

DYNAMICS OF A DUMBBELL SATELLITE UNDER THE ZONAL HARMONIC EFFECT OF AN OBLATE BODY

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ABSTRACT. The aim of the present paper is to study the dynamics of a dumbbell satellite moving in a gravity field generated by an oblate body considering the effect of the zonal harmonic parameter. We prove that the pass trajectory of the mass center of the system is periodic and different from the classical one when the effect of the zonal harmonic parameter is non zero. Moreover, we complete the classical theory showing that the equations of motion in the satellite approximation can be reduced to Beletsky's equation when the zonal harmonic parameter is zero. The main tool for proving these results is the Lindstedt–Poincaré's technique.

1. INTRODUCTION

From the end of the sixth decade of the last century, a part of the mathematical community, has directed its attention to the study the so called *dumbbell body or satellite* in central gravity, see for instance Morán [21], Schechter [25], Brereton and Modi [11], Beletsky [10, 9], Maciejewski et al. [20]; Kirchgraber et al. [18], Krupa et al. [19], Elipe et al. [15], Burov and Dugain [12] or Nakanishi et al. [23].

Recall that a *dumbbell body* is a quite simple structure composed by two masses connected by a massless rod. It is assume that this object is moving around a planet whose gravity field is approximated by the field of the attracting center. In general, the distance between the two points masses is considered to be much smaller that the distance between the satellite's center of mass and the attracting center of mass. Thus, it is common to neglect the influence of the attitude dynamics on the motion of the center of mass and treat it as an unperturbed Keplerian one.

Rodnikov [24] studied equilibrium positions of a weight on a cable fixed to a dumbbell-shaped space station moving along a circular geocentric orbit. This model is composed by two masses coupled by a weightless rod, while the cable is weightless and non-stretched. The equations of motion are stated when the motion is produced in a single plane and the center of mass of the

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system moves along a circular geocentric orbit. Moreover, the equilibrium configurations of the system are obtained and the Lyapunov stability of configurations for two situations, first when the station is composed of equal masses, second when masses at the ends of the station are different are analyzed.

For the “dumbbells–load” system with two unilateral connections, all relative equilibria on the circular Keplerian orbit were established by Munitsina [22]. Recall that a relative equilibria of the system is a point of the phase space giving an evolution which is a one–parameter orbit of the action of the symmetry group of the system. These results were interpreted for studying the relative equilibria for which both connections are stretched in geometrical terms. The necessary and sufficient conditions for stability of the relative equilibria were stated.

Celletti and Sidorenko [14] investigated the dumbbell satellite’s attitude dynamics, when the center of mass moves on a Keplerian trajectory. They found a stable relative equilibrium position in the case of circular orbits which disappears as far as elliptic trajectories are considered. They replaced the equilibrium position by planar periodic motions and they proved this motion is unstable with respect to out-of-plane perturbations. They also gave some numerical evidences of the existence of stable spatial periodic motions.

Burov et al. [13] considered the motion of a dumbbell–shaped body in an attractive Newtonian central field. They used the Poincaré’s theory to determine the conditions for the existence of families of system periodic motions depending on the arising small parameter and passing into some stable radial steady–state motion of the unperturbed problem as the small parameter tends to zero. They also proved that, each of the radial relative equilibria generates one family of such periodic motions, for sufficiently small parameter values. Furthermore, they studied the stability of the obtained periodic solutions in the linear approximation as well as these solutions were calculated up to terms of the first order in the small parameter.

Guirao et al. [17] gave sufficient conditions for the existence of periodic solutions of the perturbed attitude dynamics of a rigid dumbbell satellite in a circular orbit.

The statement of our main results is the following.

Theorem 1. *Consider a dumbbell satellite moving in a gravity field generated by an oblate body considering the effect of the zonal harmonic parameter A . The pass trajectory of the mass center of the system is periodic and different from the classical one. If A is equal to zero our solution coincides with the elliptical classical one.*

Finally, considering the motion in the satellite approximation we complete the classical theory, stating the following result.

Theorem 2. *The equations of motion in the satellite approximation can be reduced to Beletsky's equation when A is equal to zero.*

Note that Theorem 1 generalizes Celletti and Sidorenko [14], Burov and Dugain [12] and Nakanishi et al. [23] due to oblateness parameter.

The structure of the paper is as follows. In Section 2 we present the model description, the potential, the kinetic energy and the Lagrangian function of the system. In Section 3 we present the morphology of the equations of motion and the equation of the mass center of the system. In Sections 4 and 5 we respectively provide proof of Theorems 1 and 2. We remark that when $J_2 = 0$ is clear that the dynamics occurs on a plane, however when the coefficient J_2 is considered the effects of the gravitational potential are not the same for planes with different inclinations and a natural question is if there is an invariant plane for the dynamics. The answer of this fact is positive and it will be a key point in the proofs of our main result. In the Appendix we provide a proof of this property.

2. MODEL DESCRIPTION

2.1. Hypothesis. We assume that the dumbbell satellite is formed by massless rod of length l with two masses m_1 and m_2 placed at its ends. Let consider c the center of mass of the two masses moving in a gravity field generated by an oblate body whose mass m having mass center located at 0 where the distance between 0 and c is r and $r \gg l$.

Let us consider the orbital reference frame cxy with origin at the dumbbell's center, and the polar coordinates of the center are (r, θ) . While the rotation of the satellite relative to ray oc will be determined by an angle Θ . Furthermore we denote the reduced mass by μ and the sum of the two masses by m_s where $\mu = m_1 m_2 / m_s$ and $m_s = m_1 + m_2$, see Figure 1 for details.

Now, we assume that $\underline{r}_i$ is the position vector of m_i with respect to 0. Moreover, let the vector $\underline{n}_i$ denotes the position vector of m_i with respect to the center of mass of the dumbbell satellite, $i \in \{1, 2\}$.

Therefore, the magnitudes of the position vectors $\underline{r}_i$ are controlled by

$$(1) \quad r_i^2 = r^2 + n_i^2 + 2(-1)^{2-i} n_i r \cos \Theta$$

where

$$(2) \quad n_i = m_{3-i} l / m_s.$$

2.2. The potential of the model. From the potential theory, the gravitational potential (any object has axial symmetry m_0) experienced by the satellite m will be controlled by (see Murray and Dermott [16])

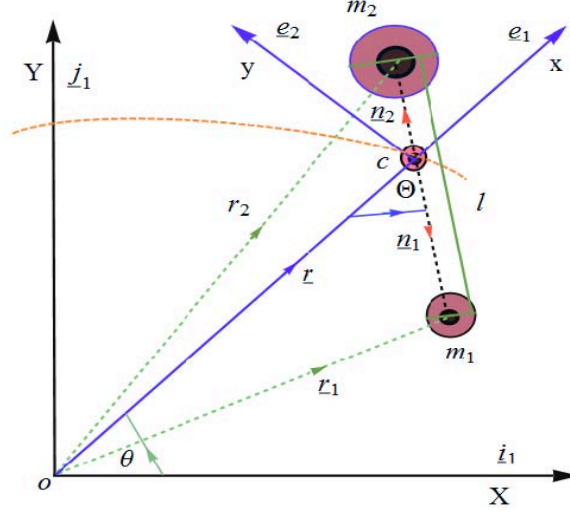


FIGURE 1. The dumbbell satellite model

$$(3) \quad V = -\frac{Gm_0m}{r_0} \left[1 - \sum_{n=2}^{+\infty} J_n \left(\frac{R}{r_0} \right) p_n(\sin(\delta)) \right]$$

where:

- (1) G is the universal constant; m_0 is the mass of the oblate object and m is the mass of the satellite;
- (2) R is the mean radius of the oblate object;
- (3) J_n is a dimensionless coefficient that characterizes the size of non-spherical components of the potential;
- (4) r_0 is the distance between m_0 and m ;
- (5) $p_n \sin(\delta)$ are the Legendre polynomials of degree n ;
- (6) δ denotes the latitude of the satellite.

If the two bodies move in the same plane, then $\delta = 0$ and equation (3) can be written as:

$$(4) \quad V = -\frac{Gm_0m}{r_0} \left[1 - \sum_{n=2}^{+\infty} J_n \left(\frac{R}{r_0} \right) p_n(0) \right],$$

where

$$p_{2n}(0) = \frac{(-1)^n 2n!}{2^{2n} (n!)^2}, \quad p_{2n+1}(0) = 0.$$

In the present model we shall consider the planar motion, for more details on it see the Appendix, and take the effect of the zonal harmonic up to J_2 , hence equation (4) can be rewritten as

$$V_0 = -Gm_0m\left(\frac{1}{r_0} + \frac{J_2R^2}{2r_0^3}\right),$$

see [1, 2, 3, 4, 5, 6, 7, 8] for more details.

If we assume that R represent the unit of distance, m_0 is also the unit mass and denote J_2 by A . We have that the potential experienced by the masses m_1 and m_2 are V_1 and V_2 such that

$$(5) \quad V_1 = -Gm_1\left(\frac{1}{r_1} + \frac{A}{2r_1^3}\right),$$

$$(6) \quad V_2 = -Gm_2\left(\frac{1}{r_2} + \frac{A}{2r_2^3}\right).$$

Therefore the total potential V can be written as

$$(7) \quad V = -k\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + A\left(\frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3}\right)\right),$$

where $k = G$ denotes the gravity parameter associated to the oblate body.

2.3. The kinetic energy of the model. Let the vectors $\underline{e}_1$ and $\underline{e}_2$ be an orthogonal set of unitary vectors with $\underline{e}_1$ corresponding to the direction from 0 to c .

Consider $\underline{i}$ and $\underline{j}$ be another orthogonal set of unitary vectors such that $\underline{i}$ is a vector in the direction of x axis. Consequently the vectors of the locations $\underline{r}_i$ and associates velocities $\underline{v}_i$ of masses m_i can be written as

$$\underline{r}_i = \underline{r} + \underline{n}_i,$$

$$\underline{v}_i = \frac{d\underline{r}_i}{dt},$$

where

$$\underline{r} = r(\cos \theta \underline{i} + \sin \theta \underline{j}),$$

$$\underline{n}_i = (-1)^i n_i (\cos \Theta \underline{e}_1 + \sin \Theta \underline{e}_2),$$

$$\underline{e}_i = (-1)^i \left(\cos \left(\theta + \frac{\pi}{i} \right) \underline{i} + \sin \left(\theta + \frac{\pi}{i} \right) \underline{j} \right).$$

Therefore, after some calculations, we obtain

$$(8) \quad v_i^2 = \left\{ \begin{array}{l} \dot{r}^2 + r^2 \dot{\theta}^2 + n_i^2 (\dot{\theta} + \dot{\Theta})^2 \\ -2(-1)^i \dot{r} n_i (\dot{\theta} + \dot{\Theta}) \sin \Theta \\ +2(-1)^i r n_i \dot{\theta} (\dot{\theta} + \dot{\Theta}) \cos \Theta \end{array} \right\}.$$

Since the kinetic energy of the dumbbell satellite system is

$$(9) \quad T = \frac{1}{2} \sum_{i=1}^2 m_i v_i^2.$$

Substituting equation (8) into (9), the kinetic energy can be written in the form

$$T = T_s + T_r.$$

where

$$(10) \quad T_s = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2).$$

$$T_r = \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2.$$

Hence

$$(11) \quad T = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2.$$

2.4. The Lagrangian function of the model. Since the Lagrange's function is defined by $L = T - V$ from equations (7) and (11) we get

$$(12) \quad L = \frac{1}{2} m_s (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \mu l^2 (\dot{\theta} + \dot{\Theta})^2 + k \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + A \left(\frac{m_1}{2r_1^3} + \frac{m_2}{2r_2^3} \right) \right).$$

Therefore, the equations of motion will be governed by

$$(13) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\chi}} \right) - \frac{\partial L}{\partial \chi} = 0, \quad \chi \in \{r, \theta, \Theta\}.$$

3. EQUATION OF MOTION

3.1. Equations of motion for the general case. Substituting equation (12) into (13) when $\chi \in \{r, \theta, \Theta\}$ the equations of motion can be written in the following form

$$(14) \quad \begin{aligned} & (m_s r^2 + \mu l^2) \ddot{\theta} + \mu l^2 \ddot{\Theta} = p_\theta = Q, \\ & m_s (\ddot{r} - r (\frac{p_\theta - \mu l^2 \dot{\Theta}}{m_s r^2 + \mu l^2})^2) = -k \left\{ \begin{aligned} & \left(\frac{m_1 (r - n \cos \Theta)}{r_1^3} + \frac{m_2 (r + (l-n) \cos \Theta)}{r_2^3} \right) \\ & + \frac{3}{2} A \left(\frac{m_1 (r - n \cos \Theta)}{r_1^5} + \frac{m_2 (r + (l-n) \cos \Theta)}{r_2^5} \right) \end{aligned} \right\}, \\ & \ddot{\Theta} + \frac{2\dot{r}(\mu l^2 \dot{\Theta} - p_\theta)}{r(m_s r^2 + \mu l^2)} = -\frac{k(m_s r^2 + \mu l^2) \sin \Theta}{m_s l r} \left(\left(\frac{1}{r_1^3} - \frac{1}{r_2^3} \right) + \frac{3}{2} A \left(\frac{1}{r_1^5} - \frac{1}{r_2^5} \right) \right), \end{aligned}$$

where Q is constant and $n_1 = n$, $n_2 = l - n$ while p_θ is a constant expresses the angular momentum conservation.

3.2. Dumbbell's center of motion. Since (r, θ) is the coordinate of the dumbbell's center, therefore the kinetic energy T_s is given by equation (10), while the potential of the center of mass V_s is given by

$$V_s = -Gm_s\left(\frac{1}{r} + \frac{A}{2r^3}\right).$$

Consequently, the Lagrange function L_s of the center of mass can be represented in the form

$$(15) \quad L_s = \frac{1}{2}m_s(\dot{r}^2 + r^2\dot{\theta}^2) + Gm_s\left(\frac{1}{r} + \frac{A}{2r^3}\right).$$

Substituting equation (15) into (13) with $L = L_s$ and $\chi \in \{r, \theta\}$ and taking account that the equations of motion can be written on the form

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{r}}\right) - \frac{\partial L_s}{\partial r} &= 0, \\ \frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{\theta}}\right) - \frac{\partial L_s}{\partial \theta} &= 0, \end{aligned}$$

we state that the motion of dumbbell's center will be controlled by

$$(16) \quad \begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -k\left(\frac{1}{r^2} + \frac{3A}{2r^4}\right), \\ m_sr^2\dot{\theta} &= F \text{ or } r^2\dot{\theta} = h, \\ \frac{1}{2}\dot{r}^2 - \frac{k}{r} - \frac{6kA}{r^3} &= E, \end{aligned}$$

where F is a constant, h is the angular momentum which is constant too, that can be evaluated by the initial conditions and E is the preservation of the total energy for the dumbbell's center.

Let be $r = \frac{1}{u}$, consequently

$$(17) \quad \frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}\left(1 + \frac{3}{2}Au^2\right).$$

$$(18) \quad \begin{aligned} \dot{\theta} &= -hu^2, \\ \frac{1}{2}h^2\left(\frac{du}{d\theta}\right)^2 - ku - 6kAu^3 &= E. \end{aligned}$$

It is worth mentioning that the system of equations (17) does not represent only dumbbell's center motion, it represents too the motion of two-body problem under the effect of the zonal harmonic motion which can be reduced to the motion of the classical case when $A = 0$. Therefore our results on the dumbbell's center of motion can be applied it to the motion of two-body problem.

Now let us go back to dumbbell's center motion in which we can be assumed that this motion follows a Kepler's type orbit when the effect of

oblateness parameter is absent ($A = 0$). Consequently the solution can be written as

$$r_0 = \frac{h^2/k}{(1 + e_0 \cos \theta)}$$

where $r_0 = \frac{1}{u_0}$, e_0 is the orbit eccentricity such that $0 \leq e_0 < 1$, in the framework of elliptic orbits and θ is a true anomaly of the center of mass. When $\theta = 0$, $u_0 = \frac{1}{r_p} = \frac{k}{h^2}(1 + e_0)$, $r_p = a(1 - e_0)$ is the pericenter (periapsis) and a is a semi-major axis, see Figures 2 and 3.

Now we look for solutions in the form $u(\theta, \epsilon)$ under the condition $0 < \epsilon \ll 1$. Since $A = J_2$ and $J_2 \in [1 \times 10^{-3}, 1 \times 10^{-6}]$ for the most of celestial bodies then we can replace ϵ by A . In addition, this solution must hold the initial conditions

$$\begin{aligned} u(0, A) &= \frac{1}{r_p}, \\ D_\theta u(0, A) &= 0. \end{aligned}$$

Therefore, we search for straight forward expansion of an asymptotes solution as A tends to zero in the following form

$$(19) \quad u(\theta, A) = u_0(\theta) + Au_1(\theta) + o(A^2).$$

The effect of the zonal harmonic of J_2 is taking account, but the perturbation due to J_2 is of order about 10^{-3} of the unperturbed main term (m_1/r_1) or (m_2/r_2). While all other coefficients of zonal harmonic are about 10^{-6} or less. Therefore, it is sufficient from practical point of view, we take the expansion in equation (19) up to A . On the other hand, $o(A^2)$ represents the effect of the zonal harmonic J_4 while our potential does not contain the zonal harmonic J_4 . Consequently we truncate the expansion in equation (19) up to the linear term A . In this case the leading-order perturbation equations are

$$\begin{aligned} D_{\theta\theta}u_0 + u_0 &= \frac{k}{h^2}, \\ D_{\theta\theta}u_1 + u_1 &= \frac{3}{2} \frac{k}{h^2} u_0^2. \end{aligned}$$

Under the conditions $u_0(0) = \frac{1}{r_p}$, $D_\theta u_0 = 0$, $u_1(0) = 0$ and $D_\theta u_1 = 0$, hence the solution is governed by

$$u_0 = \frac{k}{h^2}(1 + e_0 \cos \theta)$$

and

$$u_1 = \frac{3k^3}{2h^6} \left[1 + \frac{1}{2}e_0^2 - \left(1 + \frac{1}{3}e_0^2\right) \cos \theta + e_0 \theta \sin \theta - \frac{1}{6}e_0^2 \cos 2\theta \right].$$

Therefore, the general expression of the dumbbell's center motion up to $o(A)$ will be governed by

$$(20) \quad u(\theta, A) = u_0(\theta) + Au_1(\theta).$$

4. PROOF OF THEOREM 1

Since equation (20) represents a solution which contains a secular term that grows in θ . As a result, the expansion is not uniformly valid in θ and breaks down when $\theta = o(A)$, furthermore Au_1 is no longer a small correction of u_0 . But convergent series approximation of the periodic solution can be determined by the continuation method known as the Lindstedt–Poincaré's technique.

Since equation (17) is a second order differential equation, it describes a dynamical system in which A is a small parameter. Consequently if $A = 0$ the system will be reduced to a harmonic oscillator which has a solution with period $T = 2\pi/\omega_0$ where $\omega_0 = 1$.

The continuation method enables us to construct a periodic solution for $A \neq 0$. If we consider that the angular velocity changes due to the non-linear terms, the asymptotic solution $u(\theta, A)$ and the angular velocity ω of the dynamical system can be expanded as

$$(21) \quad \begin{aligned} u(\theta, A) &= u_0(\theta) + Au_1(\theta) + A^2u_2(\theta) + \dots \\ \omega &= 1 + A\omega_1 + A^2\omega_2 + \dots \end{aligned}$$

To construct a uniformly valid solution, we will introduce a stretched variable $\tau = \omega\theta$, therefore

$$(22) \quad \begin{aligned} \frac{d}{d\theta} &= \omega \frac{d}{d\tau}, \\ \frac{d^2}{d\theta^2} &= \omega^2 \frac{d^2}{d\tau^2}. \end{aligned}$$

Substituting equations (22) into (17) we obtain

$$(23) \quad \omega^2 \frac{d^2 u}{d\tau^2} + u = \frac{k}{h^2} \left(1 + \frac{3}{2} Au^2\right).$$

Now, under the following conditions

$$\begin{aligned} u(0, A) &= \frac{1}{r_p}, \\ u_\tau(0, A) &= 0, \\ u(\tau + 2\pi, A) &= u(\tau, A), \end{aligned}$$

we insert the series expansion (21) into (23) and equating terms of the same order in A with keeping the terms up to first order of A , we obtain the following:

- The coefficient of A^0 gives a homogeneous equation in the form

$$\frac{d^2 u_0}{d\tau^2} + u_0 = \frac{k}{h^2}$$

where $u_0(0) = \frac{1}{r_p}$, $\frac{du_0(0)}{d\tau} = 0$ and $u_0(\tau + 2\pi, A) = u_0(\tau, A)$ with a solution

$$(24) \quad u_{0g}(\tau) = \frac{k}{h^2}(1 + e_0 \cos \tau),$$

being

$$(25) \quad r = \frac{h^2/k}{1 + e_0 \cos \tau}.$$

- The coefficient of A gives a non-homogeneous equation in the form

$$(26) \quad \frac{d^2 u_1}{d\tau^2} + u_1 = a_1 + a_2 \cos \tau + a_3 \cos 2\tau$$

where $u_1(0) = 0$, $\frac{du_1(0)}{d\tau} = 0$ and

$$\begin{aligned} a_1 &= \frac{3k^3}{2h^6}(1 + \frac{1}{2}e_0^2), \\ a_2 &= \frac{3e_0k^3}{h^6}(1 + \frac{2\omega_1 h^4}{3k^2}), \\ a_3 &= \frac{3e_0^2k^3}{4h^6} \end{aligned}$$

with a particular solution

$$u_1(\tau + 2\pi, A) = u_0(\tau, A)$$

$$u_{1p} = a_1 + \frac{1}{2}a_2 \cos \tau + \frac{1}{2}a_2 \tau \sin \tau - \frac{1}{3}a_3 \cos 2\tau.$$

This solution contain a secular term $a_2 \tau \sin \tau/2$, to avoid this term and the solution becomes periodic we have to equate it coefficient by zero, hence

$$\omega_1 = -\frac{3k^2}{2h^4}.$$

Therefore the general solution of equation (26) is controlled by

$$(27) \quad u_{1g} = \frac{3k^3}{2h^6}(1 + \frac{1}{2}e_0^2) - \frac{k^3}{2h^6}(3 + e_0^2) \cos \tau - \frac{e_0^2k^3}{4h^6} \cos 2\tau.$$

Substituting equations (24) and (27) into (21), the general solution of equation (23) becomes

$$u = \frac{k}{h^2}(1 + k_1)[1 + (\frac{e_0 - k_2}{1 + k_1}) \cos \tau + \frac{k_3}{1 + k_1} \cos 2\tau]$$

where

$$\begin{aligned} k_1 &= \frac{3Ak^2}{4h^4}(2 + e_0^2), \\ k_2 &= \frac{Ak^2}{2h^4}(3 + e_0^2), \\ k_3 &= -\frac{Ak^2e_0^2}{4h^4}, \\ \tau &= \left(1 - \frac{3Ak^2}{2h^4}\right)\theta. \end{aligned}$$

Therefore

$$(28) \quad r = \frac{h^2/\bar{k}}{(1 + e \cos \tau + \bar{e} \cos 2\tau)},$$

with

$$\begin{aligned} \bar{k} &= k(1 + k_1), \\ e &= \left(\frac{e_0 - k_2}{1 + k_1}\right), \\ \bar{e} &= \frac{k_3}{1 + k_1}. \end{aligned}$$

In short, it is clear that the trajectory of the mass center differs from that assumed by Celletti and Sidorenko [14], Burov and Dugain [12] and Nakanishi et al. [23] due to oblateness parameter. Although, this solution is periodic. While this trajectory is the same as their solutions when the effect of oblateness is ignored. Since $e_0 < 1$ and $A \ll 1$ as a result $Ae_0^2 \ll 1$ is very small. Therefore, if we neglect all terms that include Ae_0^2 , the equation (28) will be reduced to

$$(29) \quad \begin{aligned} r &= \frac{h^2/\bar{k}}{(1 + e \cos \tau)}, \\ \bar{k} &= k\left(1 + \frac{3Ak^2}{2h^4}\right), \\ e &= e_0 - \frac{3Ak^2}{2h^4}(1 + e_0). \end{aligned}$$

This means that the trajectory of the mass center is elliptic as the classical case with the decreasing of the elliptical parameter and the eccentricity, ending the proof. ■

Remark 1. *Taking account the oblateness effect we have proved that the solution is periodic, see equation (28). While for the small value of the parameter A , we have elliptical solutions as in the classical case with the decreasing in the elliptical parameter, see (29).*

Thus, there is no discontinuity and the solution varies smoothly as A approaches zero.

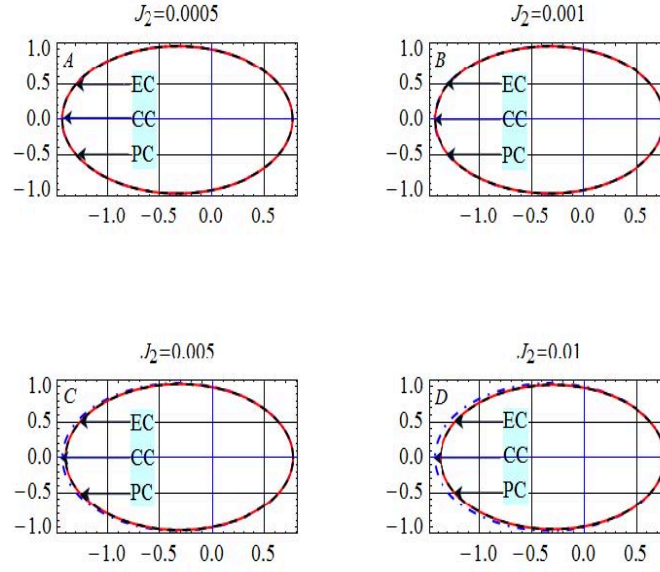


FIGURE 2. Variation in the trajectory of dumbbell's center when $e_0 = 0.3$ for different values of zonal harmonic parameter.

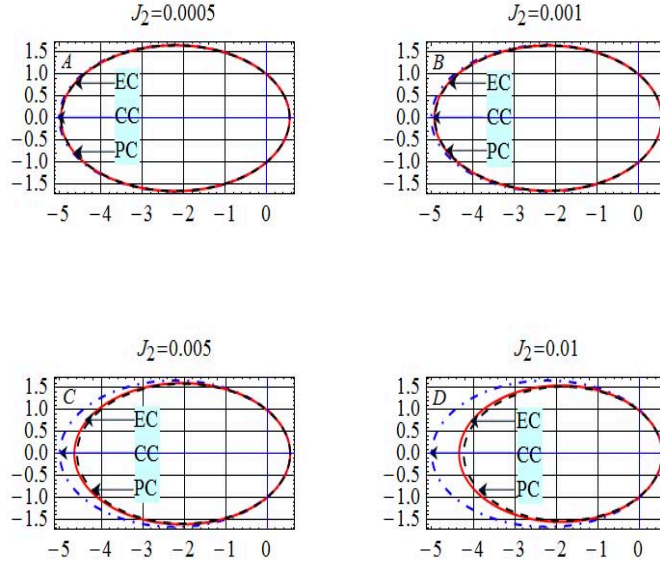


FIGURE 3. Variation in the trajectory of dumbbell's center when $e_0 = 0.8$ for different values of zonal harmonic parameter.

Note 1. *Figures 2 and 3 represent the changes in the trajectory of the dumbbell's center corresponding to the changes in the eccentricity and in the zonal harmonic parameter, here we have considered that k and h are equal to 1. We denote the curves representing the classical case (the effect of zonal harmonic is switched off) by (CC). If the effect of zonal harmonic is consider ignoring all terms with coefficients Ae_0^2 the curves will be denoted by (EC). Finally, by (PC) we denote the general trajectory.*

We observe that all trajectories are quasi-elliptical and the decreasing in the ellipse parameters is very small for small values of classical eccentricity e_0 and the zonal harmonic parameter J_2 , see some cases of Figure 2. While for some relative large values of the classical eccentricity e_0 the decreasing in the ellipse parameter is observed especially when the parameter of zonal harmonic is assigned by big value.

5. PROOF OF THEOREM 2

We shall start by the deduction of the equations of motion in satellite approximation. Indeed, substituting equations (1) and (2) into (7), the approximation of the potential energy can be written as

$$(30) \quad V = -k(m_s(\frac{1}{r} + \frac{A}{2r^3}) + \frac{\mu l^2}{2r^3}(3\cos^2 \Theta - 1)).$$

In this potential we neglect all terms that contain coefficients $(1/r)$ with power four or more, since $l \ll r$. Therefore the Lagrangian function becomes

$$(31) \quad \begin{aligned} L = & \frac{1}{2}m_s(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\mu l^2(\dot{\theta} + \dot{\Theta})^2 \\ & + k(m_s(\frac{1}{r} + \frac{A}{2r^3}) + \frac{\mu l^2}{2r^3}(3\cos^2 \Theta - 1)). \end{aligned}$$

Substituting equation (31) into (13), the approximation equations of motion are

$$(32) \quad \begin{aligned} m_s(\ddot{r} - r\dot{\theta}^2) &= -k(m_s(\frac{1}{r^2} + \frac{3A}{2r^4}) + \frac{3\mu l^2}{2r^4}(3\cos^2 \Theta - 1)), \\ (m_s r^2 + \mu l^2)\dot{\theta} + \mu l^2\dot{\Theta} &= p_\theta, \\ \mu l^2(\ddot{\theta} + \ddot{\Theta}) &= -\frac{3k\mu l^2}{r^3} \cos \Theta \sin \Theta. \end{aligned}$$

Now replacing the independent variable t with the starched variable τ where $\tau = \omega\theta$ and $r^2\dot{\theta} = h$ therefore, it is possible to write $\dot{\tau} = \Omega(\tau)$ such that

$$(33) \quad \Omega(\tau) = \frac{\omega \bar{k}^{-2}}{h^3} (1 + e \cos \tau + \bar{e} \cos 2\tau)^2.$$

Hence

$$(34) \quad \begin{aligned} \frac{d}{dt} &= \Omega \frac{d}{d\tau}, \\ \frac{d^2}{dt^2} &= \Omega^2 \frac{d^2}{d\tau^2} + \Omega \Omega' \frac{d}{d\tau}, \end{aligned}$$

where $(\cdot)'$ means $\frac{d}{d\tau}$.

Inserting equations (34) into (32) and using equation (33), we obtain

$$(35) \quad (1 + e \cos \tau + \bar{e} \cos 2\tau) \Theta'' - 2(e \sin \tau + 2\bar{e} \sin 2\tau) \left(\frac{1}{\omega} + \Theta' \right) + \frac{3k}{\omega^2 \bar{k}} \cos \Theta \sin \Theta = 0.$$

Since $\tau = \omega\theta$, we can rewrite equation (35) in the form

$$(36) \quad \begin{aligned} (1 + e \cos \omega\theta + \bar{e} \cos 2\omega\theta) \frac{d^2 \Theta}{d\theta^2} - 2\omega(e \sin \omega\theta + 2\bar{e} \sin 2\omega\theta) \left(1 + \frac{d\Theta}{d\theta} \right) \\ + \frac{3k}{\bar{k}} \cos \Theta \sin \Theta = 0 \end{aligned}$$

where

$$\begin{aligned} \omega &= 1 - \frac{3Ak^2}{2h^4}, \\ \bar{e} &= \frac{Ak^2 e_0^2}{4h^4}, \end{aligned}$$

and

$$\bar{k} = k \left[1 + \frac{3Ak^2}{4h^4} (2 + e_0^2) \right].$$

Now, for finishing only remark that equation (36) can be reduced to Beletsky's equation, see [9] for more details, if we assume the oblateness effect is not consider (i.e., $A = 0$) obtaining the relation

$$(1 + e \cos \theta) \frac{d^2 \Theta}{d\theta^2} - 2e \sin \theta \frac{d\Theta}{d\theta} + 3 \cos \Theta \sin \Theta = 2e \sin \theta,$$

which ends the proof. ■

APPENDIX

Let us introduce now an inertial reference frame $\mathcal{I}(\mathbf{O}; \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$. The coordinates of a generic vector in this reference system are denoted by $\mathbf{x} = (x, y, z)_{\mathcal{I}}$. Recall that we are considering a dumbbell formed by two material points $\mathbf{M}_1$, of mass m_1 and $\mathbf{M}_2$ of mass m_2 rigidly connected by a segment of constant length l and negligible mass mutually attracted by

a gravitational potential due to nearly spherical body $\mathbf{M}$. Recall that the potential is given by

$$\mathcal{V}(\mathbf{x}) = -\frac{GM}{\|\mathbf{x}\|} \left(1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{\|\mathbf{x}\|} \right)^n P_n \left(\frac{z}{\|\mathbf{x}\|} \right) \right)$$

with G is the gravitational constant, M the mass of the body, and R the equatorial radius. $P_n(u)$ is the Legendre polynomial of degree n and argument u , and the J_n are constant coefficients characterizing the potential of the body M . We can define a rotating frame $\mathcal{R}(\mathcal{G}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with $\mathcal{G}$ be the center of masses of the dumbbell, such that the unitary vector $\mathbf{e}_3$ is directed along the dumbbell towards the point $\mathbf{M}_2$ and $\mathbf{e}_1, \mathbf{e}_2$ are two orthonormal vectors, perpendicular to $\mathbf{e}_3$. In this frame, the principal moments of inertia (I_1, I_2, I_3) of the dumbbell are

$$I_1 = I_2 = \mu l^2, \quad I_3 = 0$$

with

$$\mu = \frac{m_1 m_2}{m_s}$$

where $m_s = m_1 + m_2$ and

$$l_1 = n = \frac{m_2 l}{m_s}, \quad l_2 = l - n = \frac{m_1 l}{m_s}$$

the distances from $\mathbf{M}_1$ and $\mathbf{M}_2$ to $\mathcal{G}$.

The attitude of the dumbbell is given by two angles, namely nutation Θ and precession ϕ . The coordinates of points M_1 and M_2 in the space frame $\mathcal{S}$ are

$$\mathbf{M}_1 \equiv -l_1 (\sin \Theta \sin \phi, -\sin \Theta \cos \phi, \cos \phi)_{\mathcal{S}}$$

$$\mathbf{M}_2 \equiv l_2 (\sin \Theta \sin \phi, -\sin \Theta \cos \phi, \cos \phi)_{\mathcal{S}}.$$

The coordinates of $\mathcal{G}$ respect to the inertial frame $\mathcal{I}$, using cylindrical coordinates are

$$\mathcal{G} \equiv (r \cos \theta, r \sin \theta, z)_{\mathcal{I}}.$$

Using the Koenig's Theorem, the Lagrangian of the dumbbell is $\mathcal{L}(\mathbf{P}; \mathbf{V}_{\mathbf{P}})$ equal to

$$\frac{m_s}{2} \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\theta}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) + \frac{\mu l^2}{2} \left(\left(\frac{d\phi}{dt} \right)^2 + \left(\frac{d\Theta}{dt} \right)^2 \sin^2 \phi \right) - \mathcal{U}(\mathbf{P})$$

with $(\mathbf{P}; \mathbf{V}_{\mathbf{P}}) = \left(r, z, \theta, \Theta, \phi; \frac{dr}{dt}, \frac{dz}{dt}, \frac{d\theta}{dt}, \frac{d\Theta}{dt}, \frac{d\phi}{dt} \right)$, and

$$\mathcal{U}(\mathbf{P}) = \mathcal{V}(\mathbf{x}_{\mathbf{M}_1}) + \mathcal{V}(\mathbf{x}_{\mathbf{M}_2}).$$

The coordinates of $\mathbf{x}_{\mathbf{M}_1}$ and $\mathbf{x}_{\mathbf{M}_2}$ are

$$\mathbf{x}_{\mathbf{M}_1} \equiv (r \cos \theta - l_1 \sin \Theta \sin \phi, r \sin \theta + l_1 \sin \Theta \cos \phi, z - l_1 \cos \phi)_{\mathcal{I}}$$

$$\mathbf{x}_{\mathbf{M}_2} \equiv (r \cos \theta + l_2 \sin \Theta \sin \phi, r \sin \theta - l_2 \sin \Theta \cos \phi, z + l_2 \cos \phi)_{\mathcal{I}}$$

and

$$\begin{aligned}\|\mathbf{x}_{M_1}\|^2 &= r^2 + z^2 + l_1^2 - 2l_1(z \cos \phi + r \sin \phi \sin(\Theta - \theta)) \\ \|\mathbf{x}_{M_2}\|^2 &= r^2 + z^2 + l_2^2 + 2l_2(z \cos \phi + r \sin \phi \sin(\Theta - \theta))\end{aligned}$$

The potential of the system has the following expression

$$\begin{aligned}\mathcal{U}(r, z, \Theta - \theta, \phi) &= - \left(\frac{GMm_1}{\|\mathbf{x}_{M_1}\|} \left(1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{\|\mathbf{x}_{M_1}\|} \right)^n P_n \left(\frac{z - l_1 \cos \phi}{\|\mathbf{x}_{M_1}\|} \right) \right) + \right. \\ &\quad \left. \frac{GMm_2}{\|\mathbf{x}_{M_2}\|} \left(1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{\|\mathbf{x}_{M_2}\|} \right)^n P_n \left(\frac{z + l_2 \cos \phi}{\|\mathbf{x}_{M_2}\|} \right) \right) \right)\end{aligned}$$

A.1 Hamiltonian expressions. From the expressions of the kinetic energy and the potential, we can derive the Hamiltonian

$$\mathcal{H}(\mathbf{P}; T \mathbf{V}_P) = \frac{1}{2m_s} \left(P_r^2 + \frac{P_\theta^2}{r^2} + P_z^2 \right) + \frac{1}{2\mu l^2} \left(\frac{P_\Theta^2}{\sin^2 \phi} + P_\phi^2 \right) + \mathcal{U}(r, z, \Theta - \theta, \phi)$$

with

$$(\mathbf{P}; T \mathbf{V}_P) = (r, z, \theta, \Theta, \phi; P_r, P_\theta, P_z, P_\Theta, P_\phi).$$

The angles θ and Θ appear only as the difference $\Theta - \theta$, we can reduce the order of the Hamiltonian by means of the canonical transformation

$$(\theta, \Theta - \theta, P_\theta, P_\Theta) \rightarrow (\lambda, \psi, P_\psi - P_\lambda, P_\psi).$$

The new Hamiltonian is

$$(A.1) \mathcal{H} = \frac{1}{2m_s} \left(P_r^2 + \frac{(P_\psi - P_\lambda)^2}{r^2} + P_z^2 \right) + \frac{1}{2\mu l^2} \left(\frac{P_\psi^2}{\sin^2 \phi} + P_\phi^2 \right) + \mathcal{U}(r, z, \psi, \phi).$$

The variable λ is cyclic and the momentum P_λ is a constant of the motion. The Hamiltonian itself is another integral.

A.2 Equations of the motion. The Hamiltonian equations of the motion are

$$(A.2.1) \quad \begin{aligned} \frac{dr}{dt} &= \frac{P_r}{\mu}, & \frac{dP_r}{dt} &= \frac{(P_\psi - P_\lambda)^2}{\mu r^3} - \frac{\partial \mathcal{U}}{\partial r}, \\ \frac{dz}{dt} &= \frac{P_z}{\mu}, & \frac{dP_z}{dt} &= -\frac{\partial \mathcal{U}}{\partial z}, \\ \frac{d\psi}{dt} &= \frac{P_\psi - P_\lambda}{\mu r^2} + \frac{P_\psi}{\mu l^2 \sin^2 \phi}, & \frac{dP_\psi}{dt} &= -\frac{\partial \mathcal{U}}{\partial \psi}, \\ \frac{d\phi}{dt} &= \frac{P_\phi}{\mu l^2}, & \frac{dP_\phi}{dt} &= \frac{P_\psi^2 \cos \phi}{\mu l^2 \sin^3 \phi} - \frac{\partial \mathcal{U}}{\partial \phi}. \end{aligned}$$

Theorem. *The equations (A.2.1) has an invariant manifold given by $z \equiv 0, P_z \equiv 0, \phi \equiv \pi/2$ and $P_\phi \equiv 0$.*

Proof. Using the equations

$$\begin{aligned}\frac{dz}{dt} &= \frac{P_z}{\mu}, & \frac{dP_z}{dt} &= -\frac{\partial \mathcal{U}}{\partial z}, \\ \frac{d\phi}{dt} &= \frac{P_\phi}{\mu l^2}, & \frac{dP_\phi}{dt} &= \frac{P_\psi^2 \cos \phi}{\mu l^2 \sin^3 \phi} - \frac{\partial \mathcal{U}}{\partial \phi}.\end{aligned}$$

the result is immediate. ■

The Hamiltonian (A.1) restricted to the invariant manifold is

$$\mathcal{H} = \frac{1}{2m_s} \left(P_r^2 + \frac{P_\psi^2}{l^2} + \frac{(P_\psi - P_\lambda)^2}{r^2} \right) + \mathcal{U}_1(r, \psi).$$

with

$$\begin{aligned}(A.2.2) \quad \mathcal{U}_1(r, \psi) &= -GM \left[\left(\frac{m_1}{\sqrt{r^2 + l_1^2 - 2l_1 r \sin \psi}} + \frac{m_2}{\sqrt{r^2 + l_2^2 + 2l_2 r \sin \psi}} \right) + \right. \\ &\quad \left. R^2 J_2 \left(\frac{m_1}{\left(\sqrt{r^2 + l_1^2 - 2l_1 r \sin \psi} \right)^3} + \frac{m_2}{\left(\sqrt{r^2 + l_2^2 + 2l_2 r \sin \psi} \right)^3} \right) + O(J_4) \right]\end{aligned}$$

If $r \gg l$ and $R = 1$, $M = 1$, $k = G$, $J_2 = A$ we obtain

$$(A.2.3) \quad \mathcal{U}_1(r, \psi) = -k \left(m_s \left(\frac{1}{r} + \frac{A}{2r^3} \right) + \frac{\mu l^2}{2r^3} (3 \cos^2 \psi - 1) \right).$$

A.3 The Lagrangian function. The Lagrangian are

$$\mathcal{L} \left(r, \lambda, \psi, \frac{dr}{dt}, \frac{d\lambda}{dt}, \frac{d\psi}{dt} \right) = \frac{m_s}{2} \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\lambda}{dt} \right)^2 \right) + \frac{\mu l^2}{2} \left(\frac{d(\lambda + \psi)}{dt} \right)^2 - \mathcal{U}_1(r, \psi)$$

and the second order equations of the motion are given by

$$\begin{aligned}(A.3) \quad m_s \left(\frac{d^2 r}{dt^2} - r \left(\frac{d\lambda}{dt} \right)^2 \right) &= -\frac{\partial \mathcal{U}_1}{\partial r}, \\ \mu l^2 \left(\frac{d^2 \lambda}{dt^2} + \frac{d^2 \psi}{dt^2} \right) &= -\frac{\partial \mathcal{U}_1}{\partial \psi}, \\ m_s r^2 \frac{d\lambda}{dt} + \mu l^2 \left(\frac{d\lambda}{dt} + \frac{d\psi}{dt} \right) &= \text{constant}.\end{aligned}$$

It is clear that if we replace the symbols λ by θ and ψ by Θ the equation (A.2.3) is the same of equation (30). Also the system of equations (A.3) becomes into the system of equations (14) when $\mathcal{U}_1$ is represented by equation (A.2.3) and it is the same of equation (32) when $\mathcal{U}_1$ is represented by equation (A.3).

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