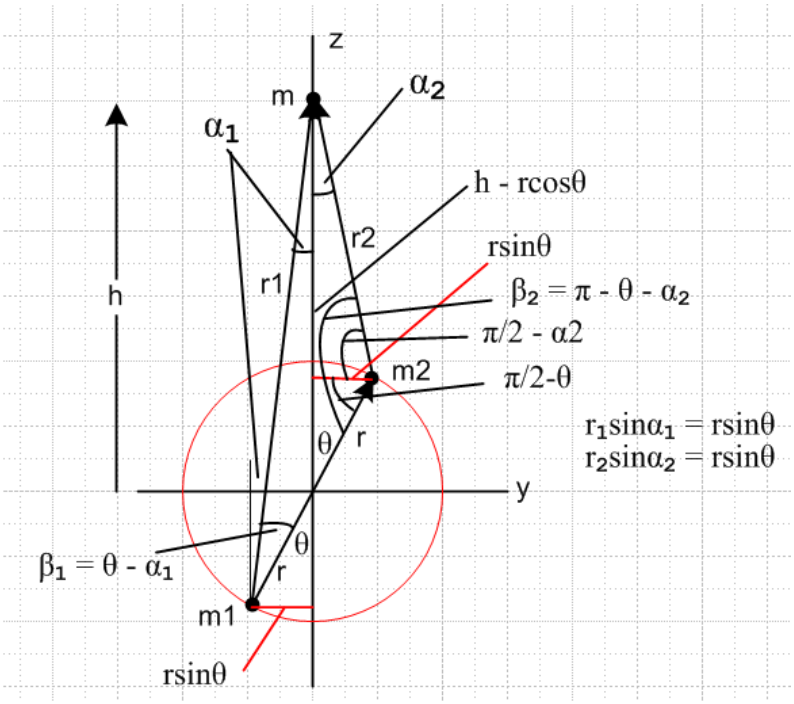
Center of gravity PhL 12.2.16

This subject just won't go away. I said in frames that the cog point is always closer to the point gravity source than the cms point, but I suspect for a sphere that is not true. Also, I see a better way to set up the 2-mass problem so I will redo that problem right here. I draw the two masses and then instrument the picture with everything I can find



$r_1\sin\alpha_1 = r\sin\theta$ $\beta_1 = \theta - \alpha_1$ $r_1\cos\alpha_1 = h + r\cos\theta$

$r_2\sin\alpha_2 = r\sin\theta$ $\beta_2 = \pi/2 - \alpha_2$ $r_2\cos\alpha_2 = h - r\cos\theta$

$\tan\alpha_1 = r\sin\theta/(h+r\cos\theta)$

$\tan\alpha_2 = r\sin\theta/(h-r\cos\theta)$

Given h,r,θ we therefore know the angles α_1 and α_2 and note that $0 < \alpha_i < \pi/2$ as long as h > r. Then r_1 and r_2 are determined by

$r_1 = r\sin\theta/\sin\alpha_1$ $r_1^2 = (r\sin\theta)^2 + (h+r\cos\theta)^2$

$r_2 = r\sin\theta/\sin\alpha_2$ $r_2^2 = (r\sin\theta)^2 + (h-r\cos\theta)^2$

Note: The origin is not at the center of mass location.

What is the *force on mass m*?

$F_y = -(Gmm_1/r_1^2)\sin\alpha_1 + (Gmm_2/r_2^2)\sin\alpha_2$

$F_z = -(Gmm_1/r_1^2)\cos\alpha_1 - (Gmm_2/r_2^2)\cos\alpha_2$

or

$F_y = -(Gmm_1/r_1^3)\, r_1 \sin\alpha_1 + (Gmm_2/r_2^3)\, r_2 \sin\alpha_2$

$F_z = -(Gmm_1/r_1^3) r_1 \cos\alpha_1 - (Gmm_2/r_2^3) r_2 \cos\alpha_2$

or

$F_y = -(Gmm_1/r_1^3)\, r\sin\theta + (Gmm_2/r_2^3)\, r\sin\theta$

$F_z = -(Gmm_1/r_1^3)(h + r\cos\theta) - (Gmm_2/r_2^3)(h - r\cos\theta)$

or

$F_y = [Gm/(r_1^3 r_2^3)] * r\sin\theta\,(m_2 r_1^3 - m_1 r_2^3)$ to right of $m_2 >> m_1$

$-F_z = [Gm/(r_1^3 r_2^3)] * [m_1 r_2^3 (h + r\cos\theta) + m_2 r_1^3 (h - r\cos\theta)]$ down

The total force acting on m is then

$\mathbf{F} = F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}} = F\,\hat{\mathbf{F}}$ = a generally downward force on mass m.

$$F = \sqrt{F_y^2 + F_z^2} \qquad \hat{\mathbf{F}} = \mathbf{F}/F = \frac{F_y}{\sqrt{F_y^2+F_z^2}}\,\hat{\mathbf{y}} + \frac{F_z}{\sqrt{F_y^2+F_z^2}}\,\hat{\mathbf{z}}$$

If $m_2 >> m_1$ then the attractive force on m is down and to the right. In this case $F_y < 0$, $F_z < 0$ so $\mathbf{F}$ and $\hat{\mathbf{F}}$ point up and to the left.

We then go a distance R along this direction away from mass m and determine R from

$Gm(m_1+m_2)/r_{cog}^2 = F$

$r_{cog}^2 = Gm(m_1+m_2)/F$

so

$$r_{cog} = \sqrt{\frac{Gm(m_1+m_2)}{F}} = \sqrt{Gm(m_1+m_2)}\; F^{-1/2}$$

Then relative to the location of mass m we have

$$\mathbf{r}_{cog} = r_{cog}\hat{\mathbf{F}} = (r_{cog}/F)\,\mathbf{F} = \sqrt{Gm(m_1+m_2)}\; F^{-3/2}\,[\,F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}\,]$$

which is a vector generally downward from the mass m.

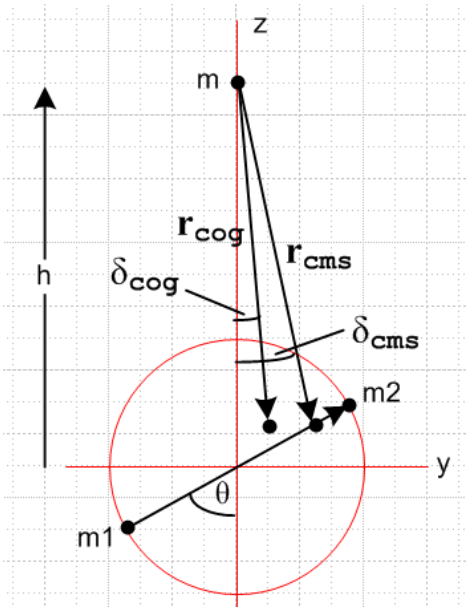
Meanwhile, the center of mass (relative to the origin) is located at,

$$\mathbf{R}_{cms} = (m_2\mathbf{r} + m_1(-\mathbf{r}))/(m_1+m_2) = \frac{m_2 - m_1}{m_1+m_2}\,\mathbf{r} \equiv f\,\mathbf{r} \qquad f = \frac{m_2 - m_1}{m_1+m_2}\ .$$

Then a vector from mass m down to this center of mass location is,

$$\mathbf{r}_{cms} = -h\hat{\mathbf{z}} + \mathbf{R}_{cms} = -h\hat{\mathbf{z}} + f\mathbf{r} = -h\hat{\mathbf{z}} + fy\hat{\mathbf{y}} + fz\hat{\mathbf{z}} = fr\sin\theta\,\hat{\mathbf{y}} - (h - fr\cos\theta)\hat{\mathbf{z}}$$

We now define two angles away from vertical as in this picture,



The drawing is appropriate for $m_2 >> m_1$ so $\mathbf{r}_{cms}$ lies on the line segment between the masses, but toward the m_2 end. We are not sure at this point where $\mathbf{r}_{cog}$ lies, so we show it at some arbitrary point.

Then the angles δ_{cog} and δ_{cms} are defined shown in the drawing, both positive as drawn.

$$\tan(\delta_{cog}(\theta)) = r_{cog,y} / (-r_{cog,z})$$
$$\tan(\delta_{rms}(\theta)) = r_{cms,y} / (-r_{cms,z})$$

Then

$$\tan(\delta_{cog}(\theta)) = r_{cog,y} / (-r_{cog,z}) = F_y/(-F_z) = \frac{r\sin\theta\,(m_2 r_1^3 - m_1 r_2^3)}{[m_1 r_2^3(h + r\cos\theta) + m_2 r_1^3(h - r\cos\theta)]}$$

$$\tan(\delta_{rms}(\theta)) = r_{cms,y} / (-r_{cms,z}) = \frac{fr\sin\theta}{h - fr\cos\theta} = \frac{\frac{m_2 - m_1}{m_1 + m_2} r\sin\theta}{h - \frac{m_2 - m_1}{m_1 + m_2} r\cos\theta} = \frac{(m_2-m_1)r\sin\theta}{(m_1+m_2)h-(m_2-m_1)r\cos\theta}$$

We now compare:

$$\tan(\delta_{cog}(\theta)) = \frac{r\sin\theta\,(m_2 r_1^3 - m_1 r_2^3)}{[m_1 r_2^3(h + r\cos\theta) + m_2 r_1^3(h - r\cos\theta)]}$$

$$\tan(\delta_{cms}(\theta)) = \frac{(m_2-m_1)r\sin\theta}{(m_1+m_2)h-(m_2-m_1)r\cos\theta}$$

$$\begin{cases} r_1^2 = (r\sin\theta)^2 + (h+r\cos\theta)^2 \\ r_2^2 = (r\sin\theta)^2 + (h-r\cos\theta)^2 \end{cases}$$

As noted above, if $m_2 >> m_1$ (and $h >> r$) then both δ angles are positive, in agreement with the drawing.

In general, these angles are *not the same*, being different dimensionless functions of m_1,m_2,r,h and θ . So in general, the line from m to the center of gravity is different from the line from m to the center of mass. And then of course the points $\mathbf{r}_{cog}$ and $\mathbf{r}_{cms}$ are also different.

If θ = 0 then the lines are the same because then $\delta_{cog}(0) = \delta_{cms}(0) = 0$. In this case $\mathbf{r}_{cog}$ and $\mathbf{r}_{cms}$ are different but they both lie on the z axis.

Equal mass for general θ. For general θ if $m_1 = m_2$ we find

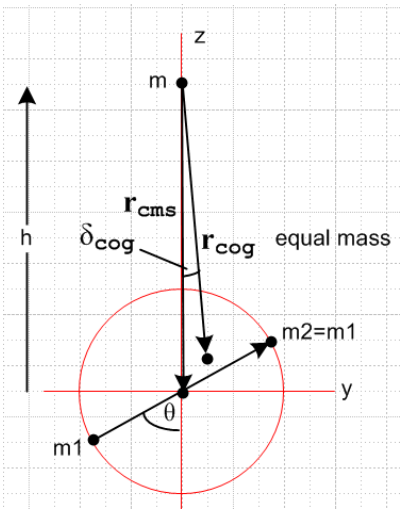
$$\tan(\delta_{cog}(\theta)) = \frac{r\sin\theta\,(r_1^3 - r_2^3)}{[r_2^3(h + r\cos\theta) + r_1^3(h - r\cos\theta)]} \qquad \begin{cases} r_1^2 = (r\sin\theta)^2 + (h+r\cos\theta)^2 \\ r_2^2 = (r\sin\theta)^2 + (h-r\cos\theta)^2 \end{cases}$$

$$\tan(\delta_{cms}(\theta)) = 0$$

and in this case $\mathbf{r}_{cms}$ points to the origin while $\mathbf{r}_{cog}$ generally does not. So even in the equal mass case the two lines are different for general θ. In this equal mass case we have from above,

$$\mathbf{r}_{cog} = \sqrt{Gm(m_1+m_2)}\; F^{-3/2}\,[\; F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}\;]$$

$$F_y = [Gmm_1/(r_1^3r_2^3)] * r\sin\theta\,(r_1^3 - r_2^3)$$
$$-F_z = [Gmm_1/(r_1^3r_2^3)] * [r_2^3(h + r\cos\theta) + r_1^3(h - r\cos\theta)]$$



Unequal mass, but θ = 0.

In this case

$$F_y = 0$$
$$-F_z = [Gm/(r_1^3r_2^3)] * [m_1r_2^3(h + r) + m_2r_1^3(h - r)]$$

But also

$r_1 = h+r$

$r_2 = h-r$

so

$$-F_z = \frac{Gm}{r_1^3 r_2^3}\ [m_1 r_2^3 (h + r)\ +\ m_2 r_1^3 (h - r)]$$

$$= \frac{Gm}{(h+r)^3 (h-r)^3}\ [m_1 (h-r)^3 (h + r)\ +\ m_2 (h+r)^3 (h - r)]$$

$$= \frac{Gm(h+r)(h-r)}{(h+r)^3 (h-r)^3}\ [m_1 (h-r)^2\ +\ m_2 (h+r)^2]$$

$$= \frac{Gm}{(h+r)^2 (h-r)^2}\ [m_1 (h-r)^2\ +\ m_2 (h+r)^2]$$

$$\mathbf{r}_{cog} = \sqrt{Gm(m_1+m_2)}\ |F_z|^{-3/2}\ [\ F_z \hat{\mathbf{z}}\ ]$$

$$= \sqrt{Gm(m_1+m_2)}\ |F_z|^{-3/2}\ [\ -|F_z| \hat{\mathbf{z}}\ ]$$

$$\mathbf{r}_{cog} = -\ \sqrt{Gm(m_1+m_2)}\ |F_z|^{-1/2}\ \hat{\mathbf{z}}$$

$$|F_z| = \frac{Gm}{(h+r)^2 (h-r)^2}\ [m_1 (h-r)^2\ +\ m_2 (h+r)^2]$$

$$|F_z|^{-1/2} = \frac{h^2-r^2}{\sqrt{Gm}}\ \frac{1}{\sqrt{m_1 (h-r)^2\ +\ m_2 (h+r)^2}}$$

$$\mathbf{r}_{cog} = -\sqrt{Gm(m_1+m_2)}\ \frac{h^2-r^2}{\sqrt{Gm}}\ \frac{1}{\sqrt{m_1 (h-r)^2\ +\ m_2 (h+r)^2}}\ \hat{\mathbf{z}}$$

$$= -\frac{(h^2-r^2)\sqrt{m_1+m_2}}{\sqrt{m_1 (h-r)^2\ +\ m_2 (h+r)^2}}\ \hat{\mathbf{z}}$$

$$r_{cog} = \frac{(h^2-r^2)\sqrt{m_1+m_2}}{\sqrt{m_1 (h-r)^2\ +\ m_2 (h+r)^2}} = \frac{(h^2-r^2)}{\sqrt{\mu_1 (h-r)^2\ +\ \mu_2 (h+r)^2}}$$ // agrees with frames doc

Meanwhile,

$$\mathbf{r}_{cms} = \ frsin\theta\ \hat{\mathbf{y}} - (h - frcos\theta)\hat{\mathbf{z}} = -(h - fr)\hat{\mathbf{z}}$$

$r_{cms} = h - fr$ $\qquad f = \dfrac{m_2 - m_1}{m_1 + m_2} = \mu_2 - \mu_1 \qquad -1 < f < 1$

Now can we show that

$r_{cog} < r_{cms}$?

According to the tether picture the r_{cms} is farther from the earth, so that means is farther from mass m and that means $r_{cms} > r_{cog}$. So let's try to verify this tether doc claim.

$$\frac{(h^2-r^2)}{\sqrt{\mu_1(h-r)^2 + \mu_2(h+r)^2}} < h - (\mu_2 - \mu_1)r \quad ?$$

$$(h^2-r^2)^2 < [h - (\mu_2 - \mu_1)r]^2 [\mu_1(h-r)^2 + \mu_2(h+r)^2] \quad ?$$

$$(h^2-r^2)^2 < [h - (\mu_2 - \mu_1)r]^2 [(\mu_1+\mu_2)(h^2+r^2) + 2(\mu_2 - \mu_1)hr] \quad ?$$

$$(h^2-r^2)^2 < [h - \Delta\mu r]^2 [(h^2+r^2) + 2\Delta\mu hr] \quad ?$$

Let $\Delta\mu = x$, then

$RHS = [h - rx]^2 [(h^2+r^2) + 2hrx] \quad = f(x)$

I would like to figure out the MIN value of f(x) and then if I can show $(h^2-r^2)^2 < f_{min}$ then I have succeeded in showing $r_{cog} < r_{cms}$.

So let's have Maple study up on this function:

```
f := (h-r*x)^2*(h^2+r^2+2*h*r*x);
```

$$f := (h - r x)^2 (h^2 + r^2 + 2 h r x)$$

```
fp := diff(f,x);
```

$$fp := -2 (h - r x) (h^2 + r^2 + 2 h r x) r + 2 (h - r x)^2 h r$$

```
s := solve(fp,x);
```

$$s := \frac{h}{r}, -\frac{1}{3}\frac{r}{h}$$

I then test these two points to see if max or min:

```
fpp := diff(f,x,x);
```

$$fpp := 2r^2(h^2+r^2+2hrx)-8(h-rx)hr^2$$

```
x := s[1]; fpp; unassign('x'):
```

$$x := \frac{h}{r}$$

$$2r^2(3h^2+r^2)$$

```
x := s[2]; simplify(fpp); unassign('x'):
```

$$x := -\frac{1}{3}\frac{r}{h}$$

$$-2r^2(3h^2+r^2)$$

Thus the zero located at s = h/r is a minimum since cups up. But I always have h > r so this zero is out of range anyway.

The zero located at s = -(1/3) r/h on the other hand is a max since it cups down. And since h > r, this zero is always in range. So this is the one I care about. So what is f(x) at this max point?

```
x := s[2]; f; unassign('x'):
```

$$x := -\frac{1}{3}\frac{r}{h}$$

$$\left(h+\frac{1}{3}\frac{r^2}{h}\right)^2\left(h^2+\frac{1}{3}r^2\right)$$

$f_{max} = (1/h)^2 (h^2 + (1/3)r^2)^3$

But I want f_{min}!! Since there is a max in range and no min in range, then min will be at one of the endpoints x = 1 or x = -1. So I need to look at both those endpoints.

```
x := 1; f; unassign('x'):
```

$$x := 1$$

$$(h-r)^2(h^2+r^2+2hr)$$

```
x := -1; f; unassign('x'):
```

$$x := -1$$

$$(h+r)^2(h^2+r^2-2hr)$$

Interestingly these are the same! I have therefore found that

$f_{min} = (h+r)^2(h-r)^2 = (h^2-r^2)^2$

so as long as I avoid the endpoint extremes with x, I have

$f < f_{min}$

So now I want to prove that

$(h^2-r^2)^2 \;<\; f(x)$ for all x

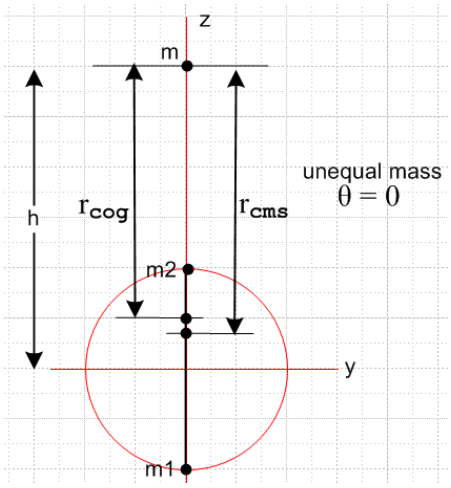
At the exact min value of the right side equals the left side, but at all other values the left side is less than the right side. That is to say, in order to show $r_{cog} \leq r_{cms}$ I need to show that

$(h^2-r^2)^2 \;\leq\; f(x)$ for all x.

But I know that

$f(x) \geq f_{min} = (h^2-r^2)^2$

but this is what I want to show, QED. This is harder than I thought. Here is the result'



Unequal mass, but $\theta = \pi/2$.

In this case $r_1 = r_2$. Then,

$\mathbf{r}_{cms} = -h\hat{\mathbf{z}} + \mathbf{R}_{cms} = -h\hat{\mathbf{z}} + f\mathbf{r} = -h\hat{\mathbf{z}} + fy\hat{\mathbf{y}} + fz\hat{\mathbf{z}} = fr\sin\theta\,\hat{\mathbf{y}} - (h - fr\cos\theta)\hat{\mathbf{z}}$

$= fr\,\hat{\mathbf{y}} - (h)\hat{\mathbf{z}}$

$\mathbf{r}_{cog} = \sqrt{Gm(m_1+m_2)}\; F^{-3/2}\,[\; F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}}\;]$

$F_y = [Gm/(r_1^3 r_2^3)] * r\sin\theta\,(m_2 r_1^3 - m_1 r_2^3)$
$-F_z = [Gm/(r_1^3 r_2^3)] * [m_1 r_2^3(h + r\cos\theta) + m_2 r_1^3(h - r\cos\theta)]$

$F_y = [Gm/(r_1^3 r_2^3)] * r\,(m_2 r_1^3 - m_1 r_2^3)$
$-F_z = [Gm/(r_1^3 r_2^3)] * [m_1 r_2^3(h) + m_2 r_1^3(h)]$

$F_y = [Gm/(r_1^3 r_1^3)] * r(m_2 r_1^3 - m_1 r_1^3)$
$-F_z = [Gm/(r_1^3 r_1^3)] * [m_1 r_1^3(h) + m_2 r_1^3(h)]$

$F_y = [Gm/(r_1^3)] * r(m_2 - m_1) = [GmM/(r_1^3)] r(\mu_2 - \mu_1)$
$-F_z = [Gm/(r_1^3)] * h(m_1 + m_2) = [GmM/(r_1^3)] h$

$F = [GmM/(r_1^3)] [h^2 + r^2(\mu_2 - \mu_1)^2]^{1/2}$

$\mathbf{r}_{cog} = \sqrt{Gm(m_1+m_2)}\ [GmM/(r_1^3)]^{-3/2} [h^2 + r^2(\mu_2 - \mu_1)^2]^{-3/4}$

$* [[GmM/(r_1^3)] r(\mu_2 - \mu_1)\hat{\mathbf{y}} - [GmM/(r_1^3)] h \hat{\mathbf{z}}]$

$= \sqrt{Gm(m_1+m_2)}\ [GmM/(r_1^3)]^{-1/2} [h^2 + r^2(\mu_2 - \mu_1)^2]^{-3/4} [r(\mu_2 - \mu_1)\hat{\mathbf{y}} - h\hat{\mathbf{z}}]$

Leading factor is

$\sqrt{Gm(m_1+m_2)}\ [GmM/(r_1^3)]^{-1/2} = r_1^{3/2}$

So get

$\mathbf{r}_{cog} = r_1^{3/2}[h^2 + r^2(\mu_2 - \mu_1)^2]^{-3/4} [r(\mu_2 - \mu_1)\hat{\mathbf{y}} - h\hat{\mathbf{z}}]$

$\mathbf{r}_{cms} = [r(\mu_2 - \mu_1)\hat{\mathbf{y}} - h\hat{\mathbf{z}}]$

The last brackets are the same! This means they are in the same direction !!!!

$\mathbf{r}_{cog} = r_1^{3/2}[h^2 + r^2(\mu_2 - \mu_1)^2]^{-3/4}\ \mathbf{r}_{cms}$

In this case $r_1^2 = h^2 + r^2$ so write as

$$\mathbf{r}_{cog} = \frac{(h^2+r^2)^{3/4}}{(h^2+r^2(\mu_2-\mu_1)^2)^{3/4}}\mathbf{r}_{cms} = \left[\frac{h^2+r^2}{h^2+(\mu_2-\mu_1)^2 r^2}\right]^{3/4}\mathbf{r}_{cms}$$

The ratio is > 1 so this says $r_{cog} > r_{cms}$.

Does this work for test cases? Looks suspicious.

$\mu_2 = 0\ \ \mu_1 = 1$

$$\mathbf{r}_{cog} = \left[\frac{h^2+r^2}{h^2+r^2}\right]^{3/4}\mathbf{r}_{cms} = \mathbf{r}_{cms}$$

Same result for $\mu_2 = 1$. If equal mass get

$$\mathbf{r}_{cog} = \left[\frac{h^2+r^2}{h^2} \right]^{3/4} \mathbf{r}_{cms}$$

which seems a very strange result

$\mathbf{r}_{cms} = [- h\hat{\mathbf{z}}]$ = tip at the origin.

so I am getting

$$\mathbf{r}_{cog} = \left[\frac{h^2+r^2}{h^2} \right]^{3/4} [- h\hat{\mathbf{z}}]$$

$$r_{cog} = \left[\frac{h^2+r^2}{h^2} \right]^{3/4} h = (r_2+h^2)^{3/4} / \sqrt{h}$$

and this DOES agree with my direct hand calculation of this equal mass case.

$$r_{cog} = (r^2+h^2)^{3/4} / \sqrt{h} = h^{3/2} (1 + (h/r)^2)^{3/4} h^{-1/2} = h (1 + (h/r)^2)^{3/4}$$

$$r_{cms} = h$$

Therefore

$$r_{cog} / r_{cms} = (1 + (h/r)^2)^{3/4} > 1$$

**********original notes below**********

The δ angle to this point from the vertical axis is

$$\tan(\delta_{cog}(\theta)) = r_{cms,y} / r_{cms,z} = y/z = \tan\theta$$

If we set $\theta = 0$ then obviously $\delta = 0$ since $F_y = 0$. In this case we have

$$Gm(m_1+m_2)/R^2 = -F_z = [Gm /(r_1^3 r_2^3)] * [m_1 r_2^3 (h + r\cos\theta) + m_2 r_1^3 (h - r\cos\theta)]$$

$$= [Gm /(r_1^3 r_2^3)] * [m_1 r_2^3 (h + r) + m_2 r_1^3 (h - r)]$$

giving

$$M/R^2 = [m_1 r_1^{-3} (h + r) + m_2 r_2^{-3} (h - r)]$$

$$= [m_1 r_1^{-3}(r_1) + m_2 r_2^{-3}(r_2)]$$

$$= [m_1 r_1^{-2} + m_2 r_2^{-2}]$$

and then

$$R^2 = q = \frac{1}{[\mu_1 r_1^{-2} + \mu_2 r_2^{-2}]}$$

which agrees with my v2 doc. If we replace h in v2 by H, then the connection to that doc for special case is $H + 2r = r_1$ and $H = r_2$ If the masses are the same, the result is

$$R^2 = \frac{2}{r_1^{-2} + r_2^{-2}} = \frac{2 r_1^2 r_2^2}{r_1^2 + r_2^2} \qquad \text{so} \qquad R = \frac{\sqrt{2}\, r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

But in this $\theta = 0$ case we have

$$r_1 = h + r$$
$$r_2 = h - r$$

so

$$R^2 = \frac{2(h+r)^2(h-r^2)}{(h+r)^2+(h-r)^2} = \frac{(h^2 - r^2)^2}{(h^2+r^2)} = h^2 \frac{(1-(r/h)^2)^2}{(1+(r/h)^2)} < h^2$$

$$R = h \frac{1-(r/h)^2}{\sqrt{1+(r/h)^2}} < h$$

What happens if $\theta = \pi/2$? Then $r_1 = r_2$ and $\delta = 0$ and

$$F_y = [Gm/(r_1^3 r_1^3)] * r\ (m_1 r_1^3 - m_2 r_1^3)$$
$$-F_z = [Gm/(r_1^3 r_1^3)] * [m_1 r_1^3(h) + m_2 r_1^3(h]$$

or

$$F_y = [Gm/(r_1^3)] * r\,(m_1 - m_2)$$
$$-F_z = [Gm/(r_1^3)] * h\,(m_1 + m_2)$$

$$F = [Gm/(r_1^3)] \sqrt{r^2 (m_1 - m_2)^2 + h^2 (m_1 + m_2)^2}$$

In this case I know that $r_1^2 = r^2 + h^2$ so

$$F = \frac{Gm}{(r^2+h^2)^{3/2}} \sqrt{r^2 (m_1 - m_2)^2 + h^2 (m_1 + m_2)^2} = GmM/R^2$$

and then

$$R^2 = M\frac{(r^2+h^2)^{3/2}}{\sqrt{r^2(m_1-m_2)^2+h^2(m_1+m_2)^2}} = \frac{(r^2+h^2)^{3/2}}{\sqrt{r^2(\mu_1-\mu_2)^2+h^2}}$$

$$= M\frac{h^3(1+(r/h)^2)^{3/2}}{h\sqrt{(r/h)^2(m_1-m_2)^2+(m_1+m_2)^2}}$$

$$= Mh^2\frac{(1+(r/h)^2)^{3/2}}{\sqrt{(r/h)^2(m_1-m_2)^2+(m_1+m_2)^2}}$$

$$= Mh^2\frac{(1+(r/h)^2)^{3/2}}{\sqrt{M^2+(r/h)^2(m_1-m_2)^2}}$$

$$= h^2\frac{(1+(r/h)^2)^{3/2}}{\sqrt{1+(r/h)^2((m_1-m_2)/M)^2}}$$

$$= h^2\frac{(1+(r/h)^2)^{3/2}}{\sqrt{1+(r/h)^2(\mu_1-\mu_2)^2}}$$

If the masses are equal we get

$$R^2 = h^2(1+(r/h)^2)^{3/2}$$

$$R = h(1+(r/h)^2)^{3/4} > h$$

If the masses are equal, then R > h so my claim that "always closer" is wrong!!!! If $\mu_1 = 0$ then $\mu_2 = 1$ and the result is then

$$R^2 = h^2\frac{(1+(r/h)^2)^{3/2}}{\sqrt{1+(r/h)^2}} = h^2(1+(r/h)^2) = h^2 + r^2 = r_1^2$$

the obvious result in this case, and again R > h.

Go back to the general result

$$F_y = [Gm/(r_1^3 r_2^3)] * r\sin\theta\,(m_1 r_2^3 - m_2 r_1^3)$$
$$-F_z = [Gm/(r_1^3 r_2^3)] * [m_1 r_2^3(h + r\cos\theta) + m_2 r_1^3(h - r\cos\theta)]$$

$$r_1^2 = (r\sin\theta)^2 + (h+r\cos\theta)^2$$
$$r_2^2 = (r\sin\theta)^2 + (h-r\cos\theta)^2$$

$$F = \sqrt{F_x^2+F_y^2} = [Gm/(r_1^3 r_2^3)] *$$

$$\{\ r^2\sin^2\theta\,(m_1r_2^3\ -\ m_2r_1^3)^2\ +\ [m_1r_2^3(h+r\cos\theta)\ +\ m_2r_1^3(h-r\cos\theta)]^2\ \}^{1/2}$$

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]\ *\ 1/\ (r_1^6r_2^6)^{1/2}$$
$$\{\ r^2\sin^2\theta\,(m_1r_2^3\ -\ m_2r_1^3)^2\ +\ [m_1r_2^3(h+r\cos\theta)\ +\ m_2r_1^3(h-r\cos\theta)]^2\ \}^{1/2}$$

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]$$
$$\{\ r^2\sin^2\theta\,(m_1r_2^3\ -\ m_2r_1^3)^2/\ (r_1^6r_2^6)\ +\ [m_1r_2^3(h+r\cos\theta)\ +\ m_2r_1^3(h-r\cos\theta)]^2/(r_1^6r_2^6)\ \}^{1/2}$$

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]$$
$$\{\ r^2\sin^2\theta\,(m_1r_2^3\ -\ m_2r_1^3)^2/\ (r_1^3r_2^3)^2\ +\ [m_1r_2^3(h+r\cos\theta)\ +\ m_2r_1^3(h-r\cos\theta)]^2/(r_1^3r_2^3)^2\ \}^{1/2}$$

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]$$
$$\{\ r^2\sin^2\theta\,(m_1r_1^{-3}\ -\ m_2r_2^{-3})^2+\ [m_1r_1^{-3}(h+r\cos\theta)\ +\ m_2r_2^{-3}(h-r\cos\theta)]^2\ \}^{1/2}$$

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]$$
$$\{\ r^2\sin^2\theta\,(m_1r_1^{-3}\ -\ m_2r_2^{-3})^2+\ [h(m_1r_1^{-3}+m_2r_2^{-3})+r\cos\theta(m_1r_1^{-3}\ -\ m_2r_2^{-3})\ ]^2\ \}^{1/2}$$

Define

$$A = m_1r_1^{-3}\ -\ m_2r_2^{-3} \qquad \dim(rA) = kg * m^{-2} = \dim(hB)$$
$$B = m_1r_1^{-3}\ +\ m_2r_2^{-3}$$

Then

$$F\ =\sqrt{F_x^2+F_y^2}\ =[Gm]\ \ \{\ \ r^2\sin^2\theta A^2+\ [hB+r\cos\theta A\ ]^2\ \}^{1/2}$$

$$=[Gm]\ \ \{\ \ r^2\sin^2\theta A^2+h^2B^2+r^2\cos^2\theta A^2\ +2hrAB\cos\theta\ \}^{1/2}$$

$$=[Gm]\ \ \{\ \ r^2A^2+h^2B^2+2hrAB\cos\theta\ \}^{1/2}$$

$$=[Gm]\ \ \{\ \ (rA)^2+(hB)^2+2\ (rA)\ (hB)\cos\theta\ \}^{1/2}$$

$$=GmM/R^2$$

The general result is then

$$R^2 = M\ /\ \{\ \ (rA)^2+(hB)^2+2\ (rA)\ (hB)\cos\theta\ \}^{1/2}\ \ =\ kg\ /\ (\ kg * m^{-2}) = m^2$$

or

$$R^2 = \frac{M}{[(rA)^2+(hB)^2+2\ (rA)\ (hB)\cos\theta]^{1/2}}$$

$$A = m_1r_1^{-3}\ -\ m_2r_2^{-3}$$
$$B = m_1r_1^{-3}\ +\ m_2r_2^{-3} \qquad M = m_1+m_2$$

Let's recheck the two special cases:

1) aligned $\theta = 0$ then

$$R^2 = \frac{M}{[(rA)^2 + (hB)^2 + 2\,(rA)\,(hB)]^{1/2}} = \frac{M}{rA + hB}$$

But $r_1 = h+r$ and $r_2 = h-r$ so

$rA = (r_1-h)[\, m_1 r_1^{-3} - m_2 r_2^{-3}]$

$hB = h[m_1 r_1^{-3} + m_2 r_2^{-3}]$

$rA+hB = (r_1-h)[\, m_1 r_1^{-3} - m_2 r_2^{-3}] + h[m_1 r_1^{-3} + m_2 r_2^{-3}]$

$= m_1 r_1^{-2} - m_2 r_1 r_2^{-3} - h m_1 r_1^{-3} + h m_2 r_2^{-3} + h m_1 r_1^{-3} + h m_2 r_2^{-3}$

$= m_1 r_1^{-2} - m_2 r_1 r_2^{-3} + 2h m_2 r_2^{-3}$

This does not look like the previous answer

$$R^2 = \frac{M}{[m_1 r_1^{-2} + m_2 r_2^{-2}]}$$

But it would be the same answer if we could show that

$- m_2 r_1 r_2^{-3} + 2h m_2 r_2^{-3} = m_2 r_2^{-2}$

or

$- r_1 r_2^{-3} + 2h r_2^{-3} = r_2^{-2}$

or

$- r_1 r_2^{-1} + 2h r_2^{-1} = 1$

or

$- r_1 + 2h = r_2$

or

$r_1 = 2h - r_2$

We do know in our special case that

But we know in this $\theta = 0$ case that

$r_1 = r_2 + 2r$
$r_1 = h + r$
$r_2 = h - r$

So the last two items tell us that

$r_1 + r_2 = 2h$

and thus I have shown that the special case here does replicate the previous result.

Now go back to the **general claim**

$$R^2 = \frac{M}{[(rA)^2 + (hB)^2 + 2\,(rA)\,(hB)\cos\theta]^{1/2}}$$

$$A = m_1 r_1^{-3} - m_2 r_2^{-3}$$
$$B = m_1 r_1^{-3} + m_2 r_2^{-3} \qquad\qquad M = m_1 + m_2$$

For $\theta = \pi/2$ this gives

$$R^2 = \frac{M}{[(rA)^2 + (hB)^2]^{1/2}}$$

and in this case we know that $r_1 = r_2$ and also $r_1^2 = h^2 + r^2$

$$rA = r[m_1 r_1^{-3} - m_2 r_2^{-3}] = r\, r_1^{-3}(m_1 - m_2)$$

$$hB = h[m_1 r_1^{-3} + m_2 r_2^{-3}] = h r_1^{-3}(m_1 + m_2)$$

Then

$$(rA)^2 + (rB)^2 = r_1^{-6}\,[\, r^2(m_1 - m_2)^2 + h^2 M^2] \;=\; r_1^{-6}\, M^2\, [\, r^2(\mu_1 - \mu_2)^2 + h^2]$$

Then

$$R^2 = \frac{r_1^3}{\sqrt{r^2(\mu_1 - \mu_2)^2 + h^2}} = \frac{(h^2 + r^2)^{3/2}}{\sqrt{r^2(\mu_1 - \mu_2)^2 + h^2}}$$

and this does match a form of my previous result. Another way to write this'

$$R^2 = \frac{r_1^3}{\sqrt{(r_1^2 - h^2)(\mu_1 - \mu_2)^2 + h^2}}$$

If the masses are equal you get

$$R^2 = r_1^3/h \;\Rightarrow\; R = r_1\sqrt{r_1/h} \;> r_1$$

I think that is enough on this problem.

What is the torque?

Relative to the sphere center, the torque on mass m_2 is (forces are on the test mass m)

Define

$\mathbf{F}_1$ = the force acting on mass m_1 due to mass m (points roughly up)
$\mathbf{F}_2$ = the force acting on mass m_2 due to mass m (points roughly up)

These are torques about the origin, not about cms or cog!

$\mathbf{N}_2 = \mathbf{r} \times \mathbf{F}_2$ $\quad \mathbf{F}_2 = (Gmm_2/r_2^2)\,\hat{\mathbf{r}}_2$ $\quad \hat{\mathbf{r}}_2$ points from m_2 up to m

$\mathbf{N}_1 = (-\mathbf{r}) \times \mathbf{F}_1$ $\quad \mathbf{F}_1 = (Gmm_1/r_1^2)\,\hat{\mathbf{r}}_1$ $\quad \hat{\mathbf{r}}_1$ points from m_1 up to m

Earlier I wrote these forces *acting on mass m*

$F_y = -(Gmm_1/r_1^3)\,r\sin\theta + (Gmm_2/r_2^3)\,r\sin\theta$ repaired

$F_z = -(Gmm_1/r_1^3)(h + r\cos\theta) - (Gmm_2/r_2^3)(h - r\cos\theta)$

which I can break down to be

$F_{1y} = -(Gmm_1/r_1^3)\,r\sin\theta$ $\quad F_{2y} = (Gmm_2/r_2^3)\,r\sin\theta$ repaired

$F_{1z} = -(Gmm_1/r_1^3)(h + r\cos\theta)$ $\quad F_{2z} = -(Gmm_2/r_2^3)(h - r\cos\theta)$

But now I want the first force to act on m_1 and the second to act on m_2, so negate all

$F_{1y} = (Gmm_1/r_1^3)\,r\sin\theta$ $\quad F_{2y} = -(Gmm_2/r_2^3)\,r\sin\theta$ repaired

$F_{1z} = +(Gmm_1/r_1^3)(h + r\cos\theta)$ $\quad F_{2z} = +(Gmm_2/r_2^3)(h - r\cos\theta)$

Since r and $\mathbf{F}_i$ are in the plane, we know that both torques are normal to paper. Staring at the picture shows that

$\mathbf{r} \times \mathbf{r}_1 = r_1 r\sin(\theta-\alpha_1)\,\hat{\mathbf{x}}$

$\mathbf{r} \times \mathbf{r}_2 = r_2 r\sin(\theta+\alpha_2)\,\hat{\mathbf{x}}$

Then

$\mathbf{N}_2 = \mathbf{r} \times \mathbf{F}_2 = \mathbf{r} \times [\,(Gmm_2/r_2^3)\,\mathbf{r}_2\,] = (Gmm_2/r_2^3)\,\mathbf{r} \times \mathbf{r}_2 = (Gmm_2/r_2^3)\,r_2 r\sin(\theta+\alpha_2)\,\hat{\mathbf{x}}$

$\mathbf{N}_1 = -\mathbf{r} \times \mathbf{F}_1 = -\mathbf{r} \times [\,(Gmm_1/r_1^3)\,\mathbf{r}_1 = -(Gmm_1/r_1^3)\,\mathbf{r} \times \mathbf{r}_1 = -(Gmm_1/r_1^3)\,r_1 r\sin(\theta-\alpha_1)\,\hat{\mathbf{x}}$

Summary:

$\mathbf{N}_1 = -(Gmm_1/r_1^2)\,r\sin(\theta-\alpha_1)\,\hat{\mathbf{x}}$

$\mathbf{N}_2 = (Gmm_2/r_2^2)\,r\sin(\theta+\alpha_2)\,\hat{\mathbf{x}}$ units are r and force, OK

Then

$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 = Gmr\,[\,-(m_1/r_1^2)\sin(\theta-\alpha_1) + (m_2/r_2^2)\sin(\theta+\alpha_2)\,]\,\hat{\mathbf{x}}$

Let's try another way to get this result. Using the second set of forces above,

$\mathbf{F_1} = (Gmm_1/r_1^3)\, r\sin\theta\ \hat{\mathbf{y}} + (Gmm_1/r_1^3)(h + r\cos\theta)\ \hat{\mathbf{z}}$

$\mathbf{F_2} = -(Gmm_2/r_2^3)\ r\sin\theta\ \hat{\mathbf{y}} + (Gmm_2/r_2^3)(h - r\cos\theta)\hat{\mathbf{z}}$ repaired

Then

$\mathbf{N} = \mathbf{r} \times (\mathbf{F_2} - \mathbf{F_1})$

But

$\mathbf{F_2} - \mathbf{F_1} = -(Gmm_2/r_2^3)\ r\sin\theta\ \hat{\mathbf{y}} + (Gmm_2/r_2^3)(h - r\cos\theta)\hat{\mathbf{z}}$
$- \{ (Gmm_1/r_1^3)\, r\sin\theta\ \hat{\mathbf{y}} + (Gmm_1/r_1^3)(h + r\cos\theta)\}\ \hat{\mathbf{z}}$ repaired

$= -(Gmm_2/r_2^3)\, r\sin\theta\ \hat{\mathbf{y}} + (Gmm_2/r_2^3)(h - r\cos\theta)\hat{\mathbf{z}}$ repaired
$- (Gmm_1/r_1^3)\, r\sin\theta\ \hat{\mathbf{y}} - (Gmm_1/r_1^3)(h + r\cos\theta)\ \hat{\mathbf{z}}$

$= [\ -(Gmm_2/r_2^3)\, r\sin\theta - (Gmm_1/r_1^3)\, r\sin\theta\ ]\ \hat{\mathbf{y}}$ repaired
$+ [\ (Gmm_2/r_2^3)(h - r\cos\theta) - (Gmm_1/r_1^3)(h + r\cos\theta)\ ]\ \hat{\mathbf{z}}$

$= -Gm\, r\sin\theta\, [m_2/r_2^3 + m_1/r_1^3]\ \hat{\mathbf{y}}$ repaired
$+ Gm\, [\ (m_2/r_2^3)(h - r\cos\theta) - (m_1/r_1^3)(h + r\cos\theta)]\ \hat{\mathbf{z}}$

$= GmC\ \hat{\mathbf{y}}$ $\quad C = -\, r\sin\theta\, (m_2/r_2^3 + m_1/r_1^3)$ repaired, red minus
$+ GmD\ \hat{\mathbf{z}}$ $\quad D = (m_2/r_2^3)(h - r\cos\theta) - (m_1/r_1^3)(h + r\cos\theta)$

Next write

$\mathbf{r} = y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}} = r\sin\theta\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}}$

Then

$\mathbf{N} = \mathbf{r} \times (\mathbf{F_2} - \mathbf{F_1})$

$= [\ r\sin\theta\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}}] \times [\ GmC\ \hat{\mathbf{y}} + GmD\ \hat{\mathbf{z}}]$

$= Gmr\ [\ \sin\theta\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}] \times [\ C\ \hat{\mathbf{y}} + D\ \hat{\mathbf{z}}]$

$= Gmr\ \{\ \sin\theta D\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} + \cos\theta\ C\ \hat{\mathbf{z}} \times \hat{\mathbf{y}}\ \}$

$= Gmr\ \{\ \sin\theta D\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} - \cos\theta C \hat{\mathbf{y}} \times \hat{\mathbf{z}}\ \}$

$= Gmr\ \{\ \sin\theta D - \cos\theta C\}\ \hat{\mathbf{x}}$

$= Gmr\ \{\ D\sin\theta - C\cos\theta\}\ \hat{\mathbf{x}}$

$= Gmr \{ \ [(m_2/r_2^3)(h - r\cos\theta) - (m_1/r_1^3)(h + r\cos\theta)]\sin\theta + \ [r \sin\theta(m_2/r_2^3 + m_1/r_1^3)]\cos\theta\} \ \hat{\mathbf{x}}$

$= Gmr \{ \ [(m_2/r_2^3)(h - r\cos\theta) - (m_1/r_1^3)(h + r\cos\theta)]\sin\theta$
$+ [r \sin\theta\cos\theta(m_2/r_2^3) + \ r \sin\theta\cos\theta(m_1/r_1^3)]\} \ \hat{\mathbf{x}}$

$= Gmr \{ \ [((m_2/r_2^3) - (m_1/r_1^3))h - ((m_2/r_2^3) +(m_1/r_1^3))r\cos\theta]\sin\theta$
$+ \ [r \sin\theta\cos\theta(m_2/r_2^3) + \ r \sin\theta\cos\theta(m_1/r_1^3)]\} \ \hat{\mathbf{x}}$

$= Gmr \{ \ [((m_2/r_2^3) - (m_1/r_1^3))h\sin\theta - ((m_2/r_2^3) +(m_1/r_1^3)) r\sin\theta\cos\theta$
$+ \ [r \sin\theta\cos\theta(m_2/r_2^3) + \ r \sin\theta\cos\theta(m_1/r_1^3)]\} \ \hat{\mathbf{x}}$

$= Gmr \{ \ [((m_2/r_2^3) - (m_1/r_1^3))h\sin\theta - ((m_2/r_2^3) +(m_1/r_1^3)) r\sin\theta\cos\theta$
$+ r \sin\theta\cos\theta(m_2/r_2^3) - \ r \sin\theta\cos\theta(m_1/r_1^3)]\} \ \hat{\mathbf{x}}$

$= Gmr \{ \ ((m_2/r_2^3) - (m_1/r_1^3))h\sin\theta - ((m_2/r_2^3) +(m_1/r_1^3)) r\sin\theta\cos\theta$
$+ r \sin\theta\cos\theta ((m_2/r_2^3) + (m_1/r_1^3)) \ \} \ \hat{\mathbf{x}}$ repaired

$= Gmr \{ \ ((m_2/r_2^3) - (m_1/r_1^3))h\sin\theta \ \} \ \hat{\mathbf{x}}$ repaired

$= - Gmr \{ \ ((m_1/r_1^3) - (m_2/r_2^3))h\sin\theta \ \} \ \hat{\mathbf{x}}$ repaired

This is a fairly compact result. Now use earlier definitions,

$A = m_1r_1^{-3} - m_2r_2^{-3}$ $\qquad\qquad$ $\dim(rA) = kg * m^{-2} = \dim(hB)$
$B = m_1r_1^{-3} + m_2r_2^{-3}$

and then we have

$\mathbf{N} = -Gmr [h\sin\theta A \ ] \ \hat{\mathbf{x}}$

$= -Gmr [hA\sin\theta \ ] \ \hat{\mathbf{x}}$

$= -Gmr\sin\theta [hA \ ] \ \hat{\mathbf{x}}$ repaired

Here is my answer for total torque about the origin applied by m to the two-mass system:

$\mathbf{N} = - Gmr\sin\theta [hA \ ] \ \hat{\mathbf{x}}$

$A = m_1r_1^{-3} - m_2r_2^{-3}$
$B = m_1r_1^{-3} + m_2r_2^{-3}$ repaired

Let's now try some special cases. For $\theta = \pi/2$,

$\mathbf{N} = - Gmr [hA \ ] \ \hat{\mathbf{x}}$

$= - GmrhA\hat{\mathbf{x}} = - Gmrh [m_1 r_1^{-3} - m_2 r_2^{-3}]\hat{\mathbf{x}}$

$= - Gmrhr_1^{-3}(m_1 - m_2)\hat{\mathbf{x}}$

If the masses are equal, then get **N** = 0 as expected.

For θ = 0 we get

$\mathbf{N} = - Gmr\, 0\, [hA]\, \hat{\mathbf{x}} = 0$

so there is zero torque regardless of all other parameters. This is because each **r** x **F** = 0

Now go back to the general answer

$\mathbf{N} = - Gmr\sin\theta [hA]\, \hat{\mathbf{x}}$

$= - Gmr\sin\theta \{ h[m_1 r_1^{-3} - m_2 r_2^{-3}] \}\, \hat{\mathbf{x}}$ repaired

For general masses, you cannot say which term is larger. For equal masses we have

$= - Gmm_1 rh\sin\theta \{ r_1^{-3} - r_2^{-3} \}\, \hat{\mathbf{x}}$ correct as was

Since $r_1 > r_2$, then first term is smaller and we get

$\mathbf{N} = + \sin\theta$ * (positive) $\hat{\mathbf{x}}$

Thus, if θ > 0 as I have drawn it, **N** points out of paper. This is the "restoring force" I have been looking for. if θ < 0, then **N** restores the other way. [restoring notion survived repair!]

At some point I will no doubt write this up in a better order.

Update 12.3.16.

I have shown that

$\mathbf{N}^{(O)} = - Gmrh\sin\theta (m_1/r_1^3 - m_2/r_2^3)\, \hat{\mathbf{x}}$

This is the torque about the origin of my picture, it is not the torque about the cms or cog points!

The torque rule is this

$\mathbf{N}^{(O)} = \Sigma\, \mathbf{R}_i \times \mathbf{F}_i$ vectors rel to origin

$\mathbf{N}^{(a)} = \Sigma_i\, (\mathbf{R}_i - \mathbf{a}) \times \mathbf{F}_i = \mathbf{N}^{(O)} - \mathbf{a} \times (\Sigma_i \mathbf{F}_i)$

where $\mathbf{F}_i$ is the force acting on mass i of the 2-mass system. In my case I have

$F_{1y} = (Gmm_1/r_1^3)\, r\sin\theta$ $F_{2y} = - (Gmm_2/r_2^3)\, r\sin\theta$ repaired

$F_{1z} = +(Gmm_1/r_1^3)(h + r\cos\theta)$ $\quad F_{2z} = +(Gmm_2/r_2^3)(h - r\cos\theta)$

Then

$$\mathbf{F_1} = (Gmm_1/r_1^3)\, r\sin\theta\, \hat{\mathbf{y}} + (Gmm_1/r_1^3)(h + r\cos\theta)\hat{\mathbf{z}}$$
$$\mathbf{F_2} = -(Gmm_2/r_2^3)\, r\sin\theta\, \hat{\mathbf{y}} + (Gmm_2/r_2^3)(h - r\cos\theta)\hat{\mathbf{z}}$$

$$\mathbf{F} = \Sigma_i \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2$$

Thus if I want to know the torque about the cog point I had to do this

$$\mathbf{N}^{(cog)} = \mathbf{N}^{(O)} - \mathbf{r}_{cog} \times \mathbf{F}$$

$$= -Gmr\sin\theta \left(m_1/r_1^3 - m_2/r_2^3 \right) \hat{\mathbf{x}} - \mathbf{r}_{cog} \times \mathbf{F}$$

where both $\mathbf{r}_{cog}$ and $\mathbf{F}$ are fairly complicated functions. F is shown just above, and

$$\mathbf{r}_{cog} = r_{cog}\hat{\mathbf{F}} = (r_{cog}/F)\,\mathbf{F} = \sqrt{Gm(m_1+m_2)}\; F^{-3/2} \left[F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}} \right]$$

But the meaning here of F_y is the force acting on m, so I do the same repair as above to get

$$\mathbf{r}_{cog} = -\sqrt{Gm(m_1+m_2)}\; F^{-3/2} \left[F_y\hat{\mathbf{y}} + F_z\hat{\mathbf{z}} \right]$$

But of course you can now write this as

$$\mathbf{r}_{cog} = -\sqrt{Gm(m_1+m_2)}\; F^{-3/2}\, \mathbf{F}$$

where F is the force shown above . But then

$$\mathbf{r}_{cog} \times \mathbf{F} = -\sqrt{Gm(m_1+m_2)}\; F^{-3/2}\, \mathbf{F} \times \mathbf{F} = 0 \text{ !!!!!}$$

So back up again to this

$$\mathbf{N}^{(a)} = \Sigma_i (\mathbf{R}_i - \mathbf{a}) \times \mathbf{F}_i = \mathbf{N}^{(O)} - \mathbf{a} \times \mathbf{F}$$

As you change your reference point, the torque does not change if **a** is aligned with **F**. But by the definition of $\mathbf{r}_{cog}$ we know that $\mathbf{r}_{cog}$ lines up with **F** exactly. Hurray. I now know that

$$\mathbf{N}^{(cog)} = \mathbf{N}^{(O)} = -Gmr\sin\theta \left(m_1/r_1^3 - m_2/r_2^3 \right) \hat{\mathbf{x}}$$

which then is the restoring force relative to $\mathbf{r}_{cog}$ which is what we want!