Show that Cartesian EOM give Angular EOM PhL 1.13.17

• In Section F.5 I use the torque equations to obtain angular equations of motion as follows

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad (*) \qquad (F.5.7)$$

• In Section F.6 I use force equations to verify each of the above equations.

• In Section F.8 I derive two EOM directly in Cartesian coordinates, they are

1 $y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}$ // when converted to θ,φ this matches (F.5.7)

2 $z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x\dot{y}$ where $r'_1{}^2 = r_1{}^2 + b^2 + 2bz$ (**)

• In Maple "show cart EOM equals ang EOM.mws" I show the first equation of (**) replicates the second equation of (*), giving then a triple derivation of this equation.

I think that the first equation of (*) is a certain linear combination of the two equations in (**). When I have Maple apply this linear combination, I obtain the first equation of (*) BUT the red term is missing. This term is part of the "constant" in that equation. I spent about 2 hours yesterday trying to find the error but failed, so today I will resume.

One bedtime idea is to write all equations assuming velocities are all 0, which should bring a focus on the constant terms! So start with (F.8.1) and examine only constant terms:

$$\mathbf{F}_{\mathbf{eff},1} = m_1\,\mathbf{a}_1 \qquad (F.6.5) \qquad (F.8.1)$$

$$\approx \underbrace{-(GM_E m_1/r'_1{}^3)\mathbf{r}'_1}_{1} \underbrace{- T\,\hat{\mathbf{r}}_1}_{2} \underbrace{- m_1\ddot{\mathbf{b}}_{S'}}_{3} \underbrace{- m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)}_{4} \underbrace{- 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1}_{5} \underbrace{- m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1}_{6}.$$

Term 1: $-(GM_E m_1/r'_1{}^3)\mathbf{r}'_1 = -(GM_E m_1/r'_1{}^3)(\mathbf{b} + \mathbf{r}_1)$

$$= -(GM_E m_1/r'_1{}^3)\,[\,b\hat{\mathbf{z}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,]$$

$$= -(GM_E m_1/r'_1{}^3)\,[\,x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (b+z)\hat{\mathbf{z}}\,] \qquad (F.8.3)$$

Term 2: $-T\,\hat{\mathbf{r}}_1 = -(T/r_1)\mathbf{r}_1 = -(T/r_1)\,[\,x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,]$ (F.8.4)

Term 3: $-m_1\ddot{\mathbf{b}}_{S'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // (F.6.3)

$= -\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // satellite in circular orbit, $\dot{\boldsymbol{\omega}} = 0$

$= -\,m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b}$ //$-\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = -(\mathbf{A}\bullet\mathbf{C})\mathbf{A} + A^2\mathbf{C}$

$= m_1\omega^2\mathbf{b} \quad = m_1\omega^2 b\hat{\mathbf{z}}$ (F.8.5)

Term 4: $-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1$ // identity shown above

$= -m_1\omega^2(\hat{\mathbf{x}}\bullet\mathbf{r}_1)\hat{\mathbf{x}} + m_1\omega^2\mathbf{r}_1 = -m_1\omega^2(x)\hat{\mathbf{x}} + m_1\omega^2[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]$

$= m_1\omega^2[\, y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]$ (F.8.6)

Terms 5 and 6 make no contribution. So unfortunately MOST terms contribute to the constant part.

I now assemble the equations

$\mathbf{F}_{\mathbf{eff},1} = m_1\,\mathbf{a}$ *(F.8.1)*

$\approx -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$

1 2 3 4 5 6

$\hat{\mathbf{x}}$: $m_1 a_x = -(GM_E m_1/r'^3_1)\,x - (T/r_1)x$ ok

1 2

$\hat{\mathbf{y}}$: $m_1 a_y = -(GM_E m_1/r'^3_1)\,y - (T/r_1)y + m_1\omega^2 y + 2m_1\omega v_z$ ok

1 2 4

$\hat{\mathbf{z}}$: $m_1 a_z = -(GM_E m_1/r'^3_1)\,(b+z) - (T/r_1)z + m_1\omega^2 b + m_1\omega^2 z - 2m_1\omega v_y$ ok (F.8.9)

1 2 3 4

I then rewrite the three equations :

$\ddot{x} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1)]x$ ok

$\ddot{y} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1) - \omega^2]y + 2\omega v_z$ ok

$\ddot{z} = -[(\omega^2 b^3/r'^3_1) - \omega^2](b+z) - (T/m_1 r_1)z - 2\omega v_y$ ok

I then define

$A \equiv (\omega^2 b^3/r'^3_1)$

$B \equiv (T/m_1 r_1)$ // rescaled tension

and equations become

1 $\ddot{x} = -(A + B)x$ ok

2 $\ddot{y} = -(A + B - \omega^2)y + 2\omega v_z$ ok

3 $\ddot{z} = -(A - \omega^2)(b+z) - Bz - 2\omega v_y$ ok

I eliminate B from the first two equations and end up with

$y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega z v_z$

When this equation is "converted to θ,φ" I get exactly the second equation of (F.5.7) so I don't have doubts about the above elimination.

I then eliminate B from the first and third equations

1*z $z\ddot{x} = -(A + B)xz$ ok

3*x $x\ddot{z} = -(A - \omega^2)(b+z)x - Bxz - 2\omega x v_y$ ok

$= -Abx - Azx + \omega^2(b+z)x - Bxz - 2\omega x v_y$ ok

$= -(A + B)xz - Abx + \omega^2(b+z)x - 2\omega x v_y$ ok

Subtract so that the -(A + B)xy terms cancel,

$z\ddot{x} - x\ddot{z} = Abx - \omega^2(b+z)x + 2\omega x v_y$ (F.8.14)

At this point then the two Cartesian equations are

1 $y\ddot{x} - x\ddot{y} = -\omega^2 xy$ // when converted to θ,φ this matches (F.5.7)

2 $z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x v_y$

A supporting equation is this

$r'_1{}^2 = (\mathbf{r}_1+\mathbf{b})^2 = r_1^2 + b^2 + 2\,\mathbf{r}_1 \bullet \mathbf{b} = r_1^2 + b^2 + 2\,\mathbf{r}_1 \bullet [b\hat{\mathbf{z}}] = r_1^2 + b^2 + 2bz\,.$ ok (F.8.11)

I then do the far approximation for the first term in 2 :

$r'_1{}^2 = r_1^2 + b^2 + 2bz = b^2[\,1 + (r_1/b)^2 + 2(z/b)] \approx b^2[\,1 + 2(z/b)\,]$ ok

$r'_1{}^{-2} \approx b^{-2}[1-2(z/b)]$ ok

$r'_1{}^{-3} \approx b^{-3}[1-2(z/b)]^{3/2} \approx b^{-3}[\,1 + (3/2)(-2(z/b)] = b^{-3}[\,1 - 3(z/b)]$ ok

so

$(b/r'_1)^3 = 1 - 3(z/b)$ ok

Equation 2 above then becomes

$z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x v_y$ ok

$= \omega^2[1 - 3(z/b)]bx - \omega^2(b+z)x + 2\omega x v_y$ ok

$= \omega^2 bx - 3(z/b)\omega^2 bx - \omega^2 bx - \omega^2 zx + 2\omega x v_y$ ok

$= -3(z/b)\omega^2 bx - \omega^2 zx + 2\omega x v_y$ ok

$= -3\omega^2 xz - \omega^2 zx + 2\omega x v_y$ ok

$= -4\omega^2 xz + 2\omega x v_y$ ok

My two equations in the far approximation are then

2 $y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega z v_z$ // when converted to θ,φ this matches (F.5.7)

1 $z\ddot{x} - x\ddot{z} = -4\omega^2 xz + 2\omega x v_y$ // renumber equations here!

I now see that I must include the velocity terms after all because they will contribute when I convert to the θ and φ coordinates! These come only from Term 5.

Term 5: $-2m_1 \boldsymbol{\omega} \times \mathbf{v}_1 = -2m_1 [\omega\hat{\mathbf{x}}] \times (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$

$= -2m_1\omega [v_y\hat{\mathbf{z}} - v_z\hat{\mathbf{y}}]$ (F.8.7)

so these add terms to the z and y equations above. Go install these now in blue. Done.

I now go to "show cart EOM....mws" and I enter equations which will do the conversion of these EOM's from Cart to Spher coordinates:

```
> x := r1*sin(theta(t))*cos(phi(t));
```

$$x := r1 \sin(\theta(t)) \cos(\phi(t))$$

```
> y := r1*sin(theta(t))*sin(phi(t));
```

$$y := r1 \sin(\theta(t)) \sin(\phi(t))$$

```
> z := r1*cos(theta(t));
>
```

$$z := r1 \cos(\theta(t))$$

```
> xd := diff(x,t);
```

$$xd := r1 \cos(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right) \cos(\phi(t)) - r1 \sin(\theta(t)) \sin(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
> yd := diff(y,t);
```

$$yd := r1 \cos(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right) \sin(\phi(t)) + r1 \sin(\theta(t)) \cos(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
> zd := diff(z,t);
```

$$zd := -r1 \sin(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right)$$

```
> xdd := diff(x,t,t);
```

$$xdd := -r1 \sin(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \cos(\phi(t)) + r1 \cos(\theta(t)) \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) \cos(\phi(t)) - 2\, r1 \cos(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right) \sin(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)$$
$$- r1 \sin(\theta(t)) \cos(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)^2 - r1 \sin(\theta(t)) \sin(\phi(t)) \left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
> ydd := diff(y,t,t);
```

$$ydd := -r1 \sin(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \sin(\phi(t)) + r1 \cos(\theta(t)) \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) \sin(\phi(t)) + 2\, r1 \cos(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right) \cos(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)$$
$$- r1 \sin(\theta(t)) \sin(\phi(t)) \left(\frac{\partial}{\partial t}\phi(t)\right)^2 + r1 \sin(\theta(t)) \cos(\phi(t)) \left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
>
> zdd := diff(z,t,t);
```

$$zdd := -r1 \cos(\theta(t)) \left(\frac{\partial}{\partial t}\theta(t)\right)^2 - r1 \sin(\theta(t)) \left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

I then use these objects in my entry of the two equations of interest.

<u>Here is how I process equation 2</u>

2 $\ddot{y}x - \ddot{x}y = -\omega^2 xy - 2\omega z v_z$

First you see it properly entered as eq2. I divide by r_1^2 , make a subs

Process equation 2

```
> eq2 := y*xdd-x*ydd +w^2*x*y + 2*w*x*zd =0: eq2 := simplify(%);
```

$$eq2 := -2\,rl^2 \sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) - rl^2\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + rl^2\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)\cos(\theta(t))^2 + w^2\,rl^2\cos(\phi(t))\sin(\phi(t))$$
$$- w^2\,rl^2\cos(\phi(t))\sin(\phi(t))\cos(\theta(t))^2 - 2\,w\,rl^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right) + 2\,w\,rl^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\theta(t))^2 = 0$$

```
> eq2_ :=lhs(eq2)/r1^2:simplify(%);
```

$$-2\sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) - \left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + \left(\frac{\partial^2}{\partial t^2}\phi(t)\right)\cos(\theta(t))^2 + w^2\cos(\phi(t))\sin(\phi(t))$$
$$- w^2\cos(\phi(t))\sin(\phi(t))\cos(\theta(t))^2 - 2\,w\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right) + 2\,w\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\theta(t))^2$$

```
> subs(cos(theta(t))^2=1-sin(theta(t))^2,eq2_);
```

$$\left(-2\,rl^2\sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) - rl^2\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + rl^2\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)(1-\sin(\theta(t))^2) + w^2\,rl^2\cos(\phi(t))\sin(\phi(t))\right.$$
$$\left. - w^2\,rl^2\cos(\phi(t))\sin(\phi(t))(1-\sin(\theta(t))^2) - 2\,w\,rl^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right) + 2\,w\,rl^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)(1-\sin(\theta(t))^2)\right) \Big/ rl^2$$

```
> expand(%);
```

$$-2\sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) - \left(\frac{\partial^2}{\partial t^2}\phi(t)\right)\sin(\theta(t))^2 + w^2\sin(\theta(t))^2\cos(\phi(t))\sin(\phi(t))$$
$$- 2\,w\sin(\theta(t))^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$

Equation 2 matches (F.5.7) second equation !!!!

I want to compare this equation to

$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0$$

For comparison, multiply the above equation by $-\sin^2\theta$

$$-\sin^2\theta\ddot{\varphi} - 2\dot{\theta}\,\dot{\varphi}\cos\theta\sin\theta + \sin^2\theta\omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0$$

or

$$-\sin^2\theta\ddot{\varphi} - 2\dot{\theta}\,\dot{\varphi}\cos\theta\sin\theta + \omega^2\sin^2\theta\cos\varphi\sin\varphi - \sin^2\theta\omega\cos\varphi\,2\dot{\theta} = 0$$

Now reorder the terms

$$-2\dot{\theta}\,\dot{\varphi}\cos\theta\sin\theta - \sin^2\theta\ddot{\varphi} + \omega^2\sin^2\theta\cos\varphi\sin\varphi - 2\omega\sin^2\theta\cos\varphi\,\dot{\theta} = 0$$

This **agrees exactly** with the last blue Maple equation above, so I conclude that the Cart equation 2 is exactly the same as the angular equation 2.

<u>Now look at eq1.</u>

Process equation 1

```
> eq1 := z*xdd-x*zdd + 4*w^2*x*z - 2*w*x*yd =0: eq1 := simplify(%);
```

$$q1 := -2\,rl^2\cos(\theta(t))^2\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) - rl^2\cos(\theta(t))\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2$$
$$- rl^2\cos(\theta(t))\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + rl^2\cos(\phi(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + 4\,w^2\,rl^2\sin(\theta(t))\cos(\phi(t))\cos(\theta(t))$$
$$- 2\,w\,rl^2\sin(\theta(t))\cos(\phi(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) - 2\,w\,rl^2\cos(\phi(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right) + 2\,w\,rl^2\cos(\phi(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)\cos(\theta(t))^2 = 0$$

(1)

You can see that I have properly entered my equation 1 which is this

$$1 \quad z\ddot{x} - x\ddot{z} = -4\omega^2 xz + 2\omega x v_y \qquad (2)$$

Here is the equation I am aiming for, taken from (*) above,

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad (3)$$

This target equation (3) has no $\dot{\theta}\,\dot{\varphi}$ term, but Maple (*) for eq1 does have such a term. This tells me that in order to get (3), I will have to do a linear combination of Maple eq2 and eq1 so as to cancel the $\dot{\theta}\,\dot{\varphi}$ term.

So at this point in Maple we have

$$eq1 := -2r_1^2\cos^2\theta\,\sin\varphi\,\dot{\theta}\,\dot{\varphi} + \text{stuff}$$

$$eq2: = -2r_1^2\sin\theta\cos\theta\,\dot{\theta}\,\dot{\varphi} + \text{stuff}$$

Consider now

$$eq1 * \sin\theta = -2r_1^2\sin\theta\cos^2\theta\,\sin\varphi\,\dot{\theta}\,\dot{\varphi} + \text{stuff}$$

$$eq2 * \cos\theta\sin\varphi = -2r_1^2\sin\theta\cos^2\theta\,\sin\varphi\,\dot{\theta}\,\dot{\varphi} + \text{stuff}$$

Therefore this $\dot{\theta}\,\dot{\varphi}$ should cancel if I make this definition

$$eq3 := \sin\theta * \text{lhs}(eq1) - \cos\theta\sin\varphi * \text{lhs}(eq2)$$

I will now try this linear combination in Maple :

```
> eq3:= (sin(theta(t))*lhs(eq1)-cos(theta(t))*sin(phi(t))*lhs(eq2))/r1^2:
> eq3a := expand(eq3);
```

$$eq3a := -\cos(\theta(t))\sin(\theta(t))^2\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 - \cos(\theta(t))\sin(\theta(t))^2\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + \sin(\theta(t))\cos(\phi(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$
$$+ 4w^2\sin(\theta(t))^2\cos(\phi(t))\cos(\theta(t)) - 2w\sin(\theta(t))^2\cos(\phi(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) - 2\sin(\theta(t))w\cos(\phi(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)$$
$$+ 2\sin(\theta(t))w\cos(\phi(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)\cos(\theta(t))^2 + \cos(\theta(t))\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) - \cos(\theta(t))^3\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$
$$- \cos(\theta(t))\sin(\phi(t))^2w^2\cos(\phi(t)) + \cos(\theta(t))^3\sin(\phi(t))^2w^2\cos(\phi(t)) + 2\cos(\theta(t))\sin(\phi(t))w\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$- 2\cos(\theta(t))^3\sin(\phi(t))w\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$

Looking through the terms we see that there is no $\dot{\theta}\,\dot{\varphi}$ term, so the desired cancellation has been achieved. A guiding term here is $\sin\theta\cos\varphi\,\ddot{\theta}$. Now we need to simplify getting rid of $\cos^3(\theta)$:

```
> eq3b := subs(cos(theta(t))^3=cos(theta(t))*(1-sin(theta(t))^2),eq3a):
> eq3c :=factor(eq3b);
```

$$eq3c := -\sin(\theta(t))\cos(\phi(t))\left(2\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)w - 2\cos(\phi(t))\cos(\theta(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)w + \sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2\cos(\theta(t))\right.$$
$$\left. - 4\sin(\theta(t))\cos(\theta(t))w^2 + \sin(\theta(t))w^2\sin(\phi(t))^2\cos(\theta(t)) - \left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\right)$$

Next divide out the leading factor $\sin\theta\cos\varphi$ and then do another subs:

```
> eq3d := eq3c/(sin(theta(t))*cos(phi(t)));
```

$$eq3d := -\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2\cos(\theta(t)) + \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + 4\sin(\theta(t))\cos(\theta(t))w^2 - 2\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)w + 2\cos(\phi(t))\cos(\theta(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)w$$
$$- \sin(\theta(t))w^2\sin(\phi(t))^2\cos(\theta(t))$$

```
> eq3e := subs(cos(theta(t))^2=1-sin(theta(t))^2,eq3d);
```

$$eq3e := -\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2\cos(\theta(t)) + \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + 4\sin(\theta(t))\cos(\theta(t))w^2 - 2\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)w + 2\cos(\phi(t))\left(1-\sin(\theta(t))^2\right)\left(\frac{\partial}{\partial t}\phi(t)\right)w$$
$$- \sin(\theta(t))w^2\sin(\phi(t))^2\cos(\theta(t))$$

I will now manually translate the final result of 6 terms

$$-\sin\theta\cos\theta\,\dot{\varphi}^2 + \ddot{\theta} + 4\omega^2\sin\theta\cos\theta - 2\omega\cos\varphi\,\dot{\varphi} + 2\omega\cos^2\theta\cos\varphi\,\dot{\varphi} - \omega^2\sin\theta\cos\theta\sin^2\varphi$$

and I now reorder these terms

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 + 4\omega^2\sin\theta\cos\theta - \omega^2\sin\theta\cos\theta\sin^2\varphi + 2\omega\cos\varphi\,(\cos^2\theta - 1)\,\dot{\varphi}$$

or

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ + $4\omega^2$sinθcosθ - ω^2sinθcosθ$\sin^2\varphi$ - 2ωcosφ$\sin^2\theta$ $\dot{\varphi}$ A = Maple

Now the target here is this from (*) above

$\ddot{\theta}$ + sinθcosθ($3\omega^2$ - $\dot{\varphi}^2$) + ωsinθcosφ (ω cosθ cosφ - 2sinθ $\dot{\varphi}$) = 0

or

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ + $3\omega^2$sinθcosθ + ω^2sinθcosθ$\cos^2\varphi$ - 2ω$\sin^2\theta$cosφ $\dot{\varphi}$ B = target

ok ok ok

If I can show that the "constant terms" are the same, then the game is won!

$4\omega^2$sinθcosθ - ω^2sinθcosθ$\sin^2\varphi$ = $3\omega^2$sinθcosθ + ω^2sinθcosθ$\cos^2\varphi$?

$4\omega^2$cosθ - ω^2cosθ$\sin^2\varphi$ = $3\omega^2$cosθ + ω^2cosθ$\cos^2\varphi$?

$4\omega^2$cosθ - ω^2cosθ(1-$\cos^2\varphi$) = $3\omega^2$cosθ + ω^2cosθ$\cos^2\varphi$?

$4\omega^2$cosθ - ω^2cosθ + ω^2cosθ$\cos^2\varphi$ = $3\omega^2$cosθ + ω^2cosθ$\cos^2\varphi$?

$3\omega^2$cosθ + ω^2cosθ$\cos^2\varphi$ = $3\omega^2$cosθ + ω^2cosθ$\cos^2\varphi$ yes !!!!!!!

So finally it works, hurray hurray !!!!!!

Now how can I do all this more efficiently! here again is the target equation:

$\ddot{\theta}$ + sinθcosθ($3\omega^2$ - $\dot{\varphi}^2$) + ωsinθcosφ (ω cosθ cosφ - 2sinθ $\dot{\varphi}$) = 0

$\ddot{\theta}$ + $3\omega^2$sinθcosθ -sinθcosθ $\dot{\varphi}^2$ + ω^2sinθcosθ$\cos^2\varphi$ - 2ω$\sin^2\theta$cosφ $\dot{\varphi}$

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ + [$3\omega^2$sinθcosθ + ω^2sinθcosθ$\cos^2\varphi$] - 2ω$\sin^2\theta$cosφ $\dot{\varphi}$

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ + ω^2sinθcosθ[3 + $\cos^2\varphi$] - 2ω$\sin^2\theta$cosφ $\dot{\varphi}$

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ - 2ω$\sin^2\theta$cosφ $\dot{\varphi}$ + ω^2sinθcosθ(3 + $\cos^2\varphi$)

$\ddot{\theta}$ - sinθcosθ $\dot{\varphi}^2$ +ωsinθ(-2sinθcosφ$\dot{\varphi}$ + ωcosθ(3+$\cos^2\varphi$))

Nothing is very elegant.