What is fictitious torque for tether situation? PhL ~12.6.16

New material, keep track of it.

Aside: What is the fictional torque for the tether situation?

Ignoring diff between cog and cms we have

$$\mathbf{b} = R_E \hat{\mathbf{r}}$$

$$\dot{\mathbf{b}}_S = \boldsymbol{\omega} \times \mathbf{b}$$

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad (7.13) \qquad \text{Special Case 1}$$

We also have **c'** = torque center in Frame S' = 0 so **c** = **b** and $\dot{\mathbf{c}} = \dot{\mathbf{b}}_S$. Then

$$\mathbf{F}_{fict} = -m[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.6)$$

$$\mathbf{N}^{(b)}{}_{fict} = (\mathbf{r}'-\mathbf{0}) \times \mathbf{F}_{fict} + (\mathbf{0} + \boldsymbol{\omega} \times \mathbf{0} + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - \mathbf{0} \times \mathbf{v}'$$

$$= \mathbf{r}' \times \mathbf{F}_{fict} + \dot{\mathbf{b}}_S \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) = \mathbf{r}' \times \mathbf{F}_{fict} + \dot{\mathbf{b}}_S \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}')$$

$$= \mathbf{r}' \times \mathbf{F}_{fict} + (\boldsymbol{\omega} \times \mathbf{b}) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}')$$

At least it can be written down! Also

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = (\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\boldsymbol{\omega})\mathbf{b} = 0 - \omega^2 b = -\omega^2 \mathbf{b}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = (\boldsymbol{\omega}\bullet\mathbf{r}')\boldsymbol{\omega} - (\boldsymbol{\omega}\bullet\boldsymbol{\omega})\mathbf{r}' = -\omega^2 \mathbf{r}'$$

so then

$$\mathbf{F}_{fict} = -m\,\dot{\boldsymbol{\omega}} \times \mathbf{b} + m\,\omega^2 \mathbf{b} + m\omega^2 \mathbf{r}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

$$= -m\,\dot{\boldsymbol{\omega}} \times (\mathbf{b}+\mathbf{r}') - m\omega^2(\mathbf{b}+\mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

$$= -m\,\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\omega^2 \mathbf{r} - 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

Euler centr Coriolis

Is $\mathbf{v'_1}$ in the plane of paper? It is constrained to be along the circle of particle 1 if it lies in the plane of paper, so that is what I will assume for the moment to get maximum simplicity in searching for an EOM. Now.

$$(\boldsymbol{\omega} \times \mathbf{b}) \times (\boldsymbol{\omega} \times \mathbf{r'}) = [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega} - [\boldsymbol{\omega},\mathbf{b},\boldsymbol{\omega}]\,\mathbf{r'} = [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega}$$

$$= \mathbf{r'} \bullet (\boldsymbol{\omega} \times \mathbf{b})$$

so then

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = \mathbf{r'} \times \mathbf{F}_{\mathbf{fict}} + (\boldsymbol{\omega} \times \mathbf{b}) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'})$$

$$= \mathbf{r'} \times \mathbf{F}_{\mathbf{fict}} + [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega} + (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v'}$$

$$= -\mathbf{r'} \times \left[m\,\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\omega^2\mathbf{r} + 2m\,\boldsymbol{\omega} \times \mathbf{v'} \right] + [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega} + (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v'}$$

But now write

$$\mathbf{r'} \times \mathbf{r} = \mathbf{r'} \times (\mathbf{b}+\mathbf{r'}) = \mathbf{r'} \times \mathbf{b}$$

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = = -\mathbf{r'} \times \left[m\,(\dot{\boldsymbol{\omega}} \times \mathbf{r}) + m\omega^2\mathbf{b} + 2m\,\boldsymbol{\omega} \times \mathbf{v'} \right] + [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega} + (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v'}$$

Now I will of course set $\dot{\boldsymbol{\omega}} = \mathbf{0}$ to get

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = = -\mathbf{r'} \times \left[m\omega^2\mathbf{b} + 2m\,\boldsymbol{\omega} \times \mathbf{v'} \right] + [\boldsymbol{\omega},\mathbf{b},\mathbf{r'}]\,\boldsymbol{\omega} + (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v'}$$

Consider now

$$\mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{v'})$$

But $\boldsymbol{\omega} \times \mathbf{v'}$ is in the $\mathbf{r'}$ direction so this term does nothing (thinking r_1 for example in my tether picture). So

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = = -\mathbf{r'} \times \left[m\omega^2\mathbf{b} \right] + [\mathbf{r'} \bullet (\boldsymbol{\omega} \times \mathbf{b})]\,\boldsymbol{\omega} + (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v'}$$

How does this compare to the non-fictitious torque?

$$\mathbf{N}^{(b)} = \mathbf{r'_1} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2}$$

We know first that

$m_1\mathbf{r'_1} + m_2\mathbf{r'_2} = 0$ since cms is center of circle, we are staying $\mathbf{r'_{cms}} = 0$

$m_1 r'_1 \hat{\mathbf{r}}'_1 + m_2 r'_2 \hat{\mathbf{r}}'_2 = 0$

$m_1 r'_1 \hat{\mathbf{r}}'_1 - m_2 r'_2 \hat{\mathbf{r}}'_1 = 0$

$m_1 r'_1 - m_2 r'_2 = 0$

$m_1 r'_1 = m_2 r'_2$

$\hat{\mathbf{r}}'_1 = - \hat{\mathbf{r}}'_2$

Second, we know that,

$\mathbf{F}_1 = -(G m_1 M_E / r_1^3)\ \mathbf{r}_1$
$\mathbf{F}_2 = -(G m_2 M_E / r_2^3)\ \mathbf{r}_2$

So then

$\mathbf{N}^{(b)} = \mathbf{r}'_1 \times (-G m_1 M_E / r_1^3)\ \mathbf{r}_1 + \mathbf{r}'_2 \times (-G m_2 M_E / r_2^3)\ \mathbf{r}_2$

$= r'_1\ \hat{\mathbf{r}}'_1 \times (-G m_1 M_E / r_1^3)\ \mathbf{r}_1 + r'_2\ \hat{\mathbf{r}}'_2 \times (-G m_2 M_E / r_2^3)\ \mathbf{r}_2$

$= -(G M_E)\ [m_1 r'_1\ \hat{\mathbf{r}}'_1 \times (1/r_1^3)\ \mathbf{r}_1 + m_2\ r'_2\ \hat{\mathbf{r}}'_2 \times (1/r_2^3)\ \mathbf{r}_2\ ]$

$= -(G M_E)\ m_1 r'_1\ [\ \hat{\mathbf{r}}'_1 \times (1/r_1^3)\ \mathbf{r}_1 + \hat{\mathbf{r}}'_2 \times (1/r_2^3)\ \mathbf{r}_2\ ]$

$= -(G M_E)\ m_1 r'_1\ [\ \hat{\mathbf{r}}'_1 \times (1/r_1^3)\ \mathbf{r}_1 - \hat{\mathbf{r}}'_1 \times (1/r_2^3)\ \mathbf{r}_2\ ]$

What about this cross products? What have I done before with them?

$\hat{\mathbf{r}}'_1 \times \mathbf{r}_1 =$ $\alpha_1\ \ \alpha_2$

I think that

$\beta_1 + (\theta' + \pi/2) + \alpha_1 = \pi$

so that

$\beta_1 = \pi/2 - (\theta' + \alpha_1)$

$\hat{\mathbf{r}}'_1 \times \hat{\mathbf{r}}_1 = \sin(\beta_1)\ \hat{\mathbf{z}}' = \sin[\pi/2 - (\theta' + \alpha_1)]\ \hat{\mathbf{z}}' = \cos(\theta' + \alpha_1)\ \hat{\mathbf{z}}'$

On the other hand

$(\pi/2 - \theta') + (\pi - \delta) + \alpha_2 = \pi$

or

$(\pi/2 - \theta') - \delta + \alpha_2 = 0$

so

$\delta = (\pi/2 - \theta') + \alpha_2 = \pi/2 - (\theta' - \alpha_2)$

$\hat{\mathbf{r}}'_1 \times \hat{\mathbf{r}}_2 = \sin(\delta)\,\hat{\mathbf{z}}' = \sin[\pi/2 - (\theta' - \alpha_2)]\,\hat{\mathbf{z}}' = \cos(\theta' - \alpha_2)\,\hat{\mathbf{z}}'$

Now we can go back:

$\mathbf{N}^{(b)} = -(GM_E)\, m_1\, r'_1 \left[\hat{\mathbf{r}}'_1 \times (1/r_1^3)\,\mathbf{r}_1 - \hat{\mathbf{r}}'_1 \times (1/r_2^3)\,\mathbf{r}_2 \right]$

$= -(GM_E)\, m_1\, r'_1 \left[\hat{\mathbf{r}}'_1 \times (1/r_1^2)\,\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}'_1 \times (1/r_2^2)\,\hat{\mathbf{r}}_2 \right]$

$= -(GM_E)\, m_1\, r'_1 \left[(1/r_1^2)\,\hat{\mathbf{r}}'_1 \times \hat{\mathbf{r}}_1 + (1/r_2^2)\hat{\mathbf{r}}'_1 \times \hat{\mathbf{r}}_2 \right]$

$= -(GM_E)\, m_1\, r'_1 \left[(1/r_1^2)\cos(\theta' + \alpha_1)\,\hat{\mathbf{z}}' - (1/r_2^2)\cos(\theta' - \alpha_2)\,\hat{\mathbf{z}}' \right]$

$= -(GM_E)\, m_1\, r'_1 \left[(1/r_1^2)\cos(\theta' + \alpha_1) - (1/r_2^2)\cos(\theta' - \alpha_2) \right]\hat{\mathbf{z}}'$

As expected, the two forces oppose each other with a small difference.

$\mathbf{b} + \mathbf{r}'_1 = \mathbf{r}_1$

$b^2 + r'^2_1 + 2br'_1 \cos(\pi/2 - \theta') = b^2 + r'^2_1 + 2br'_1 \sin(\theta') = r_1^2$

$\mathbf{b} + \mathbf{r}'_2 = \mathbf{r}_2$

$b^2 + r'^2_2 + 2br'_2 \cos[\pi - (\pi/2 - \theta')] = b^2 + r'^2_1 + 2br'_1 \cos[\pi/2 + \theta')] = b^2 + r'^2_1 - 2br'_1 \sin\theta'$

So we have

$r_1^2 = b^2 + r'^2_1 + 2br'_1 \sin(\theta')$

$r_2^2 = b^2 + r'^2_2 - 2br'_2 \sin(\theta')$

But I need to get rid of α_1 and α_2 somehow.

$\sin\alpha_1 = r'_1 \cos\theta' / r_1$

$\sin\alpha_2 = r'_2 \cos\theta' / r_2$

Here is our result so far:

$\mathbf{N}^{(b)} = -(GM_E)\, m_1\, r'_1 \left[(1/r_1^2)\cos(\theta' + \alpha_1) - (1/r_2^2)\cos(\theta' - \alpha_2) \right]\hat{\mathbf{z}}'$

$r_1^2 = b^2 + r'^2_1 + 2br'_1 \sin(\theta')$ $\qquad$ $\sin\alpha_1 = r'_1 \cos\theta' / r_1$

$r_2^2 = b^2 + r'^2_2 - 2br'_2 \sin(\theta')$ $\quad\quad$ $\sin\alpha_2 = r'_2\cos\theta' / r_2$

Now we can approximate noting the α_i are very small angles for tether, so

$\cos(\theta'+\alpha_1) \approx \cos\theta' + \alpha_1 [-\sin\theta'] = \cos\theta' - \alpha_1 \sin\theta'$

$\cos(\theta'-\alpha_2) \approx \cos\theta' - \alpha_2 [-\sin\theta'] = \cos\theta' + \alpha_2 \sin\theta'$

$\sin\alpha_1 \approx \alpha_1 = \cos\theta' \ (r'_1/ r_1) =$ very small

$\sin\alpha_2 \approx \alpha_2 = \cos\theta' \ (r'_2/ r_2) =$ very small

Then

$$\mathbf{N}^{(b)} = -(GM_E)\, m_1\, r'_1 \left[(1/r_1^2) \ [\cos\theta' - \alpha_1 \sin\theta'] - (1/r_2^2) [\cos\theta' + \alpha_2 \sin\theta']\hat{\mathbf{z}}'\right.$$

Now what is the size of

$(1/r_1^2) - (1/r_2^2) = [r_2^2 - r_1^2] / (r_1^2 r_2^2)$

$\approx [r_2^2 - r_1^2] / b^4 = [(b^2 + r'^2_2 - 2br'_2 \sin(\theta')) - (b^2 + r'^2_1 + 2br'_1 \sin(\theta'))] / b^4$

$= [b^2 + r'^2_2 - 2br'_2 \sin(\theta') - b^2 - r'^2_1 - 2br'_1 \sin(\theta')] / b^4$

$= [r'^2_2 - 2br'_2 \sin(\theta') - r'^2_1 - 2br'_2 \sin(\theta')] / b^4$

$= [(r'^2_2 - r'^2_1) - 2b(r'_2+r'_1)\sin\theta'] /b^4$

Then

$(1/r_1^2) \cos\theta' - (1/r_2^2) \cos\theta'$

$\approx \cos\theta' [(r'^2_2 - r'^2_1) - 2b(r'_2+r'_1)\sin\theta'] /b^4$

But the second term here is much larger than the first term since $b >> r'_i$ so we then get

$(1/r_1^2) \cos\theta' - (1/r_2^2) \cos\theta' \approx - 2b(r'_2+r'_1)\sin\theta' \cos\theta' / b^4 = -(r'_2+r'_1)\sin(2\theta') / b^3$

Meanwhile, the other two terms are

$(1/r_1^2)[- \alpha_1 \sin\theta'] - (1/r_2^2) [\alpha_2 \sin\theta']$

$= - \sin\theta' [\alpha_1/r_1^2 + \alpha_2/r_2^2]$

$\approx \sin\theta' [\ \cos\theta'\ \ (r'_1/ r_1) / r_1^2 + \cos\theta'\ \ (r'_2/ r_2) / r_2^2\ ]$

$= \sin\theta'\cos\theta' [\ r'_1/r_1^3 + r'_2/r_2^3\ ]$

$\approx \sin\theta'\cos\theta' [2(r'_1+ r'_2) /b^3]$

$= + (r'_2+r'_1) \sin(2\theta') /b^3$

Yikes! The two pairs of terms exactly cancel leaving no torque at all.

dimension check: $\mathbf{N}^{(b)} \sim -(GM_E)\, m_1\, r'_1 [\ (1/r_1^2)$ = F L = r x F = correct

$\mathbf{N}^{(b)} = -(GM_E)\, m_1\, r'_1 [\ -(r'_2+r'_1)\sin(2\theta') / b^3 + (r'_2+r'_1) \sin(2\theta') /b^3\ ]\hat{\mathbf{z}}'$

Here are the two terms cancelling.

Conclusion if I did this right: To order r_1'/b^3 there is no torque on the dumbbell. So if the dumbbell satellite were just hung from a string attached to some distant star (satellite does not orbit earth) this would be the situation. You would have to examine higher terms to find a non-zero torque.

Question: If the satellite does orbit, so there is a fictitious torque, how does that torque compare to what is computed above?

$$\mathbf{N}^{(b)}_{fict} = -\mathbf{r}' \times [\ m\omega^2\mathbf{b}]\ +[\mathbf{r}' \bullet (\boldsymbol{\omega} \times \mathbf{b})]\ \boldsymbol{\omega}\ +\ (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'$$

Well add these for the two particles

$$\mathbf{N}^{(b)}_{fict} = -\mathbf{r}_1' \times [\ m_1\omega^2\mathbf{b}]\ +[\mathbf{r}'_1 \bullet (\boldsymbol{\omega} \times \mathbf{b})]\ \boldsymbol{\omega}\ +\ (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_1$$

$$-\mathbf{r}_2' \times [\ m_1\omega^2\mathbf{b}]\ +[\mathbf{r}'_2 \bullet (\boldsymbol{\omega} \times \mathbf{b})]\ \boldsymbol{\omega}\ +\ (\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_2$$

Now we have to throw in the fact that

$GM_E/r_0^2 = \omega^2 r_0$

or roughly

$GM_E/b^2 = \omega^2 b$

or

$\omega^2 = GM_E/b^3$ dim =(F/m)/L = (ma/m)/L = a/L = T^{-2} OK

Then we have

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = -\,\mathbf{r}_1' \times \left[\, m_1(GM_E/b^3)\ \mathbf{b}\right] + (GM_E/b^3)[\mathbf{r}'_1 \bullet (\hat{\boldsymbol{\omega}} \times \mathbf{b})]\ \hat{\boldsymbol{\omega}} + \sqrt{GM_E/b^3}\ (\hat{\boldsymbol{\omega}} \times \mathbf{b}) \times \mathbf{v}'_1$$
$$-\,\mathbf{r}_2' \times \left[\, m_2(GM_E/b^3)\mathbf{b}\right] + (GM_E/b^3)[\mathbf{r}'_2 \bullet (\hat{\boldsymbol{\omega}} \times \mathbf{b})]\ \hat{\boldsymbol{\omega}} + \sqrt{GM_E/b^3}\ (\hat{\boldsymbol{\omega}} \times \mathbf{b}) \times \mathbf{v}'_2$$

or

$$\mathbf{N}^{(b)}_{\mathbf{fict}} = -\,\mathbf{r}_1' \times \left[\, m_1(GM_E/b^2)\ \hat{\mathbf{b}}\right] + (GM_E/b^2)[\mathbf{r}'_1 \bullet (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}})]\ \hat{\boldsymbol{\omega}} + \sqrt{GM_E/b}\ (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}}) \times \mathbf{v}'_1$$
$$-\,\mathbf{r}_2' \times \left[\, m_2(GM_E/b^2)\ \hat{\mathbf{b}}\right] + (GM_E/b^2)[\mathbf{r}'_2 \bullet (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}})]\ \hat{\boldsymbol{\omega}} + \sqrt{GM_E/b^3}\ (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}}) \times \mathbf{v}'_2$$

$$= -\,(\ m_1\mathbf{r}_1' + m_2\mathbf{r}'_2) \times \left[(GM_E/b^2)\ \hat{\mathbf{b}}\right] + (GM_E/b^2)[(\ (\mathbf{r}_1' + \mathbf{r}'_2) \bullet (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}})]\ \hat{\boldsymbol{\omega}} + \sqrt{GM_E/b}\ (\mathbf{v}'_1 + \mathbf{v}'_2)$$