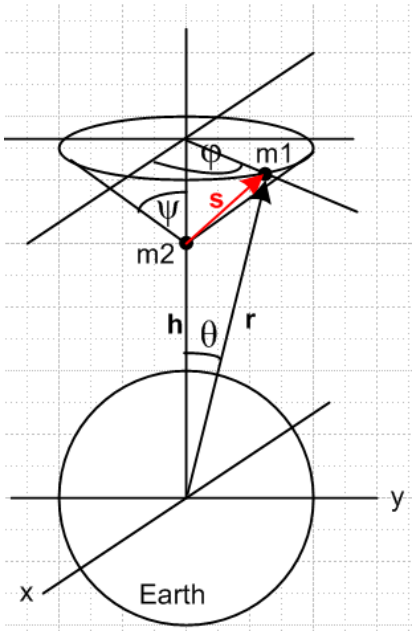
Center of Gravity for Satellite of two masses on massless stick PhL 1.11.15

The Earth is treated as a point particle of mass M located at its center. Here is the picture:



The "satellite" consists of masses m_1 and m_2 at the ends of a massless rigid stick. Mass m_2 is located on the z axis at z = h, while mass m_1 is located relative to mass m_2 by the stick vector **s**. By this means we can have the two masses be in an arbitrary orientation for this problem.

Now in spherical coordinates we know that

$$\hat{\mathbf{r}} = \cos\varphi \sin\theta\, \hat{\mathbf{x}} + \sin\varphi \sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}}$$

Therefore for our little stick unit vector $\hat{\mathbf{s}}$ we can write

$$\hat{\mathbf{s}} = \cos\varphi \sin\psi\, \hat{\mathbf{x}} + \sin\varphi \sin\psi\, \hat{\mathbf{y}} + \cos\psi\, \hat{\mathbf{z}}$$

I think of the Earth as the test mass m_E, a point mass located at its center. The force on this test mass due to the two masses is this

$$\mathbf{F} = Gm_E \left[(m_1/r^2)\hat{\mathbf{r}} + (m_2/h^2)\hat{\mathbf{z}} \right]$$

$$= Gm_E \left[(m_1/r^3)\, \mathbf{r} + (m_2/h^3)\, \mathbf{h} \right]$$

$$= F\, \hat{\mathbf{F}}$$

This vector certainly lies in the plane φ. If $m_1 = 0$ it lies in the **h** direction and if $m_2 = 0$ it lies in the **r** direction, so for general masses it lies in the φ plane between the vectors **r** and **h**.

Let **R** be the solution center of gravity vector (point) which lies in this plane and which has length R so

$$\mathbf{R} = R\hat{\mathbf{F}}$$

Then the main act is this

$$Gm_E M/R^2 = F \qquad\qquad M = m_1 + m_2$$

We know something about θ from the law of cosines

$$s^2 = r^2+h^2 - 2rh\cos\theta$$

$$\cos\theta = (r^2+h^2-s^2)/(2rh)$$

Now

$$\hat{\mathbf{r}} \bullet \hat{\mathbf{z}} = \cos\theta = (r^2+h^2-s^2)/(2rh)$$

We know that

$$F^2 = (Gm_E)^2[\ (m_1/r^2)^2 + (m_2/h^2)^2 + 2(m_1/r^2)(m_2/h^2) * (r^2+h^2-s^2)/(2rh)\ ]$$

$$= (Gm_E)^2[\ m_1^2/r^4 + m_2^2/h^4 + m_1m_2(r^2+h^2-s^2)/(r^3h^3)\ ]$$

$$= (Gm_E/r^2h^2)^2\ [\ m_1^2h^4 + m_2^2r^4 + m_1m_2(r^2+h^2-s^2)\ rh\ ]$$

$$F = (Gm_E/r^2h^2)\sqrt{\ m_1^2h^4 + m_2^2r^4 + m_1m_2(r^2+h^2-s^2)\ rh}$$

$$= (Gm_E/r_1^2r_2^2)\sqrt{\ m_1^2r_2^4 + m_2^2r_1^4 + m_1m_2(r_1^2+r_2^2-s^2)\ r_1r_2}$$

So then the distance to the COG point would be

$$R^2 = Gm_E(m_1+m_2)/F$$

It seems unlikely that this COG point lies on the stick!

Special case.

Suppose ψ = 0 in which case we have h + s = r. The stick is then vertical and on the z axis. Then

$$\mathbf{F} = Gm_E\ [\ (m_1/r^3)\ \mathbf{r} + (m_2/h^3)\ \mathbf{h}\ ] = Gm_E\ [\ (m_1/r^2)\ \hat{\mathbf{z}} + (m_2/h^2)\ \hat{\mathbf{z}}\ ]$$

$$= Gm_E[(m_1/r^2) + (m_2/h^2)]\hat{\mathbf{z}}$$

$$F = Gm_E[(m_1/r^2) + (m_2/h^2)]$$

$$R_{cog}^2 = Gm_E(m_1+m_2)/F = \frac{m_1+m_2}{(m_1/r^2) + (m_2/h^2)} = \frac{m_1+m_2}{(m_1/r_1^2) + (m_2/r_2^2)}$$

The center of mass is this

$$R_{cms} = (r_1m_1 + r_2m_2)/(m_1+m_2) = \frac{r_1m_1 + r_2m_2}{m_1+m_2}$$

If the masses are equal we find

$$R_{cog}^2 = \frac{2}{(1/r_1^2) + (1/r_2^2)} = \frac{2r_1^2r_2^2}{r_2^2 + r_1^2} \qquad \text{and} \qquad R_{cog} = \sqrt{2}\, r_1r_2/\sqrt{r_1^2+r_2^2}$$

$$R_{cms} = (r_1+r_2)/2 \qquad R_{cog} = \frac{r_1r_2}{\sqrt{(r_1^2+r_2^2)/2}}$$

Here then are the results for the Special Case (tethered satellite):

general mass	equal mass
$R_{cms} = \frac{r_1m_1 + r_2m_2}{m_1+m_2}$	$R_{cms} = \frac{r_1 + r_2}{2}$
$R_{cog} = \sqrt{\frac{m_1+m_2}{(m_1/r_1^2) + (m_2/r_2^2)}}$	$R_{cog} = \sqrt{\frac{2}{(1/r_1^2) + (1/r_2^2)}}$

I know these both lie in the range (r_2, r_1). I would guess that $R_{cog} < R_{cms}$ since gravity is stronger on the closer mass. Let's check

$$R_{cog}^2 < R_{cms}^2 \quad ?$$

$$\frac{m_1+m_2}{(m_1/r_1^2) + (m_2/r_2^2)} < \left(\frac{r_1m_1 + r_2m_2}{m_1+m_2}\right)^2 \quad ?$$

$$\frac{(m_1+m_2)^3}{(m_1/r_1^2) + (m_2/r_2^2)} < (r_1m_1 + r_2m_2)^2 \quad ?$$

$$\frac{1}{(\mu_1/r_1^2) + (\mu_2/r_2^2)} < \left(\frac{r_1\mu_1 + r_2\mu_2}{1}\right)^2 \quad ? \qquad \mu_i = m_i/(m_1+m_2)$$

$$\frac{1}{(\mu_1/r_1^2) + (\mu_2/r_2^2)} < (r_1\mu_1 + r_2\mu_2)^2 \quad ?$$

$$(r_1\mu_1 + r_2\mu_2)^2 [(\mu_1/r_1^2) + (\mu_2/r_2^2)] > 1 \quad ?$$

$$(r_1\mu_1 + r_2\mu_2)^2 (\mu_1 r_2^2 + \mu_2 r_1^2) > r_1^2 r_2^2 \quad ?$$

$$(r_1\mu_1 + r_2\mu_2)(r_1\mu_1 + r_2\mu_2)(\mu_1 r_2^2 + \mu_2 r_1^2) > r_1^2 r_2^2 \quad ?$$

Let $r_2/r_1 = x$. Then

$$(\mu_1 + x\mu_2)(\mu_1 + x\mu_2)(\mu_1 + \mu_2 x^{-2}) > 1 \quad ?$$

This is just not very obvious!

$$(\mu_1 + x[1-\mu_1])(\mu_1 + x[1-\mu_1])(\mu_1 + [1-\mu_1]x^{-2}) > 1 \ ?$$

Now only two variables μ_1 in (0,1) and x in $(0,\infty)$. Write $\mu_1 = y$ to simplify

$$(y + x[1-y])(y + x[1-y])(y + [1-y]x^{-2}) > 1 \ ?$$

I bet it is not true!

$$(y + x[1-y])^2(y + [1-y]x^{-2}) > 1 \quad ?$$

Plot this in Maple. Maple shows that it is really true! Wow.

$$(y + x[1-y])^2(y + [1-y]x^{-2}) > 1 \quad ?$$

$$(y + x[1-y])^2(x^2y + [1-y]) > x^2 \quad ?$$

$$(y + x[1-y])^2(x^2y + [1-y]) - x^2 > 0 \ ?$$

Maple can expand then factor this thing

$$y(x-1)^2(y-1)(x^2y-y-x^2-2x) > 0 \quad ? \quad 0 < y < 1$$

$$y(x-1)^2(1-y)(-x^2y+y+x^2+2x) > 0 \quad ? \quad 0 < y < 1$$

$$+ \quad + \quad + \quad ?$$

I would be done if I could show that

$$-x^2y+y+x^2+2x > 0$$

$x^2(y-1) + x^2 + 2x > 0$?

The *smallest* this can be occurs with y = 0. At that point the question is

$-x^2 + x^2 + 2x > 0$?

$2x > 0$?

Yes, since $0 < x < \infty$, this is always true. Thus I have shown that

$$(\mu_1 + x[1-\mu_1]) (\mu_1 + x[1-\mu_1])(\mu_1 + [1-\mu_1]x^{-2}) > 1$$

and therefore

$$\frac{m_1+m_2}{(m_1/r_1^2) + (m_2/r_2^2)} < \left(\frac{r_1m_1 + r_2m_2}{m_1+m_2} \right)^2$$

and therefore

$$R_{cog} < R_{cms}$$

as I hoped would be the case. This confirms that tether picture.

Question: How do you interpret the COG point in terms of an orbit?

You have a 1-mass M system orbiting at radius r in circular orbit. You replace it with a 2-mass system of the same total mass, and you do it in such a way that the 2-mass system is swapped in so its COG point lies on the same orbit of radius r. This system then has the same total force toward Earth center that the single mass had, so it has the same acceleration a, so it has the same orbit. Both masses of the 2-mass system have this same acceleration a toward earth center. There is tension in the stick.

Exercise. Suppose masses are equal. Then write

$$r_1 = r_0 + \varepsilon$$
$$r_2 = r_0 - \varepsilon \qquad r_0 = r_{cms}$$

If you use these in the r_{cog} equation you get

$$r_{cog} = \sqrt{\frac{2}{(1/r_1^2) + (1/r_2^2)}} = \sqrt{\frac{2}{\frac{1}{(r_0+\varepsilon)^2} + \frac{1}{(r_0-\varepsilon)^2}}} = \sqrt{\frac{2}{\frac{(r_0-\varepsilon)^2 + (r_0+\varepsilon)^2}{(r_0-\varepsilon)^2(r_0+\varepsilon)^2}}}$$

$$= (r_0-\varepsilon)(r_0+\varepsilon)\sqrt{\frac{2}{(r_0-\varepsilon)^2 + (r_0+\varepsilon)^2}} = (r_0^2 - \varepsilon^2)\sqrt{\frac{2}{2r_0^2 + 2\varepsilon^2}}$$

$$= (r_0^2 - \varepsilon^2)\,(r_0^2+\varepsilon^2)^{-1/2} = r_0^2\,(1 - (\varepsilon/r_0)^2)\,(1/r_0)\,(1 +(\varepsilon/r_0)^2)^{-1/2}$$

$$= r_0\ (1 - (\varepsilon/r_0)^2)\ (1 +(\varepsilon/r_0)^2)^{-1/2} \qquad (1+a)^n = 1 + na + \text{order}(a^2)$$

$$\approx r_0\ (1 - (\varepsilon/r_0)^2)\ (1 -1/2(\varepsilon/r_0)^2$$

$$\approx r_{cms}\,[\ 1 - (3/2)\,(\varepsilon/r_0)^2 + \text{order}\ (\varepsilon/r_0)^4\ ]$$

In the case that $\varepsilon \sim r_0/10$ say, you then have an estimate for the difference

$$r_{cog} \approx r_{cms}\,[\ 1 - (3/2)\,(\varepsilon/r_{cms})^2\ ]$$

which is accurate to an error on the order of 10^{-4} inside the bracket.

Appendix A

$$\mathbf{r} = \mathbf{h} + \mathbf{s} = h\hat{\mathbf{z}} + s\hat{\mathbf{s}} = h\hat{\mathbf{z}} + s\cos\varphi\sin\psi\,\hat{\mathbf{x}} + s\sin\varphi\sin\psi\,\hat{\mathbf{y}} + s\cos\psi\,\hat{\mathbf{z}}$$
$$= s\cos\varphi\sin\psi\,\hat{\mathbf{x}} + s\sin\varphi\sin\psi\,\hat{\mathbf{y}} + (s\cos\psi+h)\,\hat{\mathbf{z}}$$

Then

$$r^2 = [s\cos\varphi\sin\psi\,\hat{\mathbf{x}} + s\sin\varphi\sin\psi\,\hat{\mathbf{y}} + (s\cos\psi+h)\,\hat{\mathbf{z}}]^2$$

$$= s^2\cos^2\varphi\sin^2\psi + s^2\sin^2\varphi\sin^2\psi + (s\cos\psi+h)^2$$
$$+ 2s^2\sin\psi\sin\varphi\cos\varphi + 2ss\sin\psi\cos\varphi(s\cos\psi+h) + 2ss\sin\psi\sin\varphi(s\cos\psi+h)$$

$$= s^2\sin^2\psi + s^2\cos^2\psi + 2hs\cos\psi + h^2$$
$$+ 2s^2\sin\psi\sin\varphi\cos\varphi + 2s^2\sin\psi\cos\varphi\cos\psi + 2s^2\sin\psi\sin\varphi\cos\psi$$

$$+\ 2sh\sin\psi\cos\varphi + 2sh\sin\psi\sin\varphi$$

$$= s^2 + 2hs\cos\psi + h^2$$
$$+ 2s^2\ \sin\psi\ (\sin\varphi\cos\varphi + \cos\varphi\cos\psi + \sin\varphi\cos\psi)$$
$$+\ 2sh\ \sin\psi(\cos\varphi + \sin\varphi)$$

It is a real mess. But for specific numbers, I can write an expression for F and for $\hat{\mathbf{F}}$.
Assume then that we have these two quantities. We then want to do this: