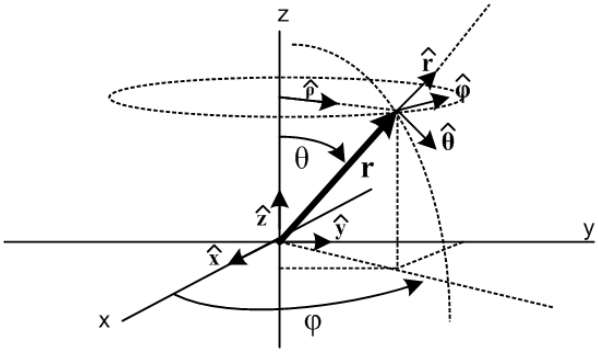
This was installed on 1.29.17, do not edit here.

E.1 Angle Conventions 1
E.2 Matrix Approach 2
E.3 The motion of a particle in spherical coordinates 5
E.4 Curvilinear coordinates approach 6
E.5 Polar Coordinates 7

Appendix E: Spherical Coordinates and Unit Vectors

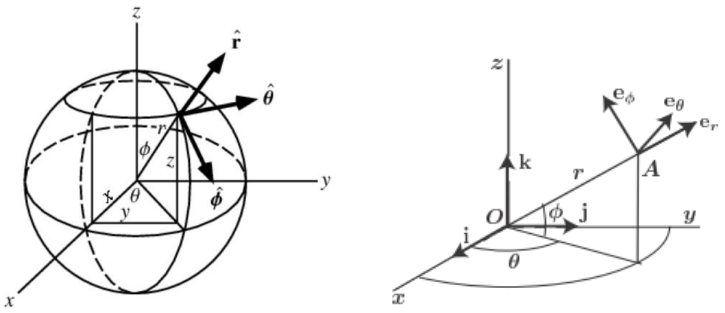
E.1 Angle Conventions

Here is our "physics" convention for spherical coordinates angles θ and φ :



(E.1.1)

The reader is advised that the results below are often quoted in the literature with θ↔φ which is the "math" convention (eg, Wolfram, left below). Sometimes after doing θ↔φ some sources replace polar angle φ with latitude π/2 - φ (right below) :



Two spherical angle conventions we do not use.

In the right picture, φ and θ might be longitude and latitude, or right ascension and declination.

If you are in charge of getting a spacecraft to Vulcan, please pay attention to these conventions.

E.2 Matrix Approach

The usual matrices for actively rotating a vector about the x, y, or z axis are these:

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{E.2.1}$$

The Cartesian unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ can be rotated into the spherical unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ as follows

$$\hat{\boldsymbol{\theta}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{x}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{x}} \\ \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} \end{pmatrix}$$

$$\hat{\boldsymbol{\varphi}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{y}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{r}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{E.2.2}$$

One can interpret the elements of the vectors on the right as dot products as we show on the first line. Notice that these three vectors on the right are just the columns of the matrix shown, so one can say

$$(\,\hat{\boldsymbol{\theta}}\ \ \hat{\boldsymbol{\varphi}}\ \ \hat{\mathbf{r}}\,) = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta) \equiv \boldsymbol{\mathcal{R}}. \tag{E.2.3}$$

It is then obvious that

$$\begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta) = \boldsymbol{\mathcal{R}} \tag{E.2.4}$$

from which one can read off any desired dot product. As verification, consider an example,

Example: $\hat{\mathbf{y}} \bullet \hat{\boldsymbol{\theta}} = \hat{\mathbf{y}} \bullet (\boldsymbol{\mathcal{R}}\hat{\mathbf{x}}) = \Sigma_i\ (\hat{\mathbf{y}})_i(\boldsymbol{\mathcal{R}}\hat{\mathbf{x}})_i = \Sigma_{ij}(\hat{\mathbf{y}})_i\boldsymbol{\mathcal{R}}_{ij}(\hat{\mathbf{x}})_j = \Sigma_{ij}\delta_{2i}\boldsymbol{\mathcal{R}}_{ij}\delta_{1j} = \boldsymbol{\mathcal{R}}_{21}$.

Looking at the last equation of (E.2.2) multiplied by r, since $\mathbf{r} = r\,\hat{\mathbf{r}}$ one sees that

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \qquad \text{(E.2.5)}$$

which is the "inverse transformation" associated with spherical coordinates. See Section E.4 below.

The three equations (E.2.2) can be trivially written out as follows (changing to a standard r,θ,φ order)

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\varphi\ \hat{\mathbf{x}} + \sin\theta\sin\varphi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\ \hat{\mathbf{x}} + \cos\theta\sin\varphi\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}} \end{aligned} \qquad \text{(E.2.6)}$$

which can be inverted to give

$$\begin{aligned} \hat{\mathbf{x}} &= \sin\theta\cos\varphi\ \hat{\mathbf{r}} + \cos\theta\cos\varphi\ \hat{\boldsymbol{\theta}} - \sin\varphi\ \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin\theta\sin\varphi\ \hat{\mathbf{r}} + \cos\theta\sin\varphi\ \hat{\boldsymbol{\theta}} + \cos\varphi\ \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}\ . \end{aligned} \qquad \text{(E.2.7)}$$

For those not wanting to invert the 3x3 matrix $\boldsymbol{\mathcal{R}}$, (E.2.7) can be quickly verified by looking at the unit vector dot products in (E.2.4). Eq. (E.2.6) can be similarly verified .

The spherical unit vectors are orthogonal due to their construction in (E.2.3).

Example: $\hat{\mathbf{r}} \bullet \hat{\boldsymbol{\theta}} = (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) \bullet (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{x}}) = \hat{\mathbf{z}} \bullet (\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\hat{\mathbf{x}}) = \hat{\mathbf{z}} \bullet \hat{\mathbf{x}} = 0$

Example: $\hat{\mathbf{r}} \bullet \hat{\mathbf{r}} = (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) \bullet (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) = \hat{\mathbf{z}} \bullet (\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\hat{\mathbf{z}}) = \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1$

Here $\boldsymbol{\mathcal{R}} \equiv R_z(\varphi)\,R_y(\theta)$ has the property $\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}} = 1$ since it is a rotation. So

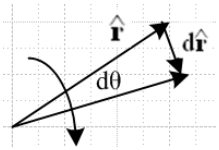
$$\hat{\mathbf{r}} \bullet \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \bullet \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}} \bullet \hat{\boldsymbol{\varphi}} = 0\ . \qquad \text{(E.2.8)}$$

The spatial derivatives of the spherical unit vectors are easy to compute ($\partial_\theta\hat{\mathbf{r}}$ means $\partial\hat{\mathbf{r}}/\partial\theta$),

$$\begin{array}{lll} \partial_r\hat{\mathbf{r}} = 0 & \partial_\theta\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} & \partial_\varphi\hat{\mathbf{r}} = \sin\theta\ \hat{\boldsymbol{\varphi}} \\ \partial_r\hat{\boldsymbol{\theta}} = 0 & \partial_\theta\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} & \partial_\varphi\hat{\boldsymbol{\theta}} = \cos\theta\ \hat{\boldsymbol{\varphi}} \\ \partial_r\hat{\boldsymbol{\varphi}} = 0 & \partial_\theta\hat{\boldsymbol{\varphi}} = 0 & \partial_\varphi\hat{\boldsymbol{\varphi}} = -\sin\theta\ \hat{\mathbf{r}} - \cos\theta\ \hat{\boldsymbol{\theta}}\ . \end{array} \qquad \text{(E.2.9)}$$

Example: $\partial_\theta \hat{\boldsymbol{\theta}} = \partial_\theta \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} = -\hat{\mathbf{r}}$.

In Fig (E.1.1) one can see for example that $\partial_\theta \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$. Often one draws a little triangle to verify an equation like this:



(E.2.10)

$$d\hat{\mathbf{r}} = dr\,\hat{\boldsymbol{\theta}} \approx 1{*}d\theta\,\hat{\boldsymbol{\theta}} = d\theta\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad d\hat{\mathbf{r}} = d\theta\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \hat{\boldsymbol{\theta}} = d\hat{\mathbf{r}}/d\theta = \partial_\theta \hat{\mathbf{r}}\ .$$

Time derivatives of the unit vectors are then obtained by the chain rule ($\partial_t \hat{\mathbf{r}}$ means $d\hat{\mathbf{r}}/dt$),

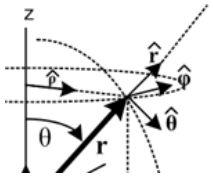
$$\begin{aligned} \partial_t \hat{\mathbf{r}} &= \dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}} \\ \partial_t \hat{\boldsymbol{\theta}} &= -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}} \\ \partial_t \hat{\boldsymbol{\varphi}} &= -\dot{\varphi}\sin\theta\,\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\theta}}\ . \end{aligned} \qquad \text{(E.2.11)}$$

Example: $\partial_t \hat{\mathbf{r}} = (\partial_r \hat{\mathbf{r}})\,\dot{r} + (\partial_\theta \hat{\mathbf{r}})\,\dot{\theta} + (\partial_\varphi \hat{\mathbf{r}})\,\dot{\varphi} = 0 + \hat{\boldsymbol{\theta}}\dot{\theta} + \sin\theta\,\hat{\boldsymbol{\varphi}}\dot{\varphi}$.

Note: If we regard r,θ,φ as coordinates of Frame S, then $\partial_t \hat{\mathbf{r}} = (d\hat{\mathbf{r}}/dt)_S = \partial_S \hat{\mathbf{r}}$ as in Section 1.7 and 1.8.

The cross products of the spherical unit vectors follow the right hand rule, so looking at Fig (E.1.1):

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \qquad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}} \qquad \hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \qquad \text{(E.2.12)}$$



The first cross product has ordering r,θ,φ and the last two are cyclic permutations. Here are some other cross products:

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{r}} &= [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\mathbf{r}} = -\cos\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} - \sin\varphi\,\hat{\boldsymbol{\theta}} \\ \hat{\mathbf{x}} \times \hat{\boldsymbol{\theta}} &= [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\theta}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} + \sin\varphi\,\hat{\mathbf{r}} \\ \hat{\mathbf{x}} \times \hat{\boldsymbol{\varphi}} &= [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\varphi}} = -\sin\theta\cos\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\varphi\,\hat{\mathbf{r}} \end{aligned} \qquad \text{(E.2.13)}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{r}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\mathbf{r}} = -\cos\theta\sin\varphi\,\hat{\boldsymbol{\varphi}} + \cos\varphi\,\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{y}} \times \hat{\boldsymbol{\theta}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\theta}} = \sin\theta\sin\varphi\,\hat{\boldsymbol{\varphi}} - \cos\varphi\,\hat{\mathbf{r}}$$
$$\hat{\mathbf{y}} \times \hat{\boldsymbol{\varphi}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\varphi}} = -\sin\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \sin\theta\sin\varphi\,\hat{\mathbf{r}} \qquad \text{(E.2.14)}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{r}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} \times \hat{\boldsymbol{\theta}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} \times \hat{\boldsymbol{\varphi}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\boldsymbol{\varphi}} = -\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\theta\,\hat{\mathbf{r}} \qquad \text{(E.2.15)}$$

E.3 The motion of a particle in spherical coordinates

The equations below are easily derived from (E.2.11):

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \text{// position} \qquad \text{(E.3.1)}$$

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// velocity}$$
$$v_r = \dot{r}$$
$$v_\theta = r\,\dot{\theta}$$
$$v_\varphi = r\,\dot{\varphi}\sin\theta \qquad \text{(E.3.2)}$$

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// acceleration}$$
$$a_r = \ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta$$
$$a_\theta = 2\,\dot{r}\,\dot{\theta} + r\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta$$
$$a_\varphi = 2\,\dot{r}\,\dot{\varphi}\sin\theta + 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\ . \qquad \text{(E.3.3)}$$

Example: $\mathbf{v} = \dot{\mathbf{r}} = \partial_t(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r(\partial_t\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r[\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}\,] = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\hat{\boldsymbol{\varphi}}$.

Notice that $\dot{v}_r \neq a_r$ and similarly for other components.

If for some reason particle motion is restricted to a spherical surface of radius r (such as in our dumbbell satellite), we can set $\dot{r} = \ddot{r} = 0$ in the above equations to get

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \text{// position} \qquad \text{(E.3.4)}$$

$$\mathbf{v} = v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}} \qquad \text{// velocity is tangent to the sphere : } \mathbf{r} \bullet \mathbf{v} = 0$$
$$v_\theta = r\,\dot{\theta}$$
$$v_\varphi = r\,\dot{\varphi}\sin\theta \qquad \text{(E.3.5)}$$

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// acceleration}$$

$$a_r = -r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2\theta$$

$$a_\theta = r\ddot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta$$

$$a_\varphi = 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta \ . \qquad \text{(E.3.6)}$$

From these equations many other useful results can be obtained, for example,

$$\hat{\mathbf{r}} \times \mathbf{v} = v_\theta \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} + v_\varphi \hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = v_\theta \hat{\boldsymbol{\varphi}} - v_\varphi \hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{r}} \times \mathbf{a} = a_\theta \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} + a_\varphi \hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = a_\theta \hat{\boldsymbol{\varphi}} - a_\varphi \hat{\boldsymbol{\theta}} \qquad \text{(E.3.7)}$$

E.4 Curvilinear coordinates approach

Spherical coordinates are defined by the "inverse transformation" on the left below:

inverse transformation	transformation			
$x = r\sin\theta\cos\varphi$	$r = +\sqrt{x^2+y^2+z^2}$	$0 \le r < \infty$		
$y = r\sin\theta\sin\varphi$	$\cos\theta = z/r$	$0 \le \theta \le \pi$	$-1 \le \cos\theta \le 1$	
$z = r\cos\theta$	$\sin\theta = +\sqrt{1-(z/r)^2}$		$0 \le \sin\theta \le 1$	
	$\sin\varphi = y/(r\sin\theta)$		$-1 \le \sin\varphi \le 1$	
	$\cos\varphi = x/(r\sin\theta)$	$0 \le \varphi < 2\pi$	$-1 \le \cos\varphi \le 1$	(E.4.1)

We quote now a set of results from our *Tensor* document which won't be needed but which we include for "completeness". Equation numbers $(...)_T$ refer to that document.

Spherical coordinates are just an example of curvilinear coordinates which fit into a general formalism:

$\mathbf{x} = (x_1, x_2, x_3) = (x,y,z)$ // x-space coordinates

$\mathbf{x}' = (x_1', x_2', x_3') = (r,\theta,\varphi)$ // x'-space coordinates

$\mathbf{x} = F^{-1}(\mathbf{x}') \leftrightarrow$ $x = r\sin\theta\cos\varphi$ // a non-linear inverse transformation

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$. $\qquad (1.6)_T$

The linearized transformation local to a point defines certain R and S matrices (the "differentials") :

$d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$ $\quad R_{ik}(\mathbf{x}) \equiv (\partial x'_i/\partial x_k)$ $\quad R = S^{-1}$ $\quad // \ dx'_i = R_{ij}\, dx_j$

$d\mathbf{x} = S(\mathbf{x}')\, d\mathbf{x}'$ $\quad S_{ik}(\mathbf{x}') \equiv (\partial x_i/\partial x'_k)$ $\quad S = R^{-1}$ $\quad // \ dx_i = S_{ij}\, dx'_j$. $\qquad (2.1.6)_T$

$$S = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$ // compute from above definition of S_{ij} $\qquad (3.4.4)_T$

$$\bar{g}' = S^T S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad \det(\bar{g}') = r^4\sin^2\theta \qquad \text{// metric tensor and its determinant} \qquad (5.13.14)_T$$

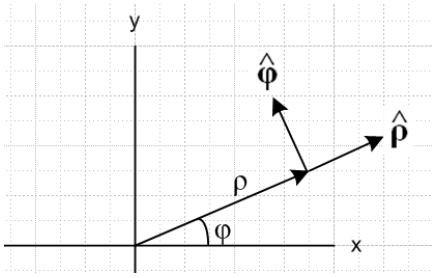
$$J(r,\theta,\varphi) = \det(S) = \sqrt{\det(\bar{g}')} = r^2\sin\theta\,. \qquad \text{// Jacobian} \qquad (5.13.16)_T$$

$$(ds)^2 = \Sigma_{ij}\bar{g}'_{ij}\,dx'_i\,dx_j = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2(d\varphi)^2 \qquad \text{// distance} \qquad (5.13.18)_T$$

Because the metric tensor $\bar{g}'$ is diagonal, the coordinates r,θ.φ are orthogonal.

E.5 Polar Coordinates

Here we use a notation common for two of the cylindrical coordinates (ρ,φ,z),



(E.5.1)

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi \qquad (E.5.2)$$

$$\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}} \qquad \hat{\boldsymbol{\rho}} = R_z(\varphi)\,\hat{\mathbf{x}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \qquad \hat{\boldsymbol{\varphi}} = R_z(\varphi)\,\hat{\mathbf{y}} \qquad (E.5.3)$$

$$\hat{\mathbf{x}} = \cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\varphi}} \qquad \hat{\mathbf{x}} = R_z(-\varphi)\,\hat{\boldsymbol{\rho}}$$
$$\hat{\mathbf{y}} = \sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} \qquad \hat{\mathbf{y}} = R_z(-\varphi)\,\hat{\boldsymbol{\theta}} \qquad (E.5.4)$$

$$\hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{x}} = \cos\varphi \qquad \hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{y}} = \sin\varphi$$
$$\hat{\boldsymbol{\varphi}} \bullet \hat{\mathbf{x}} = -\sin\varphi \qquad \hat{\boldsymbol{\varphi}} \bullet \hat{\mathbf{y}} = \cos\varphi \qquad (E.5.5)$$

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$
$$\dot{\hat{\boldsymbol{\varphi}}} = -\dot{\varphi}\,\hat{\boldsymbol{\rho}} \qquad (E.5.6)$$

Proof of (E.5.6) :

$$\dot{\hat{\boldsymbol{\rho}}} = d\hat{\boldsymbol{\rho}}/dt = d/dt(\cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}) = -\sin\varphi\,\dot{\varphi}\,\hat{\mathbf{x}} + \cos\varphi\,\dot{\varphi}\,\hat{\mathbf{y}}$$
$$= -\sin\varphi\,\dot{\varphi}\,[\cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\theta}}\,] + \cos\varphi\,\dot{\varphi}\,[\sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\theta}}]$$
$$= \dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$

$$\dot{\hat{\boldsymbol{\varphi}}} = d\hat{\boldsymbol{\theta}}/dt = d/dt(-\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}}) = -\cos\varphi\ \dot{\varphi}\ \hat{\mathbf{x}} - \sin\varphi\ \dot{\varphi}\ \hat{\mathbf{y}}$$
$$= -\cos\varphi\ \dot{\varphi}[\cos\varphi\ \hat{\boldsymbol{\rho}} - \sin\varphi\ \hat{\boldsymbol{\theta}}\ ] - \sin\varphi\ \dot{\varphi}\ [\ \sin\varphi\ \hat{\boldsymbol{\rho}} + \cos\varphi\ \hat{\boldsymbol{\theta}}]$$
$$= -\ \dot{\varphi}\ \hat{\boldsymbol{\rho}}$$

E.6 The Affine Connection

Recall from above the claim that

$$\begin{array}{lll} \partial_{\mathbf{r}}\hat{\mathbf{r}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} & \partial_{\boldsymbol{\varphi}}\hat{\mathbf{r}} = \sin\theta\ \hat{\boldsymbol{\varphi}} \\ \partial_{\mathbf{r}}\hat{\boldsymbol{\theta}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} = -\ \hat{\mathbf{r}} & \partial_{\boldsymbol{\varphi}}\hat{\boldsymbol{\theta}} = \cos\theta\ \hat{\boldsymbol{\varphi}} \\ \partial_{\mathbf{r}}\hat{\boldsymbol{\varphi}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}} = 0 & \partial_{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} = -\sin\theta\ \hat{\mathbf{r}} - \cos\theta\ \hat{\boldsymbol{\theta}} \quad . \end{array} \qquad \textit{(E.2.9)} \qquad \text{(E.6.1)}$$

We wish to put these equations into a more general framework.

In the notation of our *Tensor Analysis* document, for an *arbitrary* curvilinear coordinate system $\mathbf{x}'$ the derivatives of the tangent base vectors (called $\mathbf{e}_n$ in that document) are written

$$\partial'_j \mathbf{e}_n = \Sigma_i \Gamma'^{i}_{\ jn}\ \mathbf{e}_i \quad . \qquad \text{(E.6.2)}$$

This just says that the change in a basis vector obtained by moving a small amount in some direction is (and of course must be) some linear combination of the basis vectors. The coefficients of the linear combination are known as the **affine connection** (or **Levi-Civita connection**) $\Gamma'^{c}_{\ ab} = \Gamma'^{c}_{\ ba}$. The reason for the primes is that Cartesian coordinates are thought of as $\mathbf{x}$, while curvilinear ones of some particular type are $\mathbf{x}'$. In Cartesian x-space one has $\Gamma^{c}_{\ ab} = 0$ because basis vectors don't vary in space. Then $\Gamma'^{c}_{\ ab}$ is the affine connection in x'-space. For example, in spherical coordinates one has

$$\mathbf{x} = (x_1,x_2,x_3) = (x,y,z)$$
$$\mathbf{x}' = (x'_1,x'_2,x'_3) = (r,\theta,\varphi) \qquad \text{(E.6.3)}$$

and the inverse transformation (E.2.5) is $\mathbf{x} = F^{-1}(\mathbf{x}')$. As shown below, $\mathbf{e}_1(\mathbf{x}') = \mathbf{e}_1(r,\theta,\varphi) = \hat{\mathbf{r}}$ and then as an example of (E.6.2) we write (for j=2 and n=1, and the linear combination has only one non-zero term):

$$\partial'_2 \mathbf{e}_1 = \partial_{\mathbf{x}_2'} \mathbf{e}_1 = \partial_{\boldsymbol{\theta}} \mathbf{e}_1(r,\theta,\varphi) = \partial_{\boldsymbol{\theta}} \hat{\mathbf{r}}(r,\theta,\varphi) = \hat{\boldsymbol{\theta}} = (1/r)\mathbf{e}_{\boldsymbol{\theta}} = (1/r)\mathbf{e}_2 = \Gamma'^{\,2}_{\ 21}\ \mathbf{e}_2 = \Gamma^{\boldsymbol{\theta}}_{\ \mathbf{r}\boldsymbol{\theta}}\ \mathbf{e}_{\boldsymbol{\theta}} \quad . \qquad \text{(E.6.4)}$$

The curvilinear *unit* basis vectors $\hat{\mathbf{e}}_n$ are related to the tangent base vectors by $\hat{\mathbf{e}}_n = (1/h'_n)\ \mathbf{e}_n$ where the $h'_n = |\mathbf{e}_n|$ are the so-called scale factors. The derivatives of the $\hat{\mathbf{e}}_n$ are then given by :

$$\partial'_j \hat{\mathbf{e}}_n = \partial'_j(h'^{-1}_n \mathbf{e}_n) = (\partial'_j h'^{-1}_n)\mathbf{e}_n + h'^{-1}_n(\partial'_j \mathbf{e}_n)$$
$$= -\ h'^{-2}_n\ (\partial'_j h'_n)\ \mathbf{e}_n + h'^{-1}_n(\ \Sigma_i \Gamma'^{i}_{\ jn}\ \mathbf{e}_i)$$

$$= - h'_n{}^{-1}\,(\partial'_j h'_n)\,\hat{\mathbf{e}}_n + h'_n{}^{-1}(\Sigma_i \Gamma'^{i}_{jn}\, h'_i\,\hat{\mathbf{e}}_i)$$

$$= (1/h'_n)\,[\Sigma_i\; h'_i\,\Gamma'^{i}_{jn}\hat{\mathbf{e}}_i - (\partial'_j h'_n)\,\hat{\mathbf{e}}_n]\,. \qquad \text{(E.6.5)}$$

In *Tensor* the Cartesian basis vectors of x-space are called $\mathbf{u}_i$, but in this document they are called $\mathbf{e}_i$ so we need a different symbol $\boldsymbol{e}_i$ for the tangent base vectors. In Section 14 we use these $\boldsymbol{e}_i$ with $\hat{\boldsymbol{e}}_i$ as the unit tangent base vectors, $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$ in place of $\mathbf{x}' = (x'_1,x'_2,x'_3)$, and $\Gamma' = \Gamma$ with no prime since $\boldsymbol{\xi}$ has no prime (trying not to confuse $\Gamma = 0$ of x-space with $\Gamma \neq 0$ of ξ-space). In this notation (E.6.5) would appear as

$$(\partial\hat{\boldsymbol{e}}_n/\partial\xi_j) = (1/h_n)\,[\Sigma_i(h_i\Gamma^{i}_{jn}\hat{\boldsymbol{e}}_i) - (\partial h_n/\partial\xi_j)\hat{\boldsymbol{e}}_n]\,. \qquad \text{(E.6.6)}$$

Going back to the *Tensor* notation, the affine connection for a coordinate system is related to the system's metric tensor g according to

$$\Gamma^{d}_{ab} = (1/2)\,g^{dc}\,[\,\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}] = 0 \;\text{ since } g_{ij} = \delta_{ij} \qquad \text{x-space (Cartesian)}$$

$$\Gamma'^{d}_{ab} = (1/2)\,g'^{dc}\,[\,\partial'_a g'_{bc} + \partial'_b g'_{ca} - \partial'_c g'_{ab}]\,. \qquad \text{x'-space} \qquad \text{(E.6.7)}$$

For spherical coordinates, one has (using r,θ,φ = 1,2,3 where θ = polar, φ = azimuth)

1	2	3	
$h_r = 1$	$h_\theta = r$	$h_\varphi = r\sin\theta$	// scale factors ($h'_1 = h_r$)
$\mathbf{e}_r = \hat{\mathbf{r}}$	$\mathbf{e}_\theta = r\,\hat{\boldsymbol{\theta}}$	$\mathbf{e}_\varphi = r\sin\theta\,\hat{\boldsymbol{\varphi}}$	// tangent base vectors ($\mathbf{e}_1 = \mathbf{e}_r$)
$\hat{\mathbf{e}}_r = \hat{\mathbf{r}}$	$\hat{e}_\theta = \hat{\boldsymbol{\theta}}$	$\hat{e}_\varphi = \hat{\boldsymbol{\varphi}}$	// curvilinear unit basis vectors

$$g'_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad g'^{ab} = \text{inverse}(g'_{ab}) \qquad \text{// metric tensor} \qquad \text{(E.6.8)}$$

where for example $g'_{33} = g_{\varphi\varphi} = r^2\sin^2\theta$. For spherical coordinates only 9 of the 27 elements of Γ'^{d}_{ab} are non-zero (computed from (E.6.7)) :

$$\Gamma'^{1}_{22} = -r \qquad \Gamma'^{2}_{12} = \Gamma'^{2}_{21} = 1/r \qquad \text{// notation example: } \Gamma'^{1}_{22} = \Gamma^{r}_{\theta\theta}$$
$$\Gamma'^{1}_{33} = -r\sin^2\theta \qquad \Gamma'^{3}_{13} = \Gamma'^{3}_{31} = 1/r$$
$$\Gamma'^{2}_{33} = -\cos\theta\sin\theta \qquad \Gamma'^{3}_{23} = \Gamma'^{3}_{32} = \cot\theta\,. \qquad \text{(E.6.9)}$$

<u>Example done in Section 14 notation with (E.6.6) :</u>

$$(\partial\hat{\boldsymbol{e}}_n/\partial\xi_j) = (1/h_n)\,[\Sigma_i(h_i\Gamma^{i}_{jn}\hat{\boldsymbol{e}}_i) - (\partial h_n/\partial\xi_j)\hat{\boldsymbol{e}}_n]$$

$$(\partial\hat{\boldsymbol{e}}_3/\partial\xi_2) = (1/h_3)\,[\Sigma_i(h_i\Gamma^{i}_{23}\hat{\boldsymbol{e}}_i) - (\partial h_3/\partial\xi_2)\hat{\boldsymbol{e}}_3]$$

$$(\partial\hat{\boldsymbol{\varphi}}/\partial\theta) = (1/r\sin\theta)\,[(h_3\Gamma^3{}_{23}\hat{\boldsymbol{e}}_3) - (\partial[r\sin\theta]/\partial\theta)\hat{\boldsymbol{e}}_3]$$

$$= (1/r\sin\theta)\,[(r\sin\theta * \cot\theta * \hat{\boldsymbol{\varphi}}) - r\cos\theta * \hat{\boldsymbol{\varphi}}]$$

$$= (1/r\sin\theta)\,[r\cos\theta\,\hat{\boldsymbol{\varphi}} - r\cos\theta\,\hat{\boldsymbol{\varphi}}]$$

$$= 0 \qquad \text{(E.6.10)}$$

and with some effort we have verified that $\partial_\theta\hat{\boldsymbol{\varphi}} = 0$ as appears in (E.6.1). This fact is of course obvious just looking at Fig (E.1.1), but things can be less obvious in obscure coordinate systems.