Appendix D questions PhL 12.6.16

I have now finished a first draft of 18-page Appendix D on the subject of the center of gravity and the tether. Not surprisingly, trying to answer questions raises more and new questions. So right now here are some of those:

1. I argue that $\mathbf{N}^{(rcog)} = 0$ and $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$ as part of the definition of $\mathbf{r}_{cog}$. The other part is the distance rule which sets magnitude r_{cog}. Question: does this apply only in an inertial frame, or does it apply also in a rotating frame?

2. I compute the torque about tether center point ("origin") and I show "restoring force". What does this torque have to do with the torques of item 1 above? And what about the fact that this "origin" is not the center of gravity or of mass?

3. Does it make sense to say anything about a freeze-frame of the tether since you know it must be in motion? Am I apply statics to a case where there is motion and so statics is wrong?

Question 1: I am reviewing section D.1 right now. The cms definitions are OK. I then talk about $\mathbf{F}_i$ as being a force "experienced by" mass m_i. There is no Newton's law yet, so I can assume this is a real force even if I am in a non-inertial frame. Then the definitions of (D.1.2) are all valid. And so is (D.1.3). So I am arguing that if you are in a non-inertial frame, you should NOT include fictitious forces in $\mathbf{F}_i$ or $\mathbf{F}$ or any of these torques. So (D.1.4) is valid for cog. So is (D.1.5).

Conclusion: Section D.1 is valid in any frame of reference where all F and N are real and do not include fictitious components. Thus in the case of the asteroid, if you hop on the asteroid you are in a non-inertial frame, yes. If you take a freeze frame of it as I have done, you compute $\mathbf{r}_{cog}$ as I have done. So I think this section is all OK. Inertial/non-inertial only matters when you apply $\mathbf{F} = m\mathbf{a}$ or $\mathbf{N} = I\boldsymbol{\alpha}$.

Now section D.2. Again I have a freeze-frame of the tether at angle θ. The forces F_i are only those of gravity (need a comment about tension force -- DONE). So this entire section is a freeze-frame statics analysis. I think it is all valid.

Then D.3 and D.4 are just special cases.

D.5 on the sphere is another statics analysis. The sphere could be accelerating, but the result is not affected. It is a freeze-frame statics analysis.

Section D.6 is more of a concern. (again, comment on tension forces needed!). As I do things, the stick tension won't enter the torque calcs since $\hat{\mathbf{r}} \times \mathbf{T} = 0$. I select an arbitrary origin which is the center of my circle. More static calculations. I think things are OK down through (D.5.8) where I talk about the direction as being restoring. I have no Newton's law yet.

The Tether Oscillation section brings up Newton's Law $\mathbf{N} = I\ddot{\boldsymbol{\theta}}$ for the first time, and so finally I have to worry about things. Remember that the cog and cms are different and neither is at circle center and we are in a rotating frame, lots of complications at once!

Start with fictitious torques which have to enter the above law! I have this most general case

$$\mathbf{N}'^{(c')}_{fict} = -(\mathbf{c}' - \mathbf{c} + \mathbf{b}) \times m\mathbf{a}' + \dot{\mathbf{c}}_S \times [m\mathbf{v}' + m\boldsymbol{\omega} \times \mathbf{r}' + m\dot{\mathbf{b}}_S] - \dot{\mathbf{c}}'_{S'} \times m\mathbf{v}' + (\mathbf{r}' - \mathbf{c} + \mathbf{b}) \times \mathbf{F}'_{fict} \quad . \tag{11.13}$$

If I pick $\mathbf{c} = \mathbf{c}' = 0$ for both frames, things simplify:

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{b} \times m\mathbf{a}' + (\mathbf{r}' + \mathbf{b}) \times \mathbf{F'_{fict}} \ . \qquad // \ \mathbf{c} = \mathbf{c}' = 0 \qquad (11.14)$$

where

$$\mathbf{F'_{fict}} = -m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.6) \qquad (11.12)$$

frame centrifugal Coriolis Euler

Note that **a**' appears, the natural acceleration of particle m in Frame S'.

Our two frames are in Special Case #1 as in Fig 4.3. Frame S is Earth center and is assumed fixed and the rotation axis passes through it. It is like the turntable. So let's try to apply some Special 1 casing but I won't assume we are on the surface of the Earth as I do in Section (e). Section 8 (d) does do this general special case. I claim there that

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{Special Case \#1} \qquad (7.12)$$

$$\mathbf{F'_{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \ . \qquad \text{Special Case \#1} \qquad (8.12)$$

so therefore (Frame S' is the rotating frame attached to the satellite).

$$\mathbf{F'_{fict}} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad // \text{ non-swap}$$

centrifugal Coriolis Euler

Note that **r** is the long vector here. The first term is $-m\omega^2 r\,\hat{\mathbf{r}}$ for a circular orbit whose center is Earth center. I will set any Euler term to 0, so then we have

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{b} \times m\mathbf{a}' + (\mathbf{r}' + \mathbf{b}) \times \mathbf{F'_{fict}} \ . \qquad // \ \mathbf{c} = \mathbf{c}' = 0 \qquad (11.14)$$

$$\mathbf{F_{fict}} = -m\omega^2 r\,\hat{\mathbf{r}} + 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

centrifugal Coriolis

$$\mathbf{N_{real}}^{(0 \text{ for Frame S})} = \mathbf{r} \times \mathbf{F_{real}}$$

Warning: confusion now since r and **r** have two meanings in the tether analysis!

Resolved: I will replace the radius of the "circle" with ρ instead of r.

Question: Where is the origin of Frame S' ? It can be wherever I want and I see three candidate locations all different!

I think I have to take $\mathbf{b} = \mathbf{r}_{cog}$ because only then do we have b = constant for the orbit! Remember that you can replace your satellite with a point mass there and it goes in a circle. So for mass m_1 the above reads

$$\mathbf{N'}^{(0)}_{fict} = -\mathbf{r}_{cog} \times m_1 \mathbf{a}_1' + (\mathbf{r}_1' + \mathbf{r}_{cog}) \times \mathbf{F'}_{fict,1}$$

$$\mathbf{F}_{fict,1} = -m_1 \omega^2 r_1 \hat{\mathbf{r}}_1 + 2m_1 \boldsymbol{\omega} \times \mathbf{v}_1'$$

This is immensely complicated if you really track all the distinctions of radii. We are going to need approximations, but certain things are more delicate than others for said approximations.

Plan A. Suppose I take the tether circle center as a reference point for the torque oscillation calculation. It is true that as the masses move, this circle center moves in some strange but generally small manner relative to the $\mathbf{r}_{cog}$ point. I really don't care about this motion -- it happens, fine.

Suppose I also pick circle center as the origin of Frame S'. Then Frame S' as a whole makes this same small motion of $\mathbf{r}_{cog} - \mathbf{r}_{cc}$ as time goes by in the orbit. The $\mathbf{r}_{cc}$ is tied to $\mathbf{r}_{cms}$ really.

Would it be better to use $\mathbf{r}_{cms}$ as torque origin? Locally you get a cleaner picture since then masses move at different radii relative to the cm. The idea would be this: we just ignore the $\mathbf{r}_{cog} - \mathbf{r}_{cms}$ small motion after acknowledging that it is present.

How big is this difference in practice? The general expressions for r_{cog} is very complicated, so I might restrict my estimate to the two angular special cases.

For the aligned $\theta = 0$ case I show that

$$r_{cog} = \frac{h^2 - r^2}{\sqrt{h^2 + r^2}} = h \frac{1 - (r/h)^2}{\sqrt{1 + (r/h)^2}}$$

$$r_{cms} = h \qquad \text{(D.3.13)}$$

If $r/h << 1$ as is the case even for low earth orbit, we write

$$r_{cog} = h \frac{1 - (r/h)^2}{\sqrt{1 + (r/h)^2}} = h\,[1 - (r/h)^2]\,[1 + (r/h)^2]^{-1/2}$$

$$\approx = h\,[1 - (r/h)^2]\,[1 - (1/2)(r/h)^2]$$

$$\approx h\,[1 - (3/2)(r/h)^2]$$

$$= h - (3/2)h\,(r/h)^2$$

$$= r_{cms} - (3/2)h\,(r/h)^2$$

So then

$r_{cms} - r_{cog} = (3/2)h\,(r/h)^2 = (3/2)\,r^2 / h$

$(r_{cms} - r_{cog})/h = (3/2)(r/h)^2$

$(r_{cms} - r_{cog})/(2r) = (3/4)\,(r/h)$

where r is the circle radius and h is distance to circle center.

Let's evaluate this result several ways. I will do it in Maple and assume low earth orbit. I will assume that R_E = 6371 km = 6.371 x 10^6 m and 2r = stick length = 1 km = 10^3. Then:

r = 500 meters

```
restart;Digits := 20;
                    Digits := 20
                    Digits := 20
RE := 6.371e6;
                  RE := .6371 10^7
h := RE;
                   h := .6371 10^7
r := 0.5e3;
                     r := 500.
rcms := h;
                 rcms := .6371 10^7
rcog := rcms - (3/2)* h*(r/h)^2;
          rcog := .63709999411395385340 10^7
rcms-rcog;
                  .0588604614660
(rcms-rcog)/h;
            .92388104639773975828 10^-8
(rcms-rcog)/(2*r);
            .00005886046146600000000
```

r = 5 meters

```
restart;Digits := 20;
                    Digits := 20
                    Digits := 20
RE := 6.371e6;
                  RE := .6371 10^7
h := RE;
                   h := .6371 10^7
r := 0.5e1;
                      r := 5.
rcms := h;
                 rcms := .6371 10^7
rcog := rcms - (3/2)* h*(r/h)^2;
          rcog := .63709999999941139539 10^7
rcms-rcog;
                  .58860461 10^-5
(rcms-rcog)/h;
            .92388103908334641344 10^-12
(rcms-rcog)/(2*r);
            .58860461000000000000 10^-6
```

So this shows that:

r = 500 m

$r_{cms} - r_{cog}$ = 6 cm

$(r_{cms} - r_{cog}) / h \approx 10^{-8}$

$(r_{cms} - r_{cog}) / (2r) \approx 6 \times 10^{-5}$

r = 5 m

$r_{cms} - r_{cog}$ = 6 microns

$(r_{cms} - r_{cog}) / h \approx 10^{-12}$

$(r_{cms} - r_{cog}) / (2r) \approx 6 \times 10^{-7}$

So for a 1 km long tether the offset $r_{cms} - r_{cog}$ = 6 cm, while for a 10 m tether we get 6 μ !!

At higher earth orbit, h is larger than low earth orbit, (r/h) is smaller and all results are smaller.

Conclusion:

(1) Let's take $\mathbf{r}_{cms}$ as the origin of Frame S' and as our torque calculation origin.

(2) We acknowledge that $r_{cog} - \mathbf{r}_{cms}$ moves in some complicated but very small manner during an orbit.

Now go back to where we were,

$$\mathbf{N'}^{(0)}_{fict} = -\mathbf{b} \times m\mathbf{a'} + (\mathbf{r'} + \mathbf{b}) \times \mathbf{F'}_{fict} \quad . \qquad // \ \mathbf{c} = \mathbf{c'} = 0 \qquad (11.14)$$

$$\mathbf{F}_{fict} = \underset{\text{centrifugal}}{-m\omega^2 r\, \hat{\mathbf{r}}} + \underset{\text{Coriolis}}{2m\, \boldsymbol{\omega} \times \mathbf{v'}}$$

In the notation $\mathbf{N'}^{(0)}_{fict}$ the (0) means this torque is relative to origin of Frame S' which is at $\mathbf{r}_{cms}$ but is also at $\mathbf{r'} = 0$ in the circle frame. I will leave it then with the (0) label.

Now we set $\mathbf{b} = \mathbf{r}_{cms}$ and write the above for mass m_1 only

$$\mathbf{N'}^{(0)}_{fict,1} = -\mathbf{r}_{cms} \times m_1 \mathbf{a}_1' + (\mathbf{r}_1' + \mathbf{r}_{cms}) \times \mathbf{F'}_{fict,1}$$

$$\mathbf{F}_{fict,1} = \underset{\text{centrifugal}}{-m_1\omega^2 r_1\, \hat{\mathbf{r}}_1} + \underset{\text{Coriolis}}{2m_1\, \boldsymbol{\omega} \times \mathbf{v}_1'} \qquad \boldsymbol{\omega} \times \mathbf{v}_1' = -\omega v'_1 \hat{\mathbf{r}}'_1$$

$$\mathbf{F}_{real,1} = -Gmm_1/r_1^2\, \hat{\mathbf{r}}_1 - T\, \hat{\mathbf{r}}'_1$$

$$\mathbf{N'}^{(0)}_{real,1} = \mathbf{r}'_1 \times \mathbf{F}_{real,1} = -(Gmm_1/r_1^2)\, \mathbf{r}'_1 \times \hat{\mathbf{r}}_1 \quad \text{[wrong!]}$$

Why? I set $\mathbf{c} = 0$ so this torque is referenced to the Frame S origin and $\mathbf{r}_1 \times \mathbf{F}_{real,1} = 0$ so this torque is zero and can be ignored. But then where is the gravitational torque in this problem?? But I don't think you can add torques around different points, so this whole setup needs repair! Needs a clean start.

Here ω is the orbital rate of the satellite which is 88 minutes for low earth. Primed quantities are relative to Frame S', unprimed relative to Frame S so r_1 is a long vector while r'_1 is a short one.

$$\mathbf{N'}^{(0)}_{fict,2} = -\mathbf{r}_{cms} \times m_2 \mathbf{a}_2' + (\mathbf{r}_2' + \mathbf{r}_{cms}) \times \mathbf{F'}_{fict,2}$$

$$\mathbf{F}_{fict,2} = \underset{\text{centrifugal}}{-m_2\omega^2 r_2\, \hat{\mathbf{r}}_2} + \underset{\text{Coriolis}}{2m_1\, \boldsymbol{\omega} \times \mathbf{v}_2'} \qquad \boldsymbol{\omega} \times \mathbf{v}_2' = -\omega v'_2 \hat{\mathbf{r}}'_2$$

$$\mathbf{F}_{\mathbf{real,2}} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 - T\ \hat{\mathbf{r}}'_2$$

$$\mathbf{N}'^{(0)}_{\mathbf{real,2}} = \mathbf{r}'_2 \times \mathbf{F}_{\mathbf{real,2}} = -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$

Since the Coriolis force has a simple direction shown, I will restate all the above results

$$\mathbf{N}'^{(0)}_{\mathbf{fict,1}} = -\mathbf{r}_{\mathbf{cms}} \times m_1\mathbf{a}_1' + \mathbf{r}_1 \times \mathbf{F}'_{\mathbf{fict,1}}$$

$$\mathbf{F}_{\mathbf{fict,1}} = \underset{\text{centrifugal}}{-m_1\omega^2 r_1\ \hat{\mathbf{r}}_1} - \underset{\text{Coriolis}}{2m_1\omega v'_1\hat{\mathbf{r}}'_1}$$

$$\mathbf{F}_{\mathbf{real,1}} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{real,1}} = \mathbf{r}'_1 \times \mathbf{F}_{\mathbf{real,1}} = -(Gmm_1/r_1^2)\ \mathbf{r}'_1 \times \hat{\mathbf{r}}_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict,2}} = -\mathbf{r}_{\mathbf{cms}} \times m_2\mathbf{a}_2' + \mathbf{r}_2 \times \mathbf{F}'_{\mathbf{fict,2}}$$

$$\mathbf{F}_{\mathbf{fict,2}} = \underset{\text{centrifugal}}{-m_2\omega^2 r_2\ \hat{\mathbf{r}}_2} - \underset{\text{Coriolis}}{2m_2\omega v'_2\hat{\mathbf{r}}'_2}$$

$$\mathbf{F}_{\mathbf{real,2}} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 - T\ \hat{\mathbf{r}}'_2$$

$$\mathbf{N}'^{(0)}_{\mathbf{real,2}} = \mathbf{r}'_2 \times \mathbf{F}_{\mathbf{real,2}} = -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$

And now one more time with vigor,

$$\mathbf{N}'^{(0)}_{\mathbf{fict,1}} = -\mathbf{r}_{\mathbf{cms}} \times m_1\mathbf{a}_1' + \mathbf{r}_1 \times \mathbf{[}\ -m_1\omega^2 r_1\ \hat{\mathbf{r}}_1 - 2m_1\omega v'_1\hat{\mathbf{r}}'_1\mathbf{]}$$

$$\mathbf{F}_{\mathbf{real,1}} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{real,1}} = -(Gmm_1/r_1^2)\ \mathbf{r}'_1 \times \hat{\mathbf{r}}_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict,2}} = -\mathbf{r}_{\mathbf{cms}} \times m_2\mathbf{a}_2' + \mathbf{r}_2 \times \mathbf{[}\ -m_2\omega^2 r_2\ \hat{\mathbf{r}}_2 - 2m_2\omega v'_2\hat{\mathbf{r}}'_2\mathbf{]}$$

$$\mathbf{F}_{\mathbf{real,2}} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 - T\ \hat{\mathbf{r}}'_2$$

$$\mathbf{N}'^{(0)}_{\mathbf{real,2}} = -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$

Now one of the cross products drops a term so one more time again

$$\mathbf{N}'^{(0)}_{\mathbf{fict,1}} = -\mathbf{r}_{\mathbf{cms}} \times m_1\mathbf{a}_1' - m_1\omega v'_1\ \mathbf{r}_1 \times \hat{\mathbf{r}}'_1$$

$$\mathbf{F}_{\mathbf{real},1} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 \ - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{real},1} = -(Gmm_1/r_1^2)\ \mathbf{r}'_1 \times \hat{\mathbf{r}}_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict},2} = -\mathbf{r}_{\mathbf{cms}} \times m_2\mathbf{a}_2' \ - m_2\omega v'_2\ \mathbf{r}_2 \times \hat{\mathbf{r}}'_2$$

$$\mathbf{F}_{\mathbf{real},2} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 \ - T\ \hat{\mathbf{r}}'_2$$

$$\mathbf{N}'^{(0)}_{\mathbf{real},2} = -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$

Now I think $\mathbf{a}_1' = -a'_1\hat{\mathbf{r}}'_1$ because m_1 must go in circular motion. So here we go again

$$\mathbf{N}'^{(0)}_{\mathbf{fict},1} = m_1a'_1\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_1 \ - m_1\omega v'_1\ \mathbf{r}_1 \times \hat{\mathbf{r}}'_1$$

$$\mathbf{F}_{\mathbf{real},1} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 \ - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{real},1} = -(Gmm_1/r_1^2)\ \mathbf{r}'_1 \times \hat{\mathbf{r}}_1$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict},2} = m_2a'_2\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_2 \ - m_2\omega v'_2\ \mathbf{r}_2 \times \hat{\mathbf{r}}'_2$$

$$\mathbf{F}_{\mathbf{real},2} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 \ - T\ \hat{\mathbf{r}}'_2$$

$$\mathbf{N}'^{(0)}_{\mathbf{real},2} = -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$

Now again combining the torques

$$\mathbf{N}'^{(0)}_1 = m_1a'_1\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_1 \ - m_1\omega v'_1\ \mathbf{r}_1 \times \hat{\mathbf{r}}'_1 \ - \ (Gmm_1/r_1^2)\ \mathbf{r}'_1 \times \hat{\mathbf{r}}_1$$
$$\mathbf{F}_{\mathbf{real},1} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 \ - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_2 = m_2a'_2\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_2 \ - m_2\omega v'_2\ \mathbf{r}_2 \times \hat{\mathbf{r}}'_2 \ -(Gmm_2/r_2^2)\ \mathbf{r}'_2 \times \hat{\mathbf{r}}_2$$
$$\mathbf{F}_{\mathbf{real},2} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 \ - T\ \hat{\mathbf{r}}'_2$$

But change slightly,

$$\mathbf{N}'^{(0)}_1 = m_1a'_1\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_1 \ - m_1\omega v'_1/r'_1\ \mathbf{r}_1 \times \mathbf{r}'_1 \ - \ (Gmm_1/r_1^3)\ \mathbf{r}'_1 \times \mathbf{r}_1$$
$$\mathbf{F}_{\mathbf{real},1} = -Gmm_1/r_1^2\ \hat{\mathbf{r}}_1 \ - T\ \hat{\mathbf{r}}'_1$$

$$\mathbf{N}'^{(0)}_2 = m_2a'_2\ \mathbf{r}_{\mathbf{cms}} \times \hat{\mathbf{r}}'_2 \ - m_2\omega v'_2/r'_2\ \mathbf{r}_2 \times \mathbf{r}'_2 \ -(Gmm_2/r_2^3)\ \mathbf{r}'_2 \times \mathbf{r}_2$$
$$\mathbf{F}_{\mathbf{real},2} = -Gmm_2/r_2^2\ \hat{\mathbf{r}}_2 \ - T\ \hat{\mathbf{r}}'_2$$

Now we can combine some terms to get

$$\mathbf{N'^{(0)}}_1 = m_1 a'_1 \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + [m_1 \omega v'_1 / r'_1 - (Gmm_1/r_1^3)] \, \mathbf{r}'_1 \times \mathbf{r}_1$$
$$\mathbf{F}_{real,1} = -Gmm_1/r_1^2 \, \hat{\mathbf{r}}_1 - T \, \hat{\mathbf{r}}'_1$$

$$\mathbf{N'^{(0)}}_2 = m_2 a'_2 \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + [\, m_2 \omega v'_2 / r'_2 - (Gmm_2/r_2^3)] \, \mathbf{r}'_2 \times \mathbf{r}_2$$
$$\mathbf{F}_{real,2} = -Gmm_2/r_2^2 \, \hat{\mathbf{r}}_2 - T \, \hat{\mathbf{r}}'_2$$

I am hunting for an equation of motion for m_1. Perhaps it is this

$$\mathbf{N'^{(0)}}_1 = I_1 \ddot{\theta}' \, \hat{\mathbf{z}}' \qquad I_1 = m_1 r'^2_1$$
$$\mathbf{N'^{(0)}}_2 = I_2 \ddot{\theta}' \, \hat{\mathbf{z}}' \qquad I_2 = m_2 r'^2_2$$

These equations better be consistent! Note that both have same $\ddot{\theta}'$ polar angle accel. Write out

$$m_1 a'_1 \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + [m_1 \omega v'_1 / r'_1 - (Gmm_1/r_1^3)] \, \mathbf{r}'_1 \times \mathbf{r}_1 = (m_1 r'^2_1) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

$$m_2 a'_2 \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + [\, m_2 \omega v'_2 / r'_2 - (Gmm_2/r_2^3)] \, \mathbf{r}'_2 \times \mathbf{r}_2 = (m_2 r'^2_2) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

The constraint is that $\hat{\mathbf{r}}'_2 = -\hat{\mathbf{r}}'_1$ from cms analysis. I can do more now

$$v'_1 = r'_1 \dot{\theta}' \qquad a'_1 = r'_1 \ddot{\theta}'$$
$$v'_2 = r'_2 \dot{\theta}' \qquad a'_2 = r'_2 \ddot{\theta}'$$

So install these and cancel masses at the same time

$$r'_1 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + [\omega \, r'_1 \dot{\theta}' / r'_1 - (GM/r_1^3)] \, \mathbf{r}'_1 \times \mathbf{r}_1 = (r'^2_1) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

$$r'_2 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + [\, \omega r'_2 \dot{\theta}' / r'_2 - (GM/r_2^3)] \, \mathbf{r}'_2 \times \mathbf{r}_2 = (r'^2_2) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

or

$$r'_1 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + [\omega \, \dot{\theta}' - (GM/r_1^3)] \, \mathbf{r}'_1 \times \mathbf{r}_1 = (r'^2_1) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

$$r'_2 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + [\, \omega \dot{\theta}' - (GM/r_2^3)] \, \mathbf{r}'_2 \times \mathbf{r}_2 = (r'^2_2) \ddot{\theta}' \, \hat{\mathbf{z}}'$$

We know a lot more of course:

$$m_1 \mathbf{r}'_1 + m_2 \mathbf{r}'_2 = 0 \qquad \text{since cms is center of circle, we are staying } r'_{cms} = 0$$

$$m_1 r'_1 \hat{\mathbf{r}}'_1 + m_2 r'_2 \hat{\mathbf{r}}'_2 = 0$$

$$m_1 r'_1 \hat{\mathbf{r}}'_1 - m_2 r'_2 \hat{\mathbf{r}}'_1 = 0$$

$$m_1 r'_1 - m_2 r'_2 = 0$$

$$m_1 r'_1 = m_2 r'_2$$

$$\hat{\mathbf{r}}'_1 = -\hat{\mathbf{r}}'_2$$

Then still unclear how they will come out being the same, but I continue to eliminate r'_2 stuff. First dived things through by r'_i

$$\ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + [\omega \dot{\theta}' - (GM/r_1^3)] \, \hat{\mathbf{r}}'_1 \times \mathbf{r}_1 = r'_1 \ddot{\theta}' \, \hat{\mathbf{z}}'$$

$$\ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + [\omega \dot{\theta}' - (GM/r_2^3)] \, \hat{\mathbf{r}}'_2 \times \mathbf{r}_2 = r'_2 \ddot{\theta}' \, \hat{\mathbf{z}}'$$

And now put the masses back in

$$m_1 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + m_1[\omega \dot{\theta}' - (GM/r_1^3)] \, \hat{\mathbf{r}}'_1 \times \mathbf{r}_1 = m_1 r'_1 \ddot{\theta}' \, \hat{\mathbf{z}}'$$

$$m_2 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + m_2[\omega \dot{\theta}' - (GM/r_2^3)] \, \hat{\mathbf{r}}'_2 \times \mathbf{r}_2 = m_2 r'_2 \ddot{\theta}' \, \hat{\mathbf{z}}'$$

The right sides are now exactly the same, so to be consistent we have to be able to show that

$$m_1 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + m_1[\omega \dot{\theta}' - (GM/r_1^3)] \, \hat{\mathbf{r}}'_1 \times \mathbf{r}_1$$
$$= m_2 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_2 + m_2[\omega \dot{\theta}' - (GM/r_2^3)] \, \hat{\mathbf{r}}'_2 \times \mathbf{r}_2$$

or using $\hat{\mathbf{r}}'_2 = -\hat{\mathbf{r}}'_1$

$$m_1 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 + m_1[\omega \dot{\theta}' - (GM/r_1^3)] \, \hat{\mathbf{r}}'_1 \times \mathbf{r}_1$$
$$= -m_2 \ddot{\theta}' \, \mathbf{r}_{cms} \times \hat{\mathbf{r}}'_1 - m_2[\omega \dot{\theta}' - (GM/r_2^3)] \, \hat{\mathbf{r}}'_1 \times \mathbf{r}_2$$

Picture shows that

$$\mathbf{r}_2 - \mathbf{r}'_2 = \mathbf{r}_1 - \mathbf{r}'_1 = \mathbf{r}_{cms}$$

or

$$\mathbf{r}_2 + r'_2 \hat{\mathbf{r}}'_1 = \mathbf{r}_1 - r'_1 \hat{\mathbf{r}}'_1$$

or

$$\mathbf{r}_2 = \mathbf{r}_1 - (r'_1 + r'_2) \, \hat{\mathbf{r}}'_1$$

Since $\dot{\theta}'$ and $\ddot{\theta}'$ are really independent, we really need to show things which are not going to work! Need

$m_1 = -m_2$ for example.

And then I cannot imagine the $1/r_1^3$ terms matching.

So I have taken a long trip here and I have ended up in the dump!