Appendix D: Center of gravity and torque for a tethered satellite 1
D.1 Definition of Center of Gravity 1
D.2 Center of Gravity for a 2-mass Dumbbell Satellite 4
D.3 Dumbbell Satellite Center of Gravity with the Far Approximation: Numerical Examples 10
D.4 Dumbbell Satellite Center of Gravity for equal masses and no approximation 14
D.5 Center of Gravity for a single-sphere satellite 18

Appendix D: Center of gravity and torque for a tethered satellite

D.1 Definition of Center of Gravity

The phrase "center of gravity" is often used as a synonym for "center of mass" which complicates searching for information about the former concept. The "center of mass" of a system of particles is well known to be

$$\mathbf{r}_{cms} = \frac{\Sigma_i m_i \mathbf{r}_i}{\Sigma_i m_i} = \frac{1}{M} \Sigma_i m_i \mathbf{r}_i \qquad M = \Sigma_i m_i \qquad \text{// discrete}$$

$$\mathbf{r}_{cms} = \frac{\int dV \rho\, \mathbf{r}}{\int dV \rho} = \frac{1}{M} \int dV\, \rho\, \mathbf{r} \qquad M = \int dV\, \rho \qquad \text{// continuous} \qquad \text{(D.1.1)}$$

We use "cms" to mean center of mass even though "com" might be more reasonable. Notice that $\mathbf{r}_{cms}$ is measured with respect to the same origin used for the $\mathbf{r}_i$ or $\mathbf{r}$. For a rigid object, the center of mass is a definite point that does not move around relative to the object. It is completely determined by the spatial mass distribution of the object.

The center of gravity is a completely different animal, though it happens to align with the center of mass for a uniform gravitational field. For that reason one never deals with a distinct center of gravity concept for human-scale engineering objects on the earth's surface.

Consider a system of masses m_i each of which experiences some force $\mathbf{F}_i$. We define

$$\mathbf{F} = \Sigma_i \mathbf{F}_i \qquad \mathbf{N}^{(R)} = \Sigma_i\, (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i \qquad \mathbf{N}^{(0)} = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i\,. \qquad \text{(D.1.2)}$$

Here $\mathbf{F}$ is the sum of the forces acting on all the masses m_i, $\mathbf{N}^{(R)}$ is the total torque on the system *with respect to* some arbitrary point $\mathbf{R}$, and $\mathbf{N}^{(0)}$ is the total torque with respect to the selected origin. An obvious theorem is that

$$\mathbf{N}^{(R)} = \Sigma_i\, (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{R} \times \Sigma_i \mathbf{F}_i$$

$$= \mathbf{N}^{(0)} - \mathbf{R} \times \mathbf{F} \qquad \text{(D.1.3)}$$

which shows how the two torques are related. If $\mathbf{F} = 0$, which is often the case, then the torque is the same with respect to any point.

One characteristic of a **center of gravity** point $\mathbf{r}_{cog}$ is that the total torque on a system measured *with respect to* point $\mathbf{r}_{cog}$ vanishes, so $\mathbf{N}^{(\mathbf{r}_{cog})} = 0$. From (D.1.3) we see that this is the same as saying $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$, so

$$\mathbf{N}^{(\mathbf{r}_{cog})} = 0 \qquad \Leftrightarrow \qquad \mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F} \qquad \mathbf{r}_{cog} = \text{"center of gravity"} . \tag{D.1.4}$$

The significance of $\mathbf{r}_{cog}$ is that it is a point which allows the relation between the total system torque $\mathbf{N}^{(0)}$ and the total system force $\mathbf{F}$ to have the same form as $\mathbf{N}_i = \mathbf{r}_i \times \mathbf{F}_i$ for a single point particle. It is not obvious that such a point $\mathbf{r}_{cog}$ exists for some arbitrary system of particles.

So far this notion of "center of gravity" has nothing to do specifically with gravity, but we shall below add an additional part of the definition which does bring in gravity.

Unlike the center of mass, the center of gravity (if it exists) may not be unique and it generally moves around in an object as the object changes orientation in an external force field If we dot $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$ with the vector $\mathbf{F}$ we find

$$\mathbf{N}^{(0)} \bullet \mathbf{F} = 0 \ . \tag{D.1.5}$$

As shown above, $\mathbf{N}^{(0)}$ and $\mathbf{F}$ are well-defined computable quantities and if $\mathbf{N}^{(0)} \bullet \mathbf{F} \neq 0$, then $\mathbf{r}_{cog}$ cannot possibly exist (because if it did exist one must have $\mathbf{N}^{(0)} \bullet \mathbf{F} = 0$). So (D.1.5) is a condition for the existence of $\mathbf{r}_{cog}$.

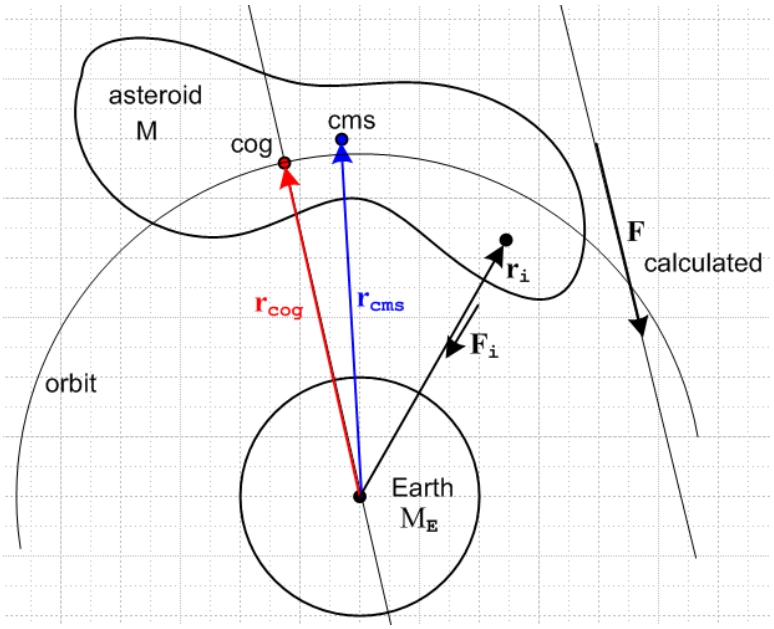
Here is a method for locating $\mathbf{r}_{cog}$, a variation of Symon p 258 which we present in the context of an asteroid of mass M near the Earth. This description includes the second characteristic of the center of gravity point which is that it is point at which gravitational action is effectively focused.

One first computes the total gravitational force $\mathbf{F} = \Sigma_i \mathbf{F}_i$ due to the Earth (summed over all points in the asteroid) and one then knows the direction of the sum vector $\mathbf{F}$. One creates a line along $\mathbf{F}$ and then translates that line parallel to $\mathbf{F}$ until the line passes through the center of the Earth. The center of gravity of the asteroid lies on that translated line a distance r_{cog} from the center of the Earth where r_{cog} is determined by,

$$GM_E M/r_{cog}^2 = F \quad \Rightarrow \quad r_{cog} = \sqrt{GM_E M/F}$$

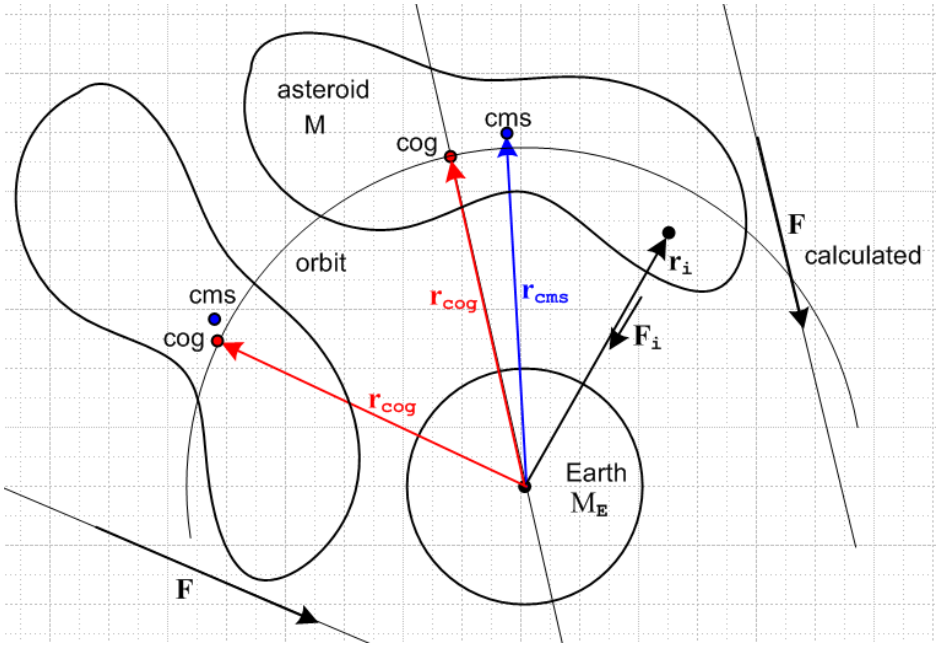
$$\mathbf{r}_{cog} = r_{cog}\,(-\hat{\mathbf{F}}) \qquad \mathbf{F} = F\,\hat{\mathbf{F}} \ . \tag{D.1.6}$$

Here is an illustration,



(D.1.7)

From the Earth's point of view, one could replace the entire asteroid with a point mass M at location $\mathbf{r}_{cog}$ and the Earth would feel the same gravitational pull from that point mass as it does from the asteroid. Furthermore, if the asteroid were in a circular orbit around the earth keeping its same aspect facing the earth (not very likely), then the orbiting characteristics of the asteroid would be the same as for its replacement point mass, and one would have for example $RM_EM/r_{cog}^2 = M\omega^2 r_{cog}$ as the balance between gravitational and centrifugal force. If the asteroid does not rotate or tumbles in some manner, at any point in its orbit $\mathbf{r}_{cog}$ will lie on the orbit shown, but the location of that point within the asteroid changes so that the new r_{cog} computed for a new position and orientation still equals the orbit radius. In this case the $\mathbf{r}_{cog}$ position will change relative to the asteroid, whereas the cms point always has the same position relative to the asteroid. Here we enhance the above drawing by adding another position of the tumbling asteroid in its orbit,



(D.1.8)

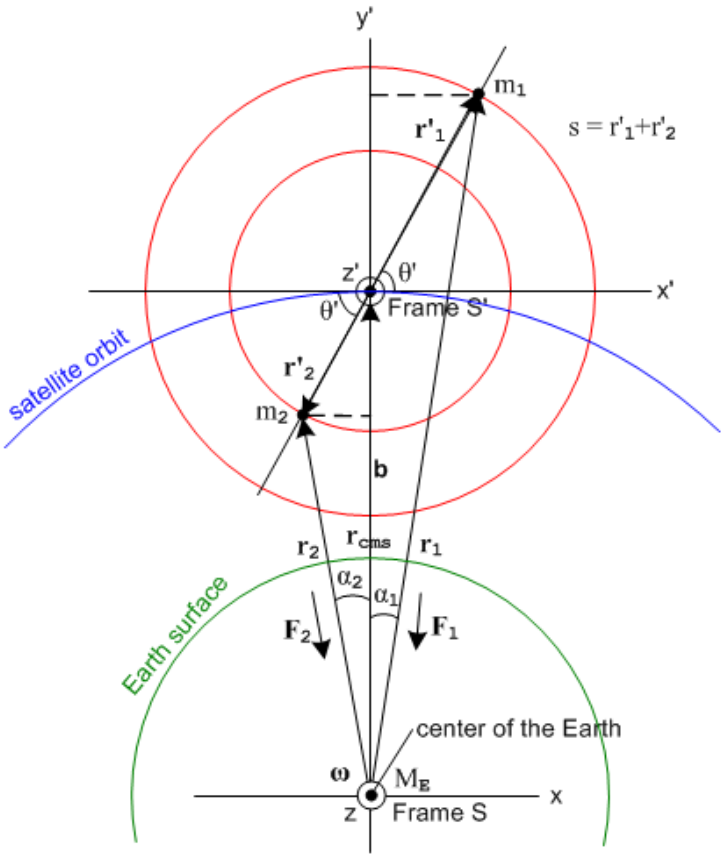
One implication is that, for any orientation of the asteroid in orbit, there exists a position of the asteroid such that the total force **F** will have the same magnitude as at any other position, since r_{cog} is the same.

One might wonder what Symon's operational prescription for computing $\mathbf{r}_{cog}$ has to do with our opening section about torque. First, for each point in the asteroid we know that $\mathbf{r}_i \times \mathbf{F}_i = 0$ ($\mathbf{r}_i$ tails are at Earth center) because each $\mathbf{r}_i$ to mass m_i is parallel to the force $\mathbf{F}_i$ to mass i. Thus according to (D.1.2) we have $\mathbf{N}^{(0)} = 0$. Therefore the condition (D.1.5) that $\mathbf{N}^{(0)} \bullet \mathbf{F} = 0$ is trivially satisfied. Then $\mathbf{r}_{cog}$ exists and is a vector which satisfies the center of gravity definition (D.1.4) that $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$. This equation is satisfied by Symon's computed $\mathbf{r}_{cog}$ because $\mathbf{r}_{cog}$ is parallel to **F**, so the equation says $\mathbf{0} = \mathbf{0}$. From (D.1.4) we then see that the $\mathbf{N}^{(R_{cog})} = 0$ so $\mathbf{r}_{cog}$ is a point with respect to which the total torque on the asteroid (due to gravitational force from the Earth) is 0. In this example it *happens* that any point **R** along the $\mathbf{r}_{cog}$ line satisfies $\mathbf{N}^{(0)} = \mathbf{R} \times \mathbf{F}$ and any such point **R** therefore is a point of zero total torque $\mathbf{N}^{(R)} = 0$. But only one point on this line gives the concentration point of the gravitational force such that $GM_EM/r_{cog}^2 = F$.

We shall now consider a 2-mass tethered satellite as a relatively simple example where we can compute the cog point and then watch it move relative to the cms point.

D.2 Center of Gravity for a 2-mass Dumbbell Satellite

Consider this sketch of the dumbbell satellite :



(D.2.1)

Frame S has its origin at the center of the Earth and is assumed fixed relative to the stars.
Frame S' has its origin at the center of *mass* of the satellite as shown.
The orbit of the satellite lies in the plane of paper.
The vector **ω** describes the angular velocity of the orbit and has nothing to do with Earth's rotation.

The picture shows things at t = 0. We make a special simplifying assumption that at t = 0 the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ lie in the plane of paper which is the plane of the orbit. This means that $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}'_1$, $\mathbf{r}'_2$ and **b** all lie in the plane of paper.

Moreover, at t = 0 the axes of both frames line up: $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$, $\hat{\mathbf{y}} = \hat{\mathbf{y}}'$ and $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$.

The two masses m_1 and m_2 are connected by a massless rigid stick of length $s = r'_1+r'_2$. Since the masses are assumed to be unequal, mass m_1 is restricted to lie on a sphere of radius r'_1 in Frame S', while mass m_2 lies on a different sphere of radius r'_2 (to be computed below). For purposes of the drawing, we have $m_2 > m_1$ so that $r'_2 < r'_1$.

The polar angle in Frame S' of m_1 (and the stick) is θ'. The angles α_1 and α_2 are positive.

Assumption: We will show below that the center of gravity is very close to the center of mass for any angle θ'. For example, if the stick is 10 m long, the distance between these two centers for a low-Earth

orbit is about 3 microns. For this reason, we shall assume that it is the center of mass point $\mathbf{r}_{cms}$ which goes in a circular orbit around the Earth, even though we know from Section D.1 that it is really $\mathbf{r}_{cog}$ that does this. So as the satellite goes around the earth and θ' varies in time, the point $\mathbf{r}_{cog}$ moves a very small amount relative to $\mathbf{r}_{cms}$, and we shall just ignore this tiny motion.

The satellite center of mass point then rotates around the Earth at some rate $\omega_0 = 2\pi/T_0$. For a low-Earth orbit, T_0 is on the order of 88 minutes. As the satellite moves in this orbit, Frame S' rotates with the satellite while Frame S stays fixed relative to the stars. This means that the Frame S' axis y' always points away from the center of the Earth.

One can generalize the discussion for an elliptical orbit, but things are complicated enough for a circular orbit so we stick with that simplification.

With the above assumption, we may identify $\mathbf{r}_{cms}$ with $\mathbf{b}$, our usual vector connecting the two Frame origins, so

$$\mathbf{b} = \mathbf{r}_{cms} \tag{D.2.2}$$

and then from the figure,

$$\begin{aligned} \mathbf{b} + \mathbf{r}'_1 &= \mathbf{r}_1 \\ \mathbf{b} + \mathbf{r}'_2 &= \mathbf{r}_2 \ . \end{aligned} \tag{D.2.3}$$

The Law of Cosines gives,

$$\begin{aligned} r_1^2 &= r'^2_1 + b^2 - 2r'_1 b \cos(\pi/2+\theta') = r'^2_1 + b^2 + 2r'_1 b \sin\theta' \\ r_2^2 &= r'^2_2 + b^2 - 2r'_2 b \cos(\pi/2-\theta') = r'^2_2 + b^2 - 2r'_2 b \sin\theta' \ . \end{aligned} \tag{D.2.4}$$

<u>Center of Mass</u>

From (D.1.1) we know that the center of mass of the satellite is given by,

$$\mathbf{r}_{cms} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} = \mathbf{b} \qquad \text{Frame S} \tag{D.2.5}$$

$$\mathbf{r}'_{cms} = \frac{m_1\mathbf{r}'_1 + m_2\mathbf{r}'_2}{m_1+m_2} = 0 \ . \qquad \text{Frame S'} \tag{D.2.6}$$

Due to the constraint of the massless stick one has,

$$\hat{\mathbf{r}}'_2 = -\,\hat{\mathbf{r}}'_1 \ . \tag{D.2.7}$$

so (D.2.6) says

$$m_1\mathbf{r}'_1 = -\,m_2\mathbf{r}'_2 \quad \Rightarrow \quad m_1 r'_1 = m_2 r'_2 \ , \quad r'_2/r'_1 = m_1/m_2 \ , \quad \hat{\mathbf{r}}'_2 = -\,\hat{\mathbf{r}}'_1 \tag{D.2.8}$$

Angles

Looking at the right triangles from the center of the Earth to the two dashes lines, one sees that

$$\begin{aligned} \sin\alpha_1 &= (r'_1 \cos\theta') / r_1 \quad \Rightarrow \quad r_1\sin\alpha_1 = r'_1\cos\theta' \\ \sin\alpha_2 &= (r'_2 \cos\theta') / r_2 \quad \Rightarrow \quad r_2\sin\alpha_2 = r'_2\cos\theta' \ . \end{aligned} \tag{D.2.9}$$

The same triangles reveal that

$$\begin{aligned} \cos\alpha_1 &= (b + r'_1\sin\theta')/r_1 \quad \Rightarrow \quad r_1\cos\alpha_1 = b + r'_1\sin\theta' \\ \cos\alpha_2 &= (b - r'_2\sin\theta')/r_2 \quad \Rightarrow \quad r_2\cos\alpha_2 = b - r'_2\sin\theta' \ . \end{aligned} \tag{D.2.10}$$

We shall need the following dot products,

$$\begin{aligned} \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{x}} &= \sin\alpha_1 \qquad & \hat{\mathbf{r}}_2 \bullet \hat{\mathbf{x}} &= -\sin\alpha_2 \\ \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{y}} &= \cos\alpha_1 \qquad & \hat{\mathbf{r}}_2 \bullet \hat{\mathbf{y}} &= \cos\alpha_2 \end{aligned} \tag{D.2.11}$$

$$\begin{aligned} \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{r}}_2 &= \cos(\alpha_1+\alpha_2) \\ &= (r_1r_2)^{-1} [(b + r'_1\sin\theta') (b - r'_2\sin\theta') - (r'_1\cos\theta')(r'_2\cos\theta')] \\ &= (r_1r_2)^{-1} [b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2 \sin^2\theta' - r'_1r'_2\cos^2\theta'] \\ &= (r_1r_2)^{-1} [b^2 + (r'_1-r'_2) b\sin\theta' - r'_1r'_2] \ . \end{aligned} \tag{D.2.12}$$

Where is Center of Gravity of the Satellite?

The gravitational forces acting on the two satellite masses due to the Earth are.

$$\begin{aligned} \mathbf{F}_1 &= - (GM_E m_1/r_1^2) \hat{\mathbf{r}}_1 = - (GM_E m_1/r_1^3) \mathbf{r}_1 \\ \mathbf{F}_2 &= - (GM_E m_2/r_2^2) \hat{\mathbf{r}}_2 = - (GM_E m_2/r_2^3) \mathbf{r}_2 \quad . \end{aligned} \tag{D.2.13}$$

Note the direction of the "long" vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in the drawing.

Following the prescription of Section D.1 for finding the location of the center of gravity, we compute the total gravitational force on the satellite,

$$\begin{aligned} \mathbf{F} &= - (GM_E m_1/r_1^2) \hat{\mathbf{r}}_1 - (GM_E m_2/r_2^2) \hat{\mathbf{r}}_2 \\ &= (-GM_E) [(m_1/r_1^2) \hat{\mathbf{r}}_1 + (m_2/r_2^2) \hat{\mathbf{r}}_2] \quad . \end{aligned} \tag{D.2.14}$$

Taking components:

$$F_x = \mathbf{F} \bullet \hat{\mathbf{x}} = (-GM_E)\ [\ (m_1/r_1^2)\ \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{x}} + (m_2/r_2^2)\ \hat{\mathbf{r}}_2 \bullet \hat{\mathbf{x}}\ ]$$

$$= (-GM_E)\ [\ (m_1/r_1^2)\sin\alpha_1 - (m_2/r_2^2)\sin\alpha_2\ ]$$

$$= (-GM_E)\ [\ (m_1/r_1^3)\ r'_1\cos\theta' - (m_2/r_2^3)\ r'_2\cos\theta'\ ] \qquad \text{// now use (D.2.8)}$$

$$= (-GM_E)\ [\ (m_1/r_1^3)\ r'_1\cos\theta' - (m_1/r_2^3)\ r'_1\cos\theta'\ ]$$

$$= (-GM_E)\ (m_1 r'_1\cos\theta')\ [\ (1/r_1^3) - (1/r_2^3)\ ]$$

$$F_y = \mathbf{F} \bullet \hat{\mathbf{y}} = (-GM_E)\ [\ (m_1/r_1^2)\ \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{y}} + (m_2/r_2^2)\ \hat{\mathbf{r}}_2 \bullet \hat{\mathbf{y}}\ ]$$

$$= (-GM_E)\ [\ (m_1/r_1^2)\cos\alpha_1 + (m_2/r_2^2)(b - r'_2\sin\theta')\ ]$$

$$= (-GM_E)\ [\ (m_1/r_1^3)(b + r'_1\sin\theta') + (m_2/r_2^3)(b - r'_2\sin\theta')\ ] \qquad \text{// now use (D.2.8)}$$

$$= (-GM_E)\ [\ b\ \{\ (m_1/r_1^3) + (m_2/r_2^3)\ \} + m_1 r'_1\ \sin\theta'\ \{\ (1/r_1^3) - (1/r_2^3)\}\ ]$$

so

$$F_x = (-GM_E)\ (m_1 r'_1\cos\theta')\ [\ (1/r_1^3) - (1/r_2^3)\ ]$$

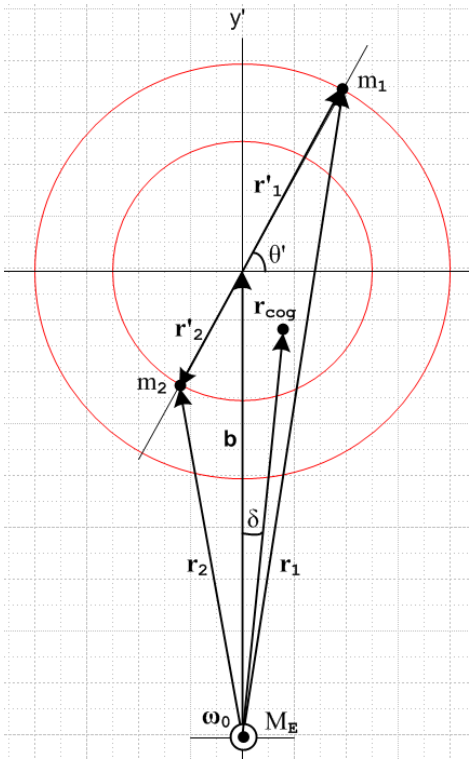
$$F_y = (-GM_E)\ [\ b\ \{\ (m_1/r_1^3) + (m_2/r_2^3)\ \}\ + m_1 r'_1\ \sin\theta'\ \{\ (1/r_1^3) - (1/r_2^3)\}\ ]\ . \qquad \text{(D.2.15)}$$

Recall now from (D.1.6) the rule for obtaining $\mathbf{r}_{cog}$:

$$GM_E M/r_{cog}^2 = F \quad \Rightarrow \quad r_{cog} = \sqrt{GM_E M/F}$$

$$\mathbf{r}_{cog} = r_{cog}\ (-\hat{\mathbf{F}}) \qquad \mathbf{F} = F\ \hat{\mathbf{F}}\ . \qquad \textit{(D.1.6)}$$

Just to have a picture, we now add $\mathbf{r}_{cog}$ to Fig (D.2.1), picking a graphic location for the point which makes things easy to draw (we don't know yet where it actually lies),



(D.2.16)

The center of gravity point lies a distance r_{cog} (to be computed) along a line which makes some angle δ relative to the vertical axis. Since $\hat{\mathbf{r}}_{cog} = -\hat{\mathbf{F}}$, we may write using (D.2.15),

$$\tan(\delta) = -F_x/F_y = -(m_1 r'_1 \cos\theta') \frac{(1/r_1^3) - (1/r_2^3)}{b\,[\,(m_1/r_1^3) + (m_2/r_2^3)\,] + m_1 r'_1 \sin\theta'\,[\,(1/r_1^3) - (1/r_2^3)]}$$

$$= -(m_1 r'_1 \cos\theta') \frac{r_2^3 - r_1^3}{b\,[\,(m_1 r_2^3) + (m_2 r_1^3)\,] + m_1 r'_1 \sin\theta'\,[\,r_2^3 - r_1^3]} \quad . \qquad \text{(D.2.17)}$$

If the masses are vertically aligned, meaning $\theta' = \pm\pi/2$, then $\cos\theta' = 0$ and we see that $\delta = 0$, as expected.

Since for general masses and angle θ' one does *not* have $r_1 = r_2$, one may conclude:

Fact 1: In general the center of gravity does not lie on the line between the center of the Earth and the center of mass. (D.2.18)

Earlier we conjectured that this was the case, and here we see it in our satellite example.

Next we wish to compute the distance r_{cog} to the center of gravity. Using (D.2.14) and (D.2.12),

$$F^2 = (GM_E)^2\,[\,(m_1/r_1^2)^2 + (m_2/r_2^2)^2 + 2\,(m_1/r_1^2)\,(m_2/r_2^2)\cos(\alpha_1+\alpha_2)\,]$$

$$= (GM_E)^2\,[\,(m_1/r_1^2)^2 + (m_2/r_2^2)^2 + 2\,(m_1/r_1^3)\,(m_2/r_2^3)\,r_1 r_2 \cos(\alpha_1+\alpha_2)\,]$$

$$= (GM_E)^2 \left[(m_1/r_1^2)^2 + (m_2/r_2^2)^2 + 2\,(m_1/r_1^3)\,(m_2/r_2^3)(b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2) \right]$$

$$= (GM_E)^2 \left[(m_1/r_1^2)^2 + (m_2/r_2^2)^2 + 2\,(m_1/r_1^3)\,(m_2/r_2^3)[b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2] \right]$$

so

$$F = GM_E \sqrt{(m_1/r_1^2)^2 + (m_2/r_2^2)^2 + 2\,(m_1/r_1^3)\,(m_2/r_2^3)(b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2)} \quad . \qquad \text{(D.2.19)}$$

Then, using $M = (m_1+m_2)$ and $\mu_i = m_i/M$, we find that

$$r_{cog} = \sqrt{GM_E M/F}$$

$$= \frac{1}{\{ (\mu_1/r_1^2)^2 + (\mu_2/r_2^2)^2 + 2\,(\mu_1/r_1^3)\,(\mu_2/r_2^3)[b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2] \}^{1/4}} \qquad \text{(D.2.20)}$$

which is somewhat more complicated than our expression for r_{cms},

$$r_{cms} = b \qquad \textit{(D.2.2)}$$

We are then led to:

Fact 2: In general the center of gravity does not lie the same distance from the center of the Earth as the center of mass. (D.2.21)

Visual check on (D.2.20) : If $m_1 = 0$, then $\mu_1 = 0$ and $\mu_2 = 1$ and the result is $r_{cog} = r_2$ which is correct.

D.3 Dumbbell Satellite Center of Gravity with the Far Approximation: Numerical Examples

A practical tethered satellite is not going to have Earth-scale dimensions so we shall now make the obvious assumption that r_1, r_2 and b are much larger than r'_1 and r'_2. We then define smallness parameters,

$$\varepsilon_1 \equiv (r'_1/b)$$
$$\varepsilon_2 \equiv (r'_2/b) = (r'_2/r'_1)\,(r'_1/b) = (m_1/m_2)\varepsilon_1 = (\mu_1/\mu_2)\varepsilon_1 \qquad \text{(D.3.1)}$$

Then

$$[b^2 + (r'_1-r'_2)b\sin\theta' - r'_1r'_2] = b^2 [\,1 + (\varepsilon_1-\varepsilon_2)\sin\theta' - \varepsilon_1\varepsilon_2] \qquad \text{(D.3.2)}$$

so that

$$r_{cog} = \frac{1}{\{ (\mu_1/r_1^2)^2 + (\mu_2/r_2^2)^2 + 2\,(\mu_1/r_1^3)\,(\mu_2/r_2^3)b^2[1 + (\varepsilon_1-\varepsilon_2)\sin\theta' - \varepsilon_1\varepsilon_2] \}^{1/4}} \qquad \text{(D.3.3)}$$

From (D.2.4) we know also that

$$r_1^2 = r'^2_1 + b^2 + 2r'_1 b \sin\theta' = b^2 [\, 1 + 2(r'_1/b)\sin\theta' + (r'_1/b)^2] = b^2 [\, 1 + 2\varepsilon_1 \sin\theta' + \varepsilon_1^2]$$

$$r_2^2 = r'^2_2 + b^2 - 2r'_2 b \sin\theta' = b^2 [\, 1 - 2(r'_2/b)\sin\theta' + (r'_2/b)^2] = b^2 [\, 1 - 2\varepsilon_2 \sin\theta' + \varepsilon_2^2]$$

so then

$$r_1 = b\sqrt{1 + 2\varepsilon_1 \sin\theta' + \varepsilon_1^2}$$
$$r_2 = b\sqrt{1 - 2\varepsilon_2 \sin\theta' + \varepsilon_2^2}\ . \qquad \text{(D.3.4)}$$

So far everything is exact, but we shall now expand r_{cog} as a series in our small parameters. Since $\varepsilon_2 = (\mu_1/\mu_2)\varepsilon_1$, we replace ε_2 by this expression so there is then only one small parameter ε_1. We employ Maple to carry out this task. We first enter our expressions of interest, using θ' = theta,

```
restart; assume(b>0);
rcog := ((mu1^2/r1^4) + (mu2^2/r2^4) +
2*b^2*(mu1/r1^3)*(mu2/r2^3)*(1+(e1-e2)*sin(theta)- e1*e2))^(-1/4),
```

$$rcog := \frac{1}{\left(\frac{\mu1^2}{r1^4} + \frac{\mu2^2}{r2^4} + 2\,\frac{b^2\,\mu1\,\mu2\,(1 + (e1 - e2)\sin(\theta) - e1\,e2)}{r1^3\,r2^3}\right)^{\left(\frac{1}{4}\right)}}$$

```
r1 := b*sqrt(1+2*e1*sin(theta)+e1^2);
```

$$r1 := b\sqrt{1 + 2\,e1\,\sin(\theta) + e1^2}$$

```
r2 := b*sqrt(1-2*e2*sin(theta)+e2^2);
```

$$r2 := b\sqrt{1 - 2\,e2\,\sin(\theta) + e2^2}$$

```
e2 := (mu1/mu2)*e1;
```

$$e2 := \frac{\mu1\,e1}{\mu2}$$

```
mu2 := 1-mu1;
```

$$\mu2 := 1 - \mu1 \qquad \text{(D.3.5)}$$

We then instruct Maple to expand rcog in a power series around $\varepsilon_1 = 0$ and we ask for the first four terms of the expansion,

```
series(rcog,e1=0,4): simplify(%);
```

$$b - \frac{3}{4}\,\frac{\mu1\,(-2 + 3\cos(\theta)^2)\,b}{-1 + \mu1}\,e1^2 + \frac{\mu1\,\sin(\theta)\,(2 - 5\cos(\theta)^2 - 4\,\mu1 + 10\,\mu1\cos(\theta)^2)\,b}{1 - 2\,\mu1 + \mu1^2}\,e1^3 + O(e1^4) \qquad \text{(D.3.6)}$$

A perhaps unexpected result is that there is no term linear in ε_1, regardless of the masses. We leave it to the energetic reader to concoct a theoretical explanation of this fact (not all odd terms vanish, just the first odd term). Our conclusion then is, to second order in smallness parameter ε_1,

$$r_{cog} \approx b\left[1+\frac{3}{4}\frac{\mu_1}{1-\mu_1}(-2+3\cos^2\theta')\,\varepsilon_1^2\right] = b\left[1+\frac{3}{4}\frac{m_1}{m_2}(-2+3\{1-\sin^2\theta'\})\,\varepsilon_1^2\right]$$

$$= b\left[1+\frac{3}{4}\frac{m_1}{m_2}(1-3\sin^2\theta')\frac{r_1'^2}{b^2}\right] = b\left[1+\frac{3}{4}\frac{r'_2}{r'_1}(1-3\sin^2\theta')\frac{r_1'^2}{b^2}\right] \qquad \text{// (D.2.8)}$$

$$= b\left[1+\frac{3}{4}(1-3\sin^2\theta')\frac{r_1'r_2'}{b^2}\right] \qquad \text{(D.3.7)}$$

and then

$$r_{cog} - r_{cms} = r_{cog} - b \approx \left[\frac{3}{4}(1-3\sin^2\theta')\frac{r_1'r_2'}{b^2}\right] b \qquad \text{(D.3.8)}$$

and

$$\frac{r_{cog} - r_{cms}}{r_{cms}} \approx \frac{3}{4}(1-3\sin^2\theta')\frac{r_1'r_2'}{b^2}. \qquad \text{(D.3.9)}$$

Fact 3: If $\theta' = \pm 35.26°$, we get $r_{cog} = r_{cms}$ through order ε_1^2, since this angle has $\sin\theta' = 1/\sqrt{3}$. But the line to the center of gravity does not in general have $\delta = 0$ since in general $r_1 \neq r_2$ at this angle. (D.3.10)

Fact 4: If the masses are vertically aligned, meaning $\theta' = \pm \pi/2$, then $\sin\theta' = 1$ and we find that

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{3}{4}(1-3)\frac{r_1'r_2'}{b^2} = -\frac{3}{2}\frac{r_1'r_2'}{b^2}. \qquad \text{(D.3.11)}$$

The center of gravity in this case is *closer* to the Earth than the center of mass and of course lies on the line to the center of mass.

Fact 5: If the masses are horizontally aligned, meaning $\theta' = 0,\pi$, then $\sin\theta' = 0$ and we find that

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{3}{4}(1-0)\frac{r_1'r_2'}{b^2} = +\frac{3}{4}\frac{r_1'r_2'}{b^2}. \qquad \text{(D.3.12)}$$

The center of gravity in this case is *farther* from the Earth than the center of mass. The line to the center of gravity does not have $\delta = 0$ unless $m_1 = m_2$.

The above cases show the extremes of the factor $(1 - 3\sin^2\theta')$ and hence of $r_{cog} - r_{cms}$. For a general angle θ' the result lies between the two limiting cases.

Numerical Examples

From (8.7.4) we have R_E = 6371 km. If we put our satellite in orbit 200 km above the Earth's surface, then b = 6371+200 = 6571m. A higher orbit of course gives a larger b and a smaller offset between r_{cms} and r_{cog}. We take the maximum displacement between r_{cog} and r_{cms} from Fact 5 to obtain

$$\left|\frac{r_{cog} - r_{cms}}{r_{cms}}\right|_{max} = \frac{3}{4}\frac{r_1' r_2'}{b^2} \qquad \text{(D.3.13)}$$

For a given mass separation (tether length) $s = r'_1 + r'_2$ the product $r'_1 r'_2$ is maximized when $r'_1 = r'_2$ (easy to show) and this in turn means $m_1 = m_2$, so to get the worst case we set $r'_1 = r'_2 = s/2$ to get

$$\left|\frac{r_{cog} - r_{cms}}{r_{cms}}\right|_{max} = \frac{3}{4}\frac{1}{4}(s/b)^2 = (3/16)(s/b)^2 \qquad s = \text{stick length} = \text{tether length} \qquad \text{(D.3.14)}$$

We now have Maple compute $dr/r = \left|\frac{r_{cog} - r_{cms}}{r_{cms}}\right|_{max}$ and $dr = |r_{cog} - r_{cms}|_{max}$ for three cases:

```
restart;
dr := b*(3/16)*(s/b)^2;
```

$$dr := \frac{3}{16}\frac{s^2}{b}$$

```
b := RE + h;
```

$$b := RE + h$$

```
h := 200*m;
```

$$h := 200\ m$$

```
RE := 6731e3*m;
```

$$RE := .6731\ 10^{7}\ m$$

(D.3.15)

```
s := 10*m;
```

$$s := 10\ m$$

```
dr/b;
```

$$.4138245797\ 10^{-12}$$

```
dr;
```

$$.2785536011\ 10^{-5}\ m$$

```
s := 1000*m;    # 1 km
```

$$s := 1000\ m$$

```
dr/b;
```

$$.4138245797\ 10^{-8}$$

```
dr;
```

$$.02785536011\ m$$

```
s := 50000*m;   # 50 km
dr/b;
```

$$s := 50000\ m$$

$$.00001034561449$$

```
dr;
```

$$69.63840028\ m$$

For a s= 10m tether length, the max offset is 2.8 microns which we feel pretty comfortable ignoring.
For a s= 1 km tether length, the max offset is 2.8 cm, still pretty small.
For a s = 50 km tether length, the max offset is about 70m. (D.3.16)

The quantities $\left|\frac{r_{cog} - r_{cms}}{r_{cms}}\right|_{max}$ for each case are shown as dr/r : 10^{-12}, 10^{-8} and 10^{-5}.

This justifies our approximation that it is the center of mass point that essentially orbits the Earth, and the variation between $\mathbf{r}_{cog}$ and $\mathbf{r}_{cms}$ is always very small.

D.4 Dumbbell Satellite Center of Gravity for equal masses and no approximation

(a) Equal Masses and general θ'

Recall the general no-approximation results (D.2.17) and (D.2.20),

$$\tan(\delta) = -(m_1 r'_1 \cos\theta') \frac{r_2^3 - r_1^3}{b\,[(m_1 r_2^3) + (m_2 r_1^3)] + m_1 r'_1 \sin\theta' [r_2^3 - r_1^3]} \quad . \qquad \textit{(D.2.17)}$$

$$r_{cog} = \frac{1}{\{(\mu_1/r_1^2)^2 + (\mu_2/r_2^2)^2 + 2(\mu_1/r_1^3)(\mu_2/r_2^3)[b^2 + (r'_1 - r'_2)b\sin\theta' - r'_1 r'_2]\}^{1/4}} \qquad \textit{(D.2.20)}$$

where

$$r_1^2 = r'^2_1 + b^2 + 2r'_1 b \sin\theta'$$
$$r_2^2 = r'^2_2 + b^2 - 2r'_2 b \sin\theta' \; . \qquad \textit{(D.2.4)} \qquad \text{(D.4.1)}$$

Setting $m_1 = m_2$ (which implies $r'_1 = r'_2 = s/2$ and also $\mu_1 = \mu_2 = 1/2$) gives these simpler forms,

$$\tan(\delta) = -(r'_1 \cos\theta') \frac{r_2^3 - r_1^3}{b\,(r_2^3 + r_1^3) + r'_1 \sin\theta' (r_2^3 - r_1^3)}$$

$$r_{cog} = \frac{\sqrt{2}\, r_1 r_2}{\{r_2^4 + r_1^4 + 2r_1 r_2 [b^2 - r'^2_1]\}^{1/4}}$$

where

$$r_1^2 = r'^2_1 + b^2 + 2r'_1 b \sin\theta'$$
$$r_2^2 = r'^2_1 + b^2 - 2r'_1 b \sin\theta' \; . \qquad \textit{(D.2.4)} \qquad \text{(D.4.2)}$$

(b) Equal Masses and θ' = π/2 (vertically aligned)

If we further specify that θ' = π/2 we get δ = 0 and moreover,

$$r_1 = (b + r'_1) \qquad r_2 = (b - r'_1) \qquad \text{(D.4.3)}$$

$$b^2 = r_1^2 + r'^2_1 - 2r_1 r'_1 \quad \Rightarrow \quad b^2 - r'^2_1 = r_1(r_1 - 2r'_1) \qquad \text{(D.4.4)}$$

$$r_{cog} = \frac{\sqrt{2}\, r_1 r_2}{\{r_2^4 + r_1^4 + 2r_1 r_2 [b^2 - r'^2_1]\}^{1/4}} = \frac{\sqrt{2}\,(b + r'_1)(b - r'_1)}{\{(b - r'_1)^4 + (b + r'_1)^4 + 2(b + r'_1)(b - r'_1)[b^2 - r'^2_1]\}^{1/4}}$$

$$= \frac{\sqrt{2}\,(b+r'_1)(b-r'_1)}{\{\,4(b^2+r'_1{}^2)^2\,\}^{1/4}} = \frac{\sqrt{2}\,(b+r'_1)(b-r'_1)}{\sqrt{2}\,\sqrt{b^2+r'_1{}^2}} = \frac{(b^2-r'_1{}^2)}{\sqrt{b^2+r'_1{}^2}} = \frac{b^2}{b}\,\frac{(1-(r'_1/b)^2)}{\sqrt{1+(r'_1/b)^2}}$$

$$= b\,\frac{1-(r'_1/b)^2}{\sqrt{1+(r'_1/b)^2}} \quad . \qquad \text{(D.4.5)}$$

The exact results are then :

$$\tan(\delta) = 0 \quad \Rightarrow \quad \delta = 0$$

$$r_{cog} = b\,\frac{1-(r'_1/b)^2}{\sqrt{1+(r'_1/b)^2}} \quad . \qquad \text{// } m_1 = m_2 \text{ and } \theta' = \pi/2 \text{ and } r'_1 = s/2 \qquad \text{(D.4.6)}$$

We then find that

$$r_{cog} - r_{cms} = b\,[\,1 - \frac{1-(r'_1/b)^2}{\sqrt{1+(r'_1/b)^2}}\,]$$

$$\frac{r_{cog}-r_{cms}}{r_{cms}} = 1 - \frac{1-(r'_1/b)^2}{\sqrt{1+(r'_1/b)^2}} \quad . \qquad \text{(D.4.7)}$$

In this vertically aligned case the center of gravity is *closer* to the Earth than the center of mass. We showed this earlier using the far-distance approximation, and now we show it with the exact result.

$$r_{cog} < r_{cms} \qquad ?$$

$$b\,\frac{1-(r'_1/b)^2}{\sqrt{1+(r'_1/b)^2}} < b \qquad ?$$

$$1-(r'_1/b)^2 < \sqrt{1+(r'_1/b)^2} \qquad ?$$

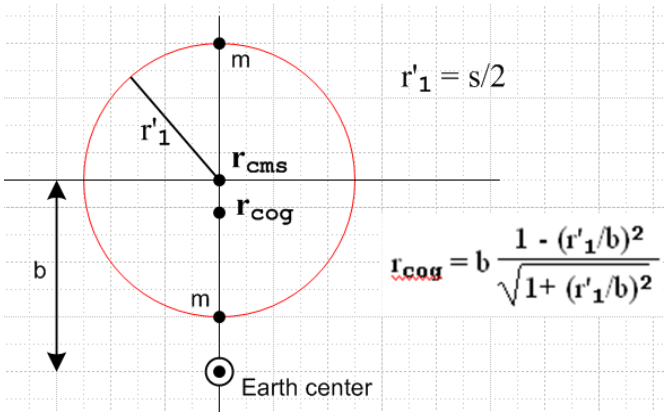
$$1 - 2\,(r'_1/b)^2 + (r'_1/b)^4 < 1 + (r'_1/b)^2 \qquad ?$$

$$(r'_1/b)^4 < 3\,(r'_1/b)^2 \qquad ?$$

$$(r'_1/b) < 3 \qquad ?$$

$$r'_1 < 3b \qquad ? \qquad \text{yes!} \quad . \qquad \text{(D.4.8)}$$

This figure summarizes our result (with $r_{cms} - r_{cog}$ exaggerated)



(D.4.9)

Looking back at the force equation (D.2.14), we can confirm result (D.4.6) fairly quickly :

$$\mathbf{F} = -(GM_E m_1/r_1^2)\,\hat{\mathbf{r}}_1 - (GM_E m_2/r_2^2)\,\hat{\mathbf{r}}_2 = -[(GM_E m_1/r_1^2) + (GM_E m_2/r_2^2)]\,\hat{\mathbf{y}}$$

$$F = GM_E m_1[\,(1/r_1^2) + (1/r_2^2)] = GM_E m_1[r_2^2 + r_1^2] / [r_1 r_2]^2$$

$$= GM_E m_1[(b-r'_1)^2 + (b+r')^2] / [(b+r'_1)(b-r'_1)]^2 =$$

$$= GM_E m_1[2b^2 + 2r'^2_1] / (b^2-r'^2_1)^2 = 2\,GM_E m_1\,(b^2 + r'^2_1) / (b^2-r'^2_1)^2$$

$$= 2\,GM_E m_1\,b^{-2}\,(1 + (r'_1/b)^2) / (1-(r'_1/b)^2)^2$$

so

$$r_{cog} = \sqrt{GM_E(2m_1)/F} = b\,(1-(r'_1/b)^2) / \sqrt{1 + (r'_1/b)^2} \qquad \text{(D.4.10)}$$

in agreement with (D.4.6).

If we now <u>further assume</u> $(r'_1/b) << 1$ then (D.4.6) becomes

$$r_{cog} \approx b\,(1 - (r'_1/b)^2)\,(1 - (1/2)\,(r'_1/b)^2) = b\,[\,1 - (3/2)(r'_1/b)^2]\ .$$

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{r_{cog} - b}{b} = -(3/2)(r'_1/b)^2 \qquad \text{(D.4.11)}$$

in agreement with (D.3.11).

<u>(c) Equal Masses and θ' = 0 (horizontally aligned)</u>

We start again with (D.4.2) for equal masses,

$$\tan(\delta) = -(r'_1\cos\theta')\,\frac{r_2^3 - r_1^3}{b\,(r_2^3 + r_1^3) + r'_1\sin\theta'\,(r_2^3 - r_1^3)}$$

$$r_{cog} = \frac{\sqrt{2}\, r_1 r_2}{\{\ r_2^4 + r_1^4 + 2r_1 r_2[b^2 - r'^2_1]\ \}^{1/4}}$$

$$r_1^2 = r'^2_1 + b^2 + 2r'_1 b \sin\theta' \qquad r_2^2 = r'^2_1 + b^2 - 2r'_1 b \sin\theta' \,. \qquad \textit{(D.4.2)}$$

When $\theta' = 0$ we not only have $r'_1 = r'_2$ but also $r_1 = r_2$ along with $\sin\theta' = 0$ and $\cos\theta' = 1$. This at once implies that $\tan(\delta) = 0$, and for r_{cog} we find

$$r_1^2 = r'^2_1 + b^2 \quad \Rightarrow \quad -r'^2_1 = b^2 - r_1^2 \quad \Rightarrow \quad [b^2 - r'^2_1] = 2b^2 - r_1^2$$

$$r_{cog} = \frac{\sqrt{2}\, r_1^2}{\{\ 2r_1^4 + 2r_1^2[b^2 - r'^2_1]\ \}^{1/4}} = \frac{\sqrt{2}\, r_1^2}{\{\ 2r_1^4 + 2r_1^2[2b^2 - r_1^2]\ \}^{1/4}} = \frac{\sqrt{2}\, r_1^2}{\{\ 4r_1^2 b^2\ \}^{1/4}}$$

$$= \left[\frac{r_1^6}{b^2}\right]^{1/4} = \left[\frac{r_1^3}{b}\right]^{1/2} = \sqrt{\frac{r_1^3}{b}} \qquad \text{(D.4.12)}$$

$$= \left[\frac{r_1^6}{b^2}\right]^{1/4} = \left[\frac{(b^2 + r'^2_1)^3}{b^2}\right]^{1/4} = b\ [1 + (r'_1/b)^2\ ]^{3/4} \,. \qquad \text{(D.4.13)}$$

The exact results are then

$$\tan(\delta) = 0$$

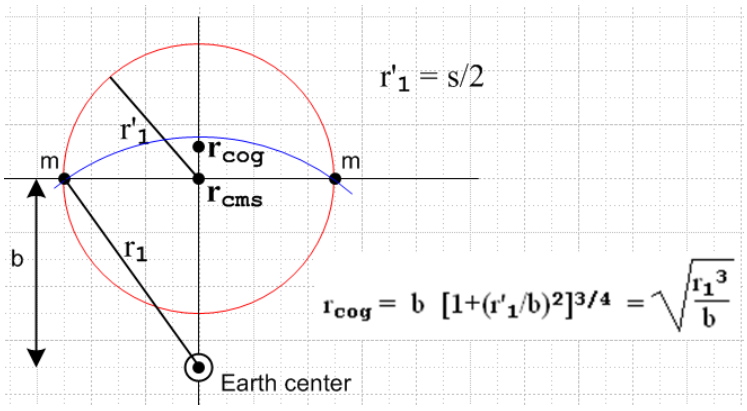
$$r_{cog} = \sqrt{\frac{r_1^3}{b}} = b\ [1 + (r'_1/b)^2\ ]^{3/4} \qquad // \ m_1 = m_2 \text{ and } \theta' = 0 \text{ and } r'_1 = s/2$$

$$r_{cog} - r_{cms} = b\ \{\ [1 + (r'_1/b)^2\ ]^{3/4} - 1\ \}$$

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = [1 + (r'_1/b)^2\ ]^{3/4} - 1 \qquad \text{(D.4.14)}$$

If is obvious by inspection that this last ratio is a positive quantity, so for the horizontally-aligned equal-mass satellite we find that the center of gravity is *farther* from the center of the Earth than the center of mass, in agreement with our earlier conclusion based on approximation.

This figure summarizes our result (with $r_{cog} - r_{ccms}$ exaggerated)



(D.4.15)

We have added in blue a portion of a circle centered at Earth center which passes through the masses. The point r_{cog} lies below this circle, as we now verify

$r_{cog} < r_1$?

$\sqrt{\frac{r_1^3}{b}} < r_1$?

$r_1^3 < r_1^2\, b$?

$r_1 < b$? yes (D.4.16)

Looking back at the force equations (D.2.15), we can confirm result (D.4.12) very quickly,

$F_y = -2GM_E m_1 (b/r_1^3)$ $\quad F_x = 0$

$r_{cog} = \sqrt{GM_E(m_1+m_2)/F} = \sqrt{\frac{r_1^3}{b}}$. (D.4.17)

If we now <u>further assume</u> $(r'_1/b) << 1$ this becomes

$r_{cog} \approx b\,[1 + (3/4)\ (r'_1/b)^2\,]$

$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{r_{cog} - b}{b} = +\,(3/4)(r'_1/b)^2$ (D.4.18)

in agreement with (D.3.12).

D.5 Center of Gravity for a single-sphere satellite

For the vertically aligned dumbbell satellite, we found that $r_{cog} < r_{cms}$. People argue that this is so because gravity acts more strongly on the mass closer to the Earth. This argument does not help much,

however, for the horizontally aligned satellite where instead one has $r_{cog} > r_{cms}$ and neither mass is closer than the other to the Earth.

One wonders if one can assemble a rigid satellite from a finite number of masses such that these two opposite effects cancel out, resulting in $\mathbf{r}_{cog} = \mathbf{r}_{cms}$, at least for some orientation of the masses. Could such a solution be found that works for any orientation of the rigid satellite? We leave these questions to the reader and return instead to the above "argument".

The argument applied to a sphere gives a wrong answer. One would argue for a spherical satellite that the near half of the sphere is closer to the Earth (where gravity is stronger) than the far half, so the center of gravity should be offset toward the Earth from the center of mass. As the reader no doubt knows, a uniform sphere or spherical shell is in fact a "rigid assembly of masses" for which $\mathbf{r}_{cog} = \mathbf{r}_{cms}$. So in such an object, the two effects found for the vertically and horizontally aligned satellites do in fact exactly cancel out.

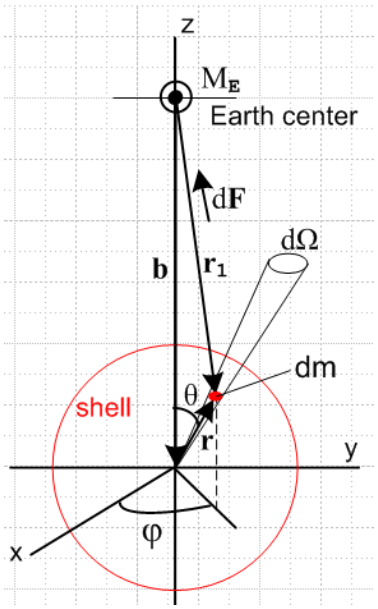
Here we demonstrate that $\mathbf{r}_{cog} = \mathbf{r}_{cms}$ for a uniform thin shell by two basic methods.

One this is shown, the result then applies for any symmetric assembly of shells such as a sphere or a thick spherical shell.

Method A. In electrostatics one of Maxwell's equations says $\text{div}\mathbf{E} = 4\pi\rho$ (cgs units). One applies the integral form of this law $\int \mathbf{E} \bullet d\mathbf{A} = 4\pi \int \rho \, dV = 4\pi Q$ to show that the electric field outside a uniform spherical shell of charge is independent of the radius of the shell and thus is the same as if the charge were all concentrated at the center of the shell. The argument is that, due to rotational invariance (or "symmetry"), the direction of the electric field can only be radial, so then $\int \mathbf{E} \bullet d\mathbf{A} = E * 4\pi R^2$ and one then finds that $E = Q/R^2$ which is indeed the electric field of a point charge Q at distance R.

The same argument can be applied to a uniform shell of mass so that the gravitational field must be in the radial direction and is independent of the shell radius R, so the shell acts as a point mass at its center. If this shell is a satellite then the Earth cannot determine from gravity alone the radius of that satellite, so one can replace the satellite by a point mass at its center. The Earth never knows, and nothing changes in the orbit. But for the point mass certainly $r_{cog} = r_{cms}$, so this fact applies as well to any uniform spherical shell or sphere satellite.

Method B. The ever-popular brute force method. This method is more direct, and is found in high-school physics texts, though the integral we obtain below is not. The drawing below is upside-down relative to our earlier drawings, to be more compatible with usual spherical coordinate notation. So the center of the Earth is now at the top and the center of *mass* of the satellite is located a distance b from Earth center, as earlier. This particular satellite is a spherical shell of radius r and very thin thickness dr. The mass of the shell is then $M_{shell} = [(4\pi r^2)dr]\rho$ where ρ is the shell's uniform mass density.



(D.5.1)

Quantity dm is a tiny chunk of mass on the spherical shell of radius r with

$$dm = \rho(dA)dr = \rho(r^2 d\Omega)dr = \rho\,(r^2 \sin\theta d\theta d\varphi)dr\,. \tag{D.5.2}$$

The force experienced by this mass dm due to mass M_E is given by

$$d\mathbf{F} = -(GM_E dm/r_1^2)\hat{\mathbf{r}}_1 = -(GM_E/r_1^3)\, dm\, \mathbf{r}_1\,. \tag{D.5.3}$$

As usual in spherical coordinates,

$$\mathbf{r} = r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} \tag{D.5.4}$$

so then

$$\mathbf{r}_1 = \mathbf{b} + \mathbf{r} = -b\hat{\mathbf{z}} + r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$$

$$= r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (-b + r\cos\theta)\hat{\mathbf{z}}\,. \tag{D.5.5}$$

Therefore,

$$d\mathbf{F} = -(GM_E/r_1^3)\, dm\, \mathbf{r}_1$$

$$= -(GM_E/r_1^3)\, dm\, [r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (-b + r\cos\theta)\hat{\mathbf{z}}\,] \tag{D.5.6}$$

Then,

$$dF_x = d\mathbf{F} \bullet \hat{\mathbf{x}} = -(GM_E/r_1^3)\,\rho\,(r^2 \sin\theta d\theta d\varphi)dr\,(r\sin\theta\cos\varphi)$$

$$dF_y = d\mathbf{F} \bullet \hat{\mathbf{y}} = -(GM_E/r_1^3)\ \rho\ (r^2 \sin\theta d\theta d\varphi) dr\ (r \sin\theta \sin\varphi)$$
$$dF_z = d\mathbf{F} \bullet \hat{\mathbf{z}} = -(GM_E/r_1^3)\ \rho\ (r^2 \sin\theta d\theta d\varphi) dr\ (-b + r\cos\theta)\ . \qquad \text{(D.5.7)}$$

Now integrate dφ from 0 to 2π to get the force on a *ring* of mass of angular width dθ on the shell at fixed θ. But sinφ has zero integral over this range and so does cosφ, so the first two terms integrate to nothing, while the integral of dφ in the last line gives 2π. We are left with this resulting force all, in the z direction,

$$dF_{ring} = (GM_E/r_1^3)\ \rho\ (r^2 \sin\theta d\theta\ 2\pi)\ dr\ (b - r\cos\theta)\ . \qquad \text{(D.5.8)}$$

Now integrate over all rings of the shell to get a total force F experienced by the spherical shell satellite,

$$F = \int_0^\pi d\theta\ (GM_E/r_1^3)\ \rho\ (r^2 \sin\theta d\theta\ 2\pi)\ dr\ (b - r\cos\theta)$$

$$= 2\pi\rho G M_E\, r^2 dr \int_0^\pi d\theta\, \frac{\sin\theta\,(b - r\cos\theta)}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}}\ . \qquad \text{(D.5.9)}$$

This is a famous discontinuous integral that we evaluate below, but for now we use Maple. Assuming that r < b which means the Earth center lies outside the shell, Maple says,

```
restart; assume(b>0,r>0,r<b);
Int(sin(theta)*(b-r*cos(theta))/ (b^2+r^2-2*b*r*cos(theta))^(3/2),theta=0..Pi);
```

$$\int_0^{\pi} \frac{\sin(\theta)\,(b - r\cos(\theta))}{\left(b^2 + r^2 - 2\,b\,r\cos(\theta)\right)^{\left(\frac{3}{2}\right)}}\, d\theta$$

```
value(%);
```

$$2\,\frac{1}{b^2} \qquad \text{(D.5.10)}$$

where $2/b^2$ is the value of the integral. Thus we have shown that

$$F = 2\pi\rho G M_E\, r^2 dr \int_0^\pi d\theta\, \frac{\sin\theta\,(b - r\cos\theta)}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} = 2\pi\rho G M_E\, r^2 dr\ (2/b^2)$$

$$= GM_E\ (4\pi r^2 dr)\rho\ /b^2 = GM_E M_{shell}\ /\ b^2\ . \qquad \text{(D.5.11)}$$

We now use the center of gravity definition from (D.1.6) to find,

$$r_{cog} = \sqrt{GM_E M_{shell}/F} = b \qquad \text{(D.5.12)}$$

Since $r_{cms} = b$ as well, we conclude that for a spherical shell satellite one has $r_{cog} = r_{cms}$.

If the mass center M_E were inside the shell so b < r, Maple gives a different answer ,

```
restart; assume(b>0,r>0,r>b);
Int(sin(theta)*(b-r*cos(theta))/ (b^2+r^2-2*b*r*cos(theta))^(3/2),theta=0..Pi);
```

$$\int_0^{\pi} \frac{\sin(\theta)\,(b - r\cos(\theta))}{\left(b^2 + r^2 - 2\,b\,r\cos(\theta)\right)^{\left(\frac{3}{2}\right)}}\,d\theta$$

```
value(%);
```

$$0$$

(D.5.13)

showing that the force experienced by the uniform shell due to a mass M_E inside the shell is zero.

It seems unsportsmanlike not to actual *do* the integral to see why it is discontinuous at r = b. Here is a brief tour:

1. Change variables from θ to r_1 and look at the endpoints in the new variable,

$$r_1^2(\theta=0) = r^2 + b^2\text{-}2rb(1) = (b\text{-}r)^2 \quad \Rightarrow \quad r_1(\theta=0) = |b\text{-}r| \equiv \alpha$$
$$r_1^2(\theta=\pi) = r^2 + b^2\text{-}2rb(-1) = (b+r)^2 \quad \Rightarrow \quad r_1(\theta=\pi) = b+r \equiv \beta$$

It is the absolute value that causes the discontinuous behavior at b = r, as we shall see.

2. Since $r_1^2 = r^2 + b^2 - 2rb\cos\theta$ one has $\sin\theta d\theta = r_1 dr_1/(rb)$. (D.5.14)

3. The factor $(b - r\cos\theta) = (b^2\text{-}r^2\text{+}r_1^2)/(2b)$, while the denominator of the integral is just r_1^3.

4. The integral reduces to two elementary power integrals to give

$$I(r) = \int_0^{\pi} d\theta \frac{\sin\theta\,(b\text{-}r\cos\theta)}{(r^2+b^2\text{-}2rb\cos\theta)^{3/2}} = \left[\ (b^2\text{-}r^2)\int_{\alpha}^{\beta} dr_1/r_1^2 \ + \int_{\alpha}^{\beta} dr_1\ 1\ \right] / (2rb^2)$$

$$= \left[\ (b\text{-}r)\ \left\{\frac{(b+r) - |b\text{-}r|}{|b\text{-}r|}\right\} \ + (b+r) - |b\text{-}r|\ \right] / (2rb^2)\ . \qquad \text{(D.5.15)}$$

For b > r one has |b-r| = b-r and the value comes out

$$I(r) = [\ 2r \ + 2r\ ] / (2rb^2) = 2/b^2 \qquad \text{(D.5.16)}$$

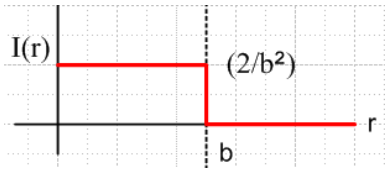
For b < r one has |b-r| = r-b and the value is

$$I(r) = [\ -2b \ + 2b\ ] / (2rb^2) = 0 \qquad \text{(D.5.17)}$$

The full result is then

$$I(r) = (2/b^2)H(b-r) \qquad \text{// Heaviside function} \qquad \text{(D.5.18)}$$

which has this appearance,



(D.5.19)

On can argue that $I(b) = (1/b^2)$ which is the average of the values at the discontinuity.

ok to here