Center of Gravity PhL 11.29.16

This document is fully reviewed, the main facts are in Section 0 Summary.

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0. Long Version Summary (written after the fact on 11.30.16).

Here (not thinking of gravity in particular) I consider a system of particles m_i having some $\mathbf{F}_i$ on each and I ask if there is some point in space **R** *relative to which* the system as a whole has zero torque. I set $\mathbf{F}_T = \Sigma_i \mathbf{F}_i$ and then I write $\mathbf{N}_T^{(R)} = \Sigma_i (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i$ and ask if **R** exists such that $\mathbf{N}_T^{(R)} = 0$. Note that $\mathbf{N}_T^{(0)} = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i$ about an Origin point. The problem is thus to find **R** such that $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$. In other words, is there some point **R** such that the *total* torque on the system (relative to the origin) and the *total* force on the system are related by an equation which has the traditional torque form $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$?

I thought surely there must always be a solution and I set out to find it. I wrote this out as a Cramer style linear matrix equation hoping to solve for the unknowns R_i, but I ran into the matrix having zero determinant so I could not solve for the solution R_i in the usual way. Dotting $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$ with $\mathbf{F}_T$ produces the equation $\mathbf{F}_T \bullet \mathbf{N}_T^{(0)} = 0$. If **R** exists such that $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$, then one must have $\mathbf{F}_T \bullet \mathbf{N}_T^{(0)} = 0$ so the latter is a *condition* without which there can be no solution **R**. Restating: for a problem having the fact $\mathbf{F}_T \bullet \mathbf{N}_T^{(0)} \neq 0$ there is no solution **R**. Notice that "gravity" has not yet been mentioned as a possible source of the forces $\mathbf{F}_i$ in this discussion. If a solution exists, some authors refer to the solution **R** as the center of gravity, but in this general scenario the forces $\mathbf{F}_i$ might have nothing to do with any "gravity". These authors then define their "**center of gravity**" **R** as a point *relative to which* $\mathbf{N}_T^{(R)} = 0$ so there is "zero torque" on the system relative to this point **R**.

We move now to an "asteroid near Earth" gravity application. Compute $\mathbf{F}_T = \Sigma_i F_i(-\hat{\mathbf{r}}_i)$ to get the total force on the asteroid as a vector. Note that $(-\hat{\mathbf{r}}_i)$ is a vector pointing from m_i of the asteroid to the center of the Earth, and $F_i = Gm_E m_i / r_i^2$. You then know the direction of $\hat{\mathbf{F}}_T$ in space. Draw a line defined by this direction. Translate this line until it passes through the center of the Earth. Then declare that $\mathbf{r}_{cog}$ lies along this line and for any point on this line we know that the vectors $\mathbf{r}_{cog}$ and $\mathbf{F}_T$ are parallel. Then set $Gm_E M / r_{cog}^2 = |F_T|$ and this determines r_{cog} for M = asteroid mass. So at point $\mathbf{r}_{cog}$ the total force on the asteroid is the same as the force on a concentrated point mass M at that point. This discussion appears in Symon.

Note that for this asteroid problem $\mathbf{N_T^{(0)}} = \Sigma_i \mathbf{r_i} \times \mathbf{F_i} = 0$ because each $\mathbf{r_i}$ is parallel to its $\mathbf{F_i}$, both pointing to Earth center as the Origin. The condition $\mathbf{F_T} \bullet \mathbf{N_T^{(0)}} = 0$ is therefore met, and thus solution(s) of $\mathbf{N_T^{(0)}} = \mathbf{R} \times \mathbf{F_T}$ may exist. One solution is $\mathbf{R} = 0$ since $\mathbf{N_T^{(R=0)}} = 0$. Had the determinant of the matrix not vanished, there could have been only one solution.

We claim that the point $\mathbf{r_{cog}}$ is another solution to $\mathbf{N_T^{(0)}} = \mathbf{R} \times \mathbf{F_T}$, so $\mathbf{N_T^{(0)}} = \mathbf{r_{cog}} \times \mathbf{F_T}$. How do we know this is so? Well we just stated above that $\mathbf{N_T^{(0)}} = 0$ *and* that $\mathbf{r_{cog}}$ is parallel to $\mathbf{F_T}$ so $\mathbf{r_{cog}} \times \mathbf{F_T} = 0$. Thus, using the point $\mathbf{R} = \mathbf{r_{cog}}$ the equation $\mathbf{N_T^{(0)}} = \mathbf{r_{cog}} \times \mathbf{F_T}$ is valid, since it just says $\mathbf{0} = \mathbf{0}$. Remember from above that *any* solution $\mathbf{R}$ to $\mathbf{N_T^{(0)}} = \mathbf{R} \times \mathbf{F_T}$ also solves $\mathbf{N_T^{(R)}} = 0$ and is thus a point relative to which the total torque on the system is 0. Therefore both $\mathbf{R} = 0$ and $\mathbf{R} = \mathbf{r_{cog}}$ are points relative to which the total torque on the system (the asteroid) is 0.

If we define "center of gravity" $\mathbf{R}$ in the general sense of "torque of system relative to $\mathbf{R}$ vanishes", then in the asteroid application we find that $\mathbf{r_{cog}}$ is such a "center of gravity". In this application the words of the phrase "center of gravity" make sense, but in the general definition they really don't make sense.

1. Short Version

Here I quickly get to the fact that there is a condition which must be satisfied for R to exist.

Suppose you have a system of masses m_i each of which has a certain force $\mathbf{F_i}$ acting on it. The total force on this system is then

$$\mathbf{F} = \Sigma_i \mathbf{F_i} .$$

If the masses are located at $\mathbf{r_i}$, then the total torque relative to some point $\mathbf{R}$ is given by

$$\mathbf{N^{(c)}} = \Sigma_i \, (\mathbf{r_i} - \mathbf{R}) \times \mathbf{F_i}$$

Can one find a location $\mathbf{R}$ for which $\mathbf{N^{(R)}} = \mathbf{0}$? That would require

$$\Sigma_i \, (\mathbf{r_i} - \mathbf{R}) \times \mathbf{F_i} = \mathbf{0}$$

That would require that

$$\Sigma_i \, \mathbf{r_i} \times \mathbf{F_i} = \mathbf{R} \times \Sigma_i \, \mathbf{F_i}$$

or simply

$$\mathbf{N^{(0)}} = \mathbf{R} \times \mathbf{F} .$$

Can you solve this for $\mathbf{R}$? I am not sure. But suppose we dot both sides with $\mathbf{F}$. We know that the object

$$\mathbf{R} \times \mathbf{F} \bullet \mathbf{F} = \mathbf{F} \times \mathbf{R} \bullet \mathbf{F} = \mathbf{F} \times \mathbf{F} \bullet \mathbf{R} = 0$$

Thus we end up with a *condition* which, if not true, means there can be no solution:

$$\mathbf{F} \bullet \mathbf{N}^{(0)} = 0$$

For some arbitrary forces and an arbitrary origin, this will not be true, so there will be no solution.

The equation $\mathbf{N}^{(0)} = \mathbf{R} \times \mathbf{F}$ asks this question: if you compute the total **F** and total torque $\boldsymbol{\tau}$ acting on a system, can you find a point **r** at which the equation $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ applies *at the system level*?

2. Corroboration

Here I finally found web and textbook support for my center of gravity concepts expressed above.

After much wasted searching, I found the above discussion corroborated at this site

https://en.wikipedia.org/wiki/Centers_of_gravity_in_non-uniform_fields

As the Talk page shows, there is much confusion regarding this topic!

Big Problem: The web search problem is that 95% of sites use the phrase "center of gravity" to be a synonym for the phrase "center of mass", so using the first as a search handle is ineffective. I found the above site by adding the phrase "non-uniform" to the search.

The wiki page has some references, and I am trying to track them down.

3. ^ Feynman, Leighton & Sands 1963, p. 19-3; Tipler & Mosca 2004, pp. 371–372; Pollard & Fletcher 2005; Rosen & Gothard 2009, pp. 75–76; Pytel & Kiusalaas 2010, pp. 442–443.

The **Feynman** Vol 1 refers to "center of gravity" 5 times, and notes there is a distinction. But almost all references in the three volumes are in the "center of mass" sense.

The **Tipler and Mosca** I have up. Here is their definition

12-2 The Center of Gravity

Figure 12-1*a* shows a rigid object in static equilibrium and a point *O*. We consider the object to be composed of many small particles. The weight of the *i*th small particle is $\vec{w}_i$, and the total weight of the object is $\vec{W} = \Sigma\vec{w}_i$. If $\vec{r}_i$ is the position vector of the *i*th particle relative to *O*, then $\vec{\tau}_i = \vec{r}_i \times \vec{w}_i$, where $\vec{\tau}_i$ the torque due to $\vec{w}_i$ about *O*. The net gravitational torque about *O* is then $\vec{\tau}_{net} = \Sigma(\vec{r}_i \times \vec{w}_i)$. Conveniently, the net torque due to gravity about any point can be calculated as if the entire weight $\vec{W}$ were applied at a single point, the center of gravity (see Figure 12-1*b*). That is,

$$\vec{\tau}_{net} = \vec{r}_{cg} \times \vec{W} \qquad 12\text{-}3$$

CENTER OF GRAVITY DEFINED

where $\vec{r}_{cg}$ is the position vector of the center of gravity relative to *O*.

In this paragraph, if one replaces the phrase "weight $\mathbf{w_i}$" with "force $\mathbf{F_i}$" and similarly **W** by $\mathbf{F_T}$, then you duplicate my Summary situation described above. However Tipler and Mosca make no comments about whether or not there is a solution. They might have added the condition as $\boldsymbol{\tau_{net}} \bullet \mathbf{W} = 0$.

The geology reference **Pollard and Fletcher** has nothing.

Rosen and Gothard Encyclopedia of Physical Science seems an unlikely source.

Pytel and Kiusalaas engineering mechanics Statics is in google books. I realized belatedly that this is the Statics book I picked up for $1 at the last library sale! They claim this

a. *Center of gravity*

The weight of a body is the most common example of a distributed force. For a body occupying the region $\mathcal{V}$, as in Fig. 8.13 , the weight of a differential volume element dV is $dW = \gamma\, dV$, where γ is the weight density (weight per unit volume). The total weight W of the body is thus the resultant of an infinite number of parallel forces dW; that is,

$$W = \int_{\mathcal{V}} dW = \int_{\mathcal{V}} \gamma\, dV$$

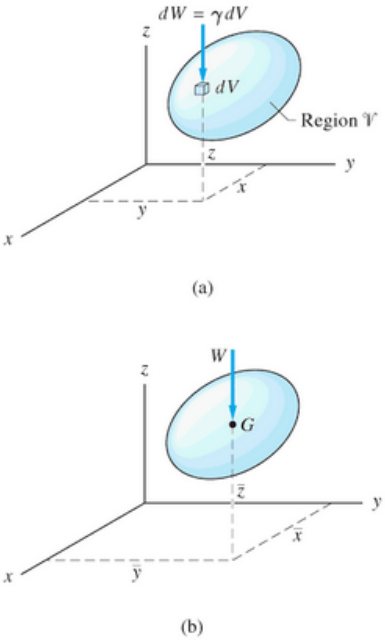
The point G through which the weight W acts is called the *center of weight*—or, more commonly—the *center of gravity of the body*—see Fig. 8.13(b).

The coordinates of G can be determined by equating the resultant moment of the distributed weight to the moment of W about the coordinate axes. Referring to Fig. 8.13(a) and (b), we obtain

$$\Sigma M_x = -W\bar{y} = -\int_{\mathcal{V}} y\, dW \qquad \Sigma M_y = W\bar{x} = \int_{\mathcal{V}} x\, dW$$

from which we obtain

$$\bar{x} = \frac{\int_{\mathcal{V}} x\, dW}{\int_{\mathcal{V}} dW} = \frac{\int_{\mathcal{V}} x\gamma\, dV}{\int_{\mathcal{V}} \gamma\, dV} \qquad \text{(8.16a)}$$



Again, if I replace the "weight = d**W** = **γ** dV (for me, both **W** and **γ** are then vectors)" with "force = $\mathbf{F_i}$" then things make sense. But they seem to have W and φ being scalars which rules out a non-uniform **g** field. For the non-uniform situation, their integrals don't make sense. They are just equations for finding the center of mass of an object.

Now back to our initial web page. They say

"Textbooks such as the *The Feynman Lectures on Physics* characterize the center of gravity as a point about which there is no torque. "

But I have now downloaded all three volumes of the lectures and have made them searchable, and none of them mentions "center of gravity" anywhere.

The fact that the equation might not have a solution is linked to

Symon, Keith R. (1971), Mechanics, Addison-Wesley, ISBN 978-0-201-07392-8

I have this book now (and added OCR). On page 258 there is a reasonable discussion of "center of gravity" which seems to have nothing to do with any torques. Here is what it says:

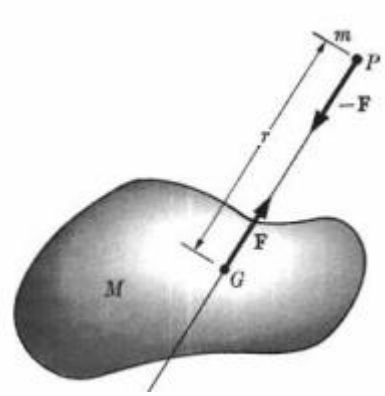
Symon. You have a distribution of masses m_i which forms some weird body of total mass M. You are observing this body from an external point test mass m located at a point $\mathbf{r}_0$. You then compute total **F** acting on your test mass m as follows,

$\mathbf{F} = \Sigma_i (m_i m G / r_{i0}^2) \hat{\mathbf{r}}_{i0}$ where $\mathbf{r}_{i0} \equiv \mathbf{r}_i - \mathbf{r}_0$, points to $\mathbf{r}_i$.

or

$\mathbf{F} = mG \Sigma_i (m_i / r_{i0}^2) \hat{\mathbf{r}}_{i0} = F \hat{\mathbf{F}}$

This is a completely well defined sum (or integral) and you end up with some **F**. This is along some line in the $\hat{\mathbf{F}}$ direction. You then need to translate this line parallel to itself until it passes through the origin of the Earth. Then the COG lies on this line (he claims).



You then define the "center of gravity point G" to be on this line determined by $\hat{\mathbf{F}}$ which is a distance s away from test mass m *such that*

$F = (MmG/s^2)$

So this is something I can actually calculate! The gravitational force on the test mass due to the distributed object is the same as if you put the entire object of mass M into a single point of mass M located at point G. As he notes, if you move your test mass to another point P', G goes to a new point G', so the point G is not like the center of mass which is a property only of the object.

Let's now apply this to the tethered satellite treating the entire earth as the test mass which acts as a point located at its center. We then compute

$\mathbf{F} = m_E G \Sigma_{i=1}^{2} (m_i / r_{i0}^2) \hat{\mathbf{r}}_{i0}$

Separate doc!

Taylor's mechanics book does not include the phrase "center of gravity".

Marion and Thornton have exactly one reference to "center of gravity" and it is in a problem:

9-5. The center of gravity of a system of particles is the point about which external gravitational forces exert no net torque. For a uniform gravitational force, show that the center of gravity is identical to the center of mass for the system of particles.

In other words, the total torque on the particles *with respect to* the center of gravity point is 0. I have the same definition of center of gravity above, but generalized for any forces, and I have then solved this problem below in Section 6.

Landau and Lifshitz has no hits on center of gravity.

Osgood has no hits on center of gravity.

Synge and Griffith have lots of hits, but they are all just in the CMS sense.

I looked earlier at Jim's mechanics books and found nothing useful.

3. Application

This section goes nowhere.

Consider a very large rigid body composed of masses m_i in orbit. Each mass has internal $\mathbf{F}_{ij}$ and also has $\mathbf{G}_i$ due to gravity. For this problem we have

$$\mathbf{F}_i = \mathbf{G}_i + \Sigma_{j \neq i} \mathbf{F}_{ij}$$

The torque about some origin is then

$$\mathbf{N}^{(0)} = \Sigma_i \, \mathbf{r}_i \times \mathbf{F}_i = \Sigma_i \, \mathbf{r}_i \times [\mathbf{G}_i + \Sigma_{j \neq i} \mathbf{F}_{ij}]$$

In order for this system to have a "center of gravity" **R** about which total torque is 0, the above condition must be met. That condition is

$$\mathbf{F} \bullet \mathbf{N}^{(0)} = 0$$

or

$$\Sigma_i [\mathbf{G}_i + \Sigma_{j \neq i} \mathbf{F}_{ij}] \bullet \Sigma_k [\mathbf{r}_k \times [\mathbf{G}_k + \Sigma_{j \neq i} \mathbf{F}_{kj}]] = 0$$

or

$$\Sigma_{i,k} [\mathbf{G}_i + \Sigma_{j \neq i} \mathbf{F}_{ij}] \bullet [\mathbf{r}_k \times (\mathbf{G}_k + \Sigma_{j \neq i} \mathbf{F}_{kj})] = 0 \qquad \text{// I know } \Sigma_j \Sigma_{j \neq i} \mathbf{F}_{kj} = 0$$

This looks pretty messy, I don't know what to do.

Conclusion: I have no idea what "center of gravity" means. I have no idea how to compute it, and I it seems likely that it does not even exist. And the entire web has nothing to say about it! [well, I have since learned a bit, see Summary above.]

4. Long Version

Suppose you have a system of masses m_i each of which has a certain force $\mathbf{F}_i$ acting on it. The total force on this system is then

$$\mathbf{F_T} = \Sigma_i \mathbf{F_i} \ .$$

If the masses are located at $\mathbf{r_i}$, then the total torque relative to some point **R** is given by

$$\mathbf{N_T^{(R)}} = \Sigma_i \ (\mathbf{r_i}-\mathbf{R}) \times \mathbf{F_i}$$

Can one find a location $\mathbf{R_0}$ where $\mathbf{N_T^{(R0)}} = \mathbf{0}$? That would require

$$\Sigma_i \ (\mathbf{r_i}-\mathbf{R_0}) \times \mathbf{F_i} \ = 0 \ .$$

Each component must be zero, so then this says

$$\Sigma_i \ [(\mathbf{r_i}-\mathbf{R_0}) \times \mathbf{F_i}]_j = 0 \qquad\qquad j = 1,2,3$$

or

$$\Sigma_i \ \varepsilon_{jab}(\mathbf{r_i}-\mathbf{R_0})_a \ (\mathbf{F_i})_b = 0$$

or

$$\varepsilon_{jab} \ \Sigma_i(\mathbf{r_i}-\mathbf{R_0})_a \ (\mathbf{F_i})_b = 0 \qquad\qquad j = 1,2,3$$

or

$$\varepsilon_{jab} \ \Sigma_i(\mathbf{R_0})_a \ (\mathbf{F_i})_b = \ \varepsilon_{jab} \ \Sigma_i(\mathbf{r_i})_a \ (\mathbf{F_i})_b$$

Write out these three equations

$j = 1$ $\quad \Sigma_i(\mathbf{R_0})_2 \ (\mathbf{F_i})_3 - \Sigma_i(\mathbf{R_0})_3 \ (\mathbf{F_i})_2 \ = \ \Sigma_i(\mathbf{r_i})_2 \ (\mathbf{F_i})_3 - \Sigma_i(\mathbf{r_i})_3 \ (\mathbf{F_i})_2$ and cyclic

Move sum on the left

$$(\mathbf{R_0})_2\Sigma_i \ (\mathbf{F_i})_3 - (\mathbf{R_0})_3\Sigma_i \ (\mathbf{F_i})_2 = \Sigma_i(\mathbf{r_i})_2 \ (\mathbf{F_i})_3 - \Sigma_i(\mathbf{r_i})_3 \ (\mathbf{F_i})_2 \ + \text{cyclic}$$

$$[\Sigma_i \ (\mathbf{F_i})_3](\mathbf{R_0})_2 - [\Sigma_i \ (\mathbf{F_i})_2] \ (\mathbf{R_0})_3 = \Sigma_i(\mathbf{r_i})_2 \ (\mathbf{F_i})_3 - \Sigma_i(\mathbf{r_i})_3 \ (\mathbf{F_i})_2$$

Now write out all three equations

$$[\Sigma_i \ (\mathbf{F_i})_3](\mathbf{R_0})_2 - [\Sigma_i \ (\mathbf{F_i})_2] \ (\mathbf{R_0})_3 = \Sigma_i(\mathbf{r_i})_2 \ (\mathbf{F_i})_3 - \Sigma_i(\mathbf{r_i})_3 \ (\mathbf{F_i})_2$$
$$[\Sigma_i \ (\mathbf{F_i})_1](\mathbf{R_0})_3 - [\Sigma_i \ (\mathbf{F_i})_3] \ (\mathbf{R_0})_1 = \Sigma_i(\mathbf{r_i})_3 \ (\mathbf{F_i})_1 - \Sigma_i(\mathbf{r_i})_1 \ (\mathbf{F_i})_3$$
$$[\Sigma_i \ (\mathbf{F_i})_2](\mathbf{R_0})_3 - [\Sigma_i \ (\mathbf{F_i})_3] \ (\mathbf{R_0})_2 = \Sigma_i(\mathbf{r_i})_1 \ (\mathbf{F_i})_2 - \Sigma_i(\mathbf{r_i})_2 \ (\mathbf{F_i})_1$$

This has the form

$$M \ \mathbf{R_0} = \mathbf{V}$$

which I guess you just solve for $\mathbf{R_0}$. This is a Cramer's Rule simple matrix inversion linear problem.

To simplify the notation, let's make up some symbols

$[\Sigma_i (\mathbf{F}_i)_3] = a_3$ so $[\Sigma_i (\mathbf{F}_i)_j] = a_j$

$(\mathbf{R}_0)_3 = x_3$ so $(\mathbf{R}_0)_j = x_j$

$\Sigma_i(\mathbf{r}_i)_2 (\mathbf{F}_i)_3 = A_{23}$ so $\Sigma_i(\mathbf{r}_i)_a (\mathbf{F}_i)_b = A_{ab}$

Then our three equations are

$a_3x_2 - a_2x_3 = A_{23} - A_{32}$
$a_1x_3 - a_3x_1 = A_{31} - A_{13}$
$a_2x_1 - a_1x_2 = A_{12} - A_{21}$

or

$0\,x_1 + a_3x_2 - a_2x_3 = A_{23} - A_{32}$
$-a_3x_1 + 0\,x_2 + a_1x_3 = A_{31} - A_{13}$
$a_2x_1 - a_1x_2 + 0\,x_3 = A_{12} - A_{21}$

or

$$\begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A_{23} - A_{32} \\ A_{31} - A_{13} \\ A_{12} - A_{21} \end{pmatrix}$$

But Maple says this matrix has determinant = 0 so has no inverse. So how solve the problem? The determinant of any 3x3 antisymmetric matrix is 0, easy to show on scratch. The three rows of the matrix are not linearly independent. Fine.

Go back to

$a_3x_2 - a_2x_3 = V_1$
$a_1x_3 - a_3x_1 = V_2$
$a_2x_1 - a_1x_2 = V_3$

Bruce force: eliminate x_3 from the first pair

$a_1a_3x_2 - a_1a_2x_3 = a_1V_1$
$a_2a_1x_3 - a_2a_3x_1 = a_2V_2$

$a_1a_3x_2 - a_2a_3x_1 = a_1V_1 + a_2V_2 = B_1$

Then we have two equations left

$a_2x_1 - a_1x_2 = V_3$

$a_2x_1 - a_1x_2 = -B_1/a_3$

The right sides must be the same to make this work, so need

$a_3V_3 = -B_1$

or

$a_3V_3 = -(a_1V_1 + a_2V_2)$

or

$\mathbf{a} \bullet \mathbf{V} = 0$

Any vector normal to V is a solution. How could I have obtained this result right away?

Start over:

$\Sigma_i (\mathbf{r}_i - \mathbf{R}_0) \times \mathbf{F}_i = 0$? // is there a solution $\mathbf{R}_0$?

or

$(\Sigma_i \mathbf{r}_i \times \mathbf{F}_i) = \Sigma_i \mathbf{R}_0 \times \mathbf{F}_i = \mathbf{R}_0 \times (\Sigma_i \mathbf{F}_i)$

Define

$\mathbf{r}_i \times \mathbf{F}_i = \mathbf{N}_i^{(0)}$ and $(\Sigma_i \mathbf{F}_i) = \mathbf{F}$

Then have

$\Sigma_i \mathbf{N}_i^{(0)} = \mathbf{N}_{Tot}^{(0)} = \mathbf{R}_0 \times \mathbf{F}$

Want to find solution R_0 to this equation

$\mathbf{R}_0 \times \mathbf{F} = \mathbf{N}_{Tot}^{(0)}$

where $\mathbf{F}$ and $\mathbf{N}_{Tot}^{(0)}$ are known.

General problem:

$\mathbf{r} \times \mathbf{a} = \mathbf{b}$ how solve for $\mathbf{r}$?

Dot both sides with **a**? But LHS = 0 so you are forced to $\mathbf{a} \bullet \mathbf{b} = 0$ as a condition for solution? In my problem that means

$\mathbf{F} \bullet \mathbf{N}_{Tot}^{(0)} = 0$

Conclusion: You must have the total force on system being normal to total torque about your selected origin, otherwise there is no solution. If total force = 0 or if total torque = 0 you are OK. Nothing is said about inertial frames here. I have used not equations of motion.

In the tether problem all forces are radial so the total torque is 0. Then $\mathbf{R}_0 \times \mathbf{F} = 0$. But any $\mathbf{R}_0$ parallel to **F** would solve this equation. It is true that any such R_0 would yield 0 total torque.

5. The center of mass point

Here I obtain the usual $\mathbf{r}_{cms}$ formula using the above COG definition of $\mathbf{r}_{cms}$.

We now have a uniform **g** field. We have a system of masses m_i in some configuration at rest. It is a rigid body with massless sticks connecting the masses. Total gravity force is $\mathbf{F} = \Sigma_i m_i \mathbf{g} = M\mathbf{g}$. The object is supported by a thread pulling up with **-F** on point some point $\mathbf{r}_{cms}$. Total force on the object is then $\mathbf{F}_T = \Sigma_i \mathbf{F}_i = 0$. Let's define $\mathbf{r}_{cms}$ as a point about which total torque $\mathbf{N}^{(rcms)}$ is 0, assuming such a point exists. The thread force makes no torque since its **r** and **F** align, so we have only the gravity forces to deal with. So we want to have

$$\mathbf{N}^{(rcms)} = \Sigma_i (\mathbf{r}_i - \mathbf{r}_{cms}) \times \mathbf{F}_i = 0$$

where the sum can be regarded as only over the grav forces. Write these as $\mathbf{F}_i = m_i\mathbf{g}$ so then want

$$\Sigma_i (\mathbf{r}_i - \mathbf{r}_{cms}) \times m_i\mathbf{g} = 0$$

or

$$[\Sigma_i (m_i\mathbf{r}_i - m_i\mathbf{r}_{cms})] \times \mathbf{g} = 0$$

Thus we need to have

$$\Sigma_i (m_i\mathbf{r}_i - m_i\mathbf{r}_{cms}) = 0$$

or

$$\Sigma_i m_i\mathbf{r}_i = \mathbf{r}_{cms} \Sigma_i m_i$$

To make this work, we must take

$$\mathbf{r}_{cms} = \frac{\Sigma_i m_i\mathbf{r}_i}{\Sigma_i m_i}$$

which is the usual result. If we take this value as $\mathbf{r}_{cms}$ when we will have

$$\mathbf{N}^{(rcms)} = \Sigma_i (\mathbf{r}_i - \mathbf{r}_{cms}) \times \mathbf{F}_i = 0$$

where the sum can be considered now to be over all forces including the thread force. With this full sum we have $\Sigma_i \mathbf{F}_i = 0$.

Digression Theorem: if $\Sigma_i \mathbf{F}_i = 0$, then torque about all reference points is the same.

Proof: $\mathbf{N}^{R1} = \Sigma_i (r_i - \mathbf{R}_1) \times \mathbf{F}_i$

$\mathbf{N}^{R2} = \Sigma_i (r_i - \mathbf{R}_2) \times \mathbf{F}_i \quad \Rightarrow$

$$(\mathbf{N}^{R1} - \mathbf{N}^{R2}) = \Sigma_i(r_i - \mathbf{R}_1) \times \mathbf{F}_i - \Sigma_i(r_i - \mathbf{R}_2) \times \mathbf{F}_i$$
$$= (-\mathbf{R}_1 - \mathbf{R}_2) \times (\Sigma_i \mathbf{F}_i) = (-\mathbf{R}_1 - \mathbf{R}_2) \times 0 = 0 \qquad \text{QED}$$

Thus in our rigid body example, since $\Sigma_i \mathbf{F}_i = 0$ we know that $\mathbf{N}^{(R)} = 0$ referenced to any point **R**, and not just to the $\mathbf{r}_{cms}$ point. So if you "hang" ("support") a rigid body at its $\mathbf{r}_{cms}$ point, the total torque on the rigid body about any reference point is zero.

6. Show that for the on-earth small situation, $r_{cog} = r_{cms}$.

It is very difficult to show this using the Symon approach due to the large size difference between the long $\mathbf{r}_i$ and the local surface $\mathbf{R}_i$ points in the rigid body. I think you could do it, but you have to work hard with approximations. It is easier to use the first and official definition of the center of gravity, namely, that it is the point **R** for which you solve $\mathbf{N}_T^{(0)} = \mathbf{R} \times \mathbf{F}_T$ (meaning it is the point with reference to which the total torque $\mathbf{N}_T^{(R)} = 0$) .

So here is the argument:

Recall that $\mathbf{r}_{cog}$ is defined as the solution of $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}_T$ or $\mathbf{N}^{(rcog)} = 0$ for a general set of forces F_i.

Recall that $\mathbf{r}_{cms}$ is defined as the solution of $\mathbf{N}^{(0)} = \mathbf{r}_{cms} \times \mathbf{F}_T$ or $\mathbf{N}^{(rcms)} = 0$ but restricted to a situation where all the forces are parallel.

In both cases we can take $\mathbf{F}_T$ to be the sum over the m_i forces $\mathbf{F}_i$ and we don't include "the string" force since it makes no contribution to $\mathbf{r}_{cms} \times \mathbf{F}_T$. For $\mathbf{r}_{cog}$ we had in mind very general forces $\mathbf{F}_i$ which result in general torques $\mathbf{r}_i \times \mathbf{F}_i$ about the origin. For $\mathbf{r}_{cms}$ we had in mind just small objects on the earth where all the forces $\mathbf{F}_i$ are parallel. So the cms *problem* is just a special case of the cog *problem*. The solution to the cms problem gives the cog solution for that particular problem.