You see the Airy formula above the 381. I will try to process this final claim.

"In period T_{osc} we have $d\varphi = (3\pi/4)(R_1R_2/a^2)$ for the precession. It must then take this number of periods T_{osc} to get a full precession period

$N = 2\pi/d\varphi = 2\pi/\{(3\pi/4)(R_1R_2/a^2)\}$

Inverting

$1/N = (3\pi/4)(R_1R_2/a^2) / 2\pi = (3/8)(R_1R_2/a^2)$

and there you have it! Their method using z is rather obscure but probably better than Airy's method which got the same result.

I wonder if the 1959 edition is better on this? $5 from Amazon. Book is well protected.

So OK I am done with this little effort. I have a formula for the precession and it is valided in a 2009 reference that I can in fact provide a link to.

How would I apply this idea in my frames doc near (C.8.14) ?

For that figure I estimate

b =