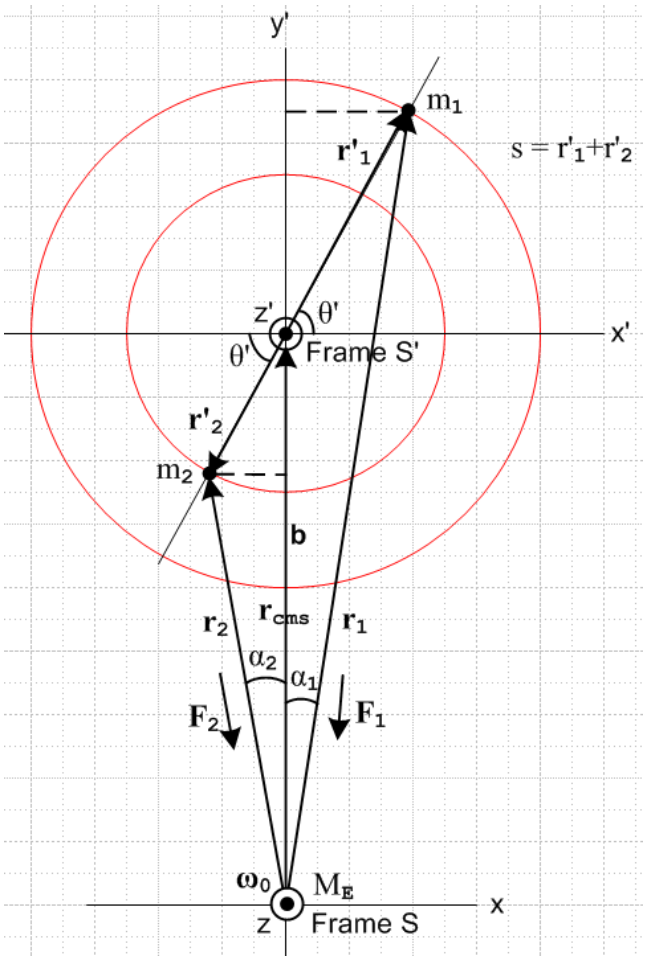
New Tether Torque Work PhL 12.13.16

At this point I have fully updated frames doc to be new frames doc with new eq nums and various changes. I think all the torque stuff is correct now. I have also written new App D and I think it is OK too. So now I return to the problem of the motion of the satellite. I will do it scratch here and then write a final section on this topic for App D.

Here is the current App D picture. I am now happy that the thing will in effect rotate about the CMS.



So let's start as if there were no fictitious forces in Frame S'. By the way, Frame S' is going to rotate, so its axes are going to move relative to the Frame S axes, they just line up at t = 0. I just added notes to that effect in Section D.2.

OK, here we go inside Frame S'. Start with

$$\mathbf{L'^{(c')}_{S'}} \equiv \mathbf{(r'-c')} \times m\mathbf{v'_{S'}} = \mathbf{(r'-c')} \times m\mathbf{v'} = \mathbf{(r'-c')} \times \mathbf{p'} \equiv \mathbf{L'^{(c')}} \qquad (1.9.3)$$

But we are going to use $\mathbf{c'} = 0$ so the torque ref point is Frame S' origin. Then

$$\mathbf{L'^{(0)}_{S'}} \equiv \mathbf{r'} \times m\mathbf{v'_{S'}} = \mathbf{r'} \times m\mathbf{v'} = \mathbf{r'} \times \mathbf{p'} \equiv \mathbf{L'^{(0)}} \qquad (1.9.3)$$

For mass m_1 we have

$$\mathbf{L'^{(0)}_1} = \mathbf{r_1'} \times m_1\mathbf{v'_1}$$

where we don't really know much about $\mathbf{v'_1}$, it might not be in the plane of paper. But fine. And

$$\mathbf{L'^{(0)}_2} = \mathbf{r_2'} \times m_2\mathbf{v'_2}$$

I do know that

$$m_1 r'_1 = m_2 r'_2 \qquad \text{or} \qquad r'_2/r'_1 = m_1/m_2 \qquad \text{or} \qquad m_1\mathbf{r'_1} = -\, m_2\mathbf{r'_2}\,. \qquad (D.2.8)$$

and it seems likely that we can apply $\partial_{S'}$ to get

$$m_1\mathbf{v'_1} = -\, m_2\mathbf{v'_2}$$

so therefore

$$\mathbf{L'^{(0)}_1} = m_1\mathbf{r_1'} \times \mathbf{v'_1}$$

STOP and resolve this paradox: Based on the above we get

$$\dot{\mathbf{L}}'^{(0)}_1 = m_1\mathbf{v_1'} \times \mathbf{v'_1} + m_1\mathbf{r_1'} \times \mathbf{a'_1} = m_1\mathbf{r_1'} \times \mathbf{a'_1}$$

Suppose Frame S' is at rest and suppose m_1 is doing planar circular motion. Then

$$\mathbf{a'_1} = -\,\omega'^2\mathbf{r'_1}$$

and this then says $\dot{\mathbf{L}}'^{(0)}_1 = 0$. I guess by assuming circular motion, you are assuming no torque! So maybe instead (assuming planar $\mathbf{v'_1}$)

$$\mathbf{v'_1} = \omega'(t)\, r'_1\, \hat{\boldsymbol{\theta}}' \qquad \text{// which } \textit{would} \text{ be true for circular motion}$$

$$\mathbf{a'_1} = \dot{\omega}'(t)\, r'_1\, \hat{\boldsymbol{\theta}}' + \omega'(t)\, r'_1\, (\partial_{S'}\hat{\boldsymbol{\theta}}')$$

I think that

$$(\partial_{S'}\hat{\boldsymbol{\theta}}') = -\omega'\, \hat{\mathbf{r}}'$$

so then we get

$$\mathbf{a'}_1 = \dot{\omega}'(t)\, r'_1\, \hat{\boldsymbol{\theta}}' - \omega'(t)^2\, r'_1\, \hat{\mathbf{r}}_1'$$

and then

$$\dot{\mathbf{L}}'^{(0)}_1 = m_1 \mathbf{r}_1' \times \mathbf{a}'_1 = m_1 \mathbf{r}_1' \times \left[\dot{\omega}'(t)\, r'_1\, \hat{\boldsymbol{\theta}}'_1 \right] = m_1 r'_1 \dot{\omega}'(t)\, \mathbf{r}_1' \times \hat{\boldsymbol{\theta}}'_1 = m_1 r'^2_1 \dot{\omega}'(t)\, \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1$$

and the paradox is resolved. Torque on this particle in forced circular motion changes ω' . The other particle will have

$$\begin{aligned}
\dot{\mathbf{L}}'^{(0)}_2 &= m_2 r'_2 \dot{\omega}'(t)\, \mathbf{r}_2' \times \hat{\boldsymbol{\theta}}'_2 \\
&= m_1 r'_1 \dot{\omega}'(t)\, r'_2\, \hat{\mathbf{r}}_2' \times \hat{\boldsymbol{\theta}}'_2 \\
&= m_1 r'_1 \dot{\omega}'(t)\, (m_1/m_2) r'_1\, (-\hat{\mathbf{r}}'_1) \times (-\hat{\boldsymbol{\theta}}'_1) \\
&= m_1 r'^2_1 \dot{\omega}'(t)\, (m_1/m_2)\, \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1
\end{aligned}$$

So add to get

$$\dot{\mathbf{L}}'^{(0)} = \dot{\mathbf{L}}'^{(0)}_1 + \dot{\mathbf{L}}'^{(0)}_2 = m_1 r'^2_1 \dot{\omega}'(t)\, \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1 + m_1 r'^2_1 \dot{\omega}'(t)\, (m_1/m_2)\, \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1$$

$$= m_1 r'^2_1 \dot{\omega}'(t) \left[1 + (m_1/m_2) \right] \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1$$

$$= (m_1/m_2)\, M\, r'^2_1 \dot{\omega}'(t)\, \hat{\mathbf{r}}_1' \times \hat{\boldsymbol{\theta}}'_1$$

$$= (m_1/m_2)\, M\, r'^2_1 \dot{\omega}'(t)\, \hat{\mathbf{z}}'$$

$$= (m_1/m_2)\, M\, r'^2_1 \ddot{\theta}'(t)\, \hat{\mathbf{z}}'$$

and finally we have our $\ddot{\theta}'$ ODE quantity.

The total ang mom of our little satellite system inside Frame S' is given by

$$\mathbf{L}'^{(0)} = \mathbf{L}'^{(0)}_1 + \mathbf{L}'^{(0)}_2$$

$$= m_1 \mathbf{r}_1' \times \mathbf{v}'_1 + (m_1^2/m_2)\, \mathbf{r}'_1 \times \mathbf{v}'_1$$

$$= \left[m_1 + (m_1^2/m_2) \right] \mathbf{r}_1' \times \mathbf{v}'_1 = m_1 \left[1 + (m_1/m_2) \right] \mathbf{r}_1' \times \mathbf{v}'_1$$

$$= m_1 \left[(m_1+m_2)/m_2 \right] \mathbf{r}_1' \times \mathbf{v}'_1 = M(m_1/m_2)\, \mathbf{r}_1' \times \mathbf{v}'_1$$

This result does not assume $\mathbf{v}'_1$ is planar.

Conclusion: The *total* satellite angular momentum in Frame S' is given by

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, \mathbf{r}_1' \times \mathbf{v}'_1 \qquad\qquad M = m_1 + m_2$$

Since $\mathbf{v}'_1$ might not be in the plane of paper, $\mathbf{L}'^{(0)}$ might not be perp to paper, so that is certainly a complication. If we make the ansatz of planar $\mathbf{v}'_1$ then we get

$$\mathbf{r}_1' \times \mathbf{v}'_1 = r'_1 v'_1\, \hat{\mathbf{z}}' = \omega'\, r'^2_1\, \hat{\mathbf{z}}' \quad \text{so in that case}$$

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, \omega'\, r'^2_1\, \hat{\mathbf{z}}'$$

Then it is pretty obvious that

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, \dot{\omega}'\, r'^2_1\, \hat{\mathbf{z}}' = (m_1/m_2)\, M\, r'^2_1 \ddot{\theta}'\, \hat{\mathbf{z}}'$$

in agreement with the above.

Then in Frame S' we can say

$$\mathbf{N}'^{(b)}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \ddot{\theta}'\, \hat{\mathbf{z}}' \quad .$$

where the b is our convention for labeling the effective torque.

Now we also have,

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \mathbf{N}^{(b)} + \mathbf{N}'^{(0)}_{\mathbf{fict}} \qquad \text{// } \textit{defines } \mathbf{N}'^{(c')}_{\mathbf{fict}} \qquad (11.3.5)$$

$$\mathbf{N}'^{(c')}_{\mathbf{fict}} = -(\mathbf{r}'-\mathbf{c}') \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\, \boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \qquad (11.3.10)$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict}} = -(\mathbf{r}'-\mathbf{0}) \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\, \boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m(\mathbf{0} + \boldsymbol{\omega} \times \mathbf{0} + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\mathbf{0} \times \mathbf{v}' \qquad (11.3.10)$$

$$\mathbf{N}'^{(0)}_{\mathbf{fict}} = -\mathbf{r}' \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\, \boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m\, \dot{\mathbf{b}}_S \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \qquad (11.3.10)$$

Now what is $\mathbf{N}^{(b)}$, the torque referenced to point $\mathbf{b} = \mathbf{r}_{\mathbf{cms}}$ in Frame S?

$$\mathbf{N}^{(b)} = \mathbf{r}'_1 \times \mathbf{F}_1 + \mathbf{r}'_2 \times \mathbf{F}_2$$

where all four of these vectors are shown in the figure. Recall that

$$\mathbf{F}_1 = -(GM_E m_1/r_1^2)\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r_1^3)\,\mathbf{r}_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r_2^2)\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r_2^3)\,\mathbf{r}_2 \quad . \qquad \text{(D.2.13)}$$

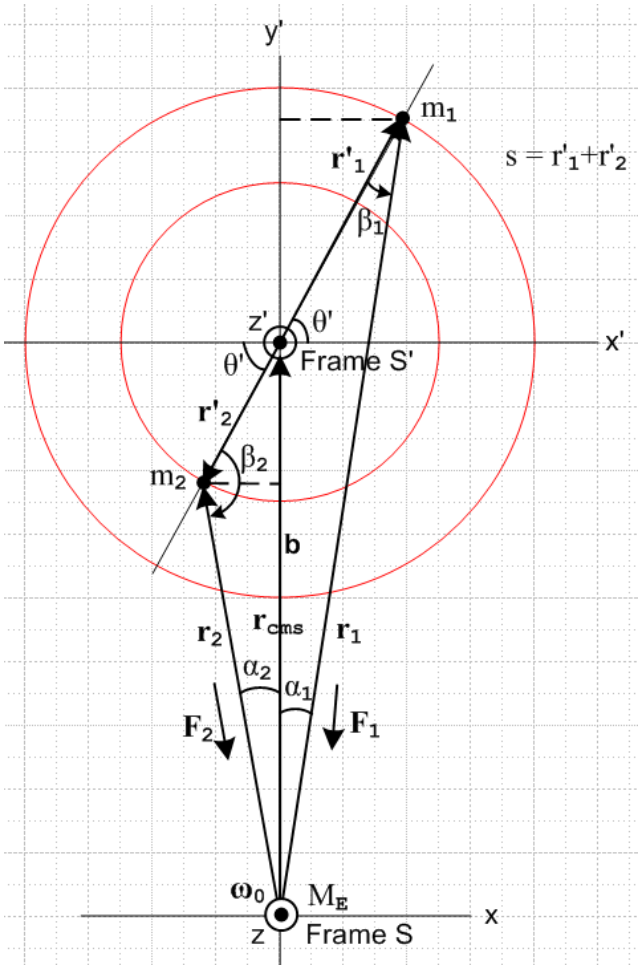
Note that we really should include the stick tension/compression **T** since these are supposed to be *total* forces, but luckily $\mathbf{r'}_1 \times \mathbf{T} = \mathbf{r'}_2 \times \mathbf{T} = \mathbf{0}$ so we can ignore **T** !

Now looking at our figure

$$\mathbf{r'}_1 \times \mathbf{F}_1 = \mathbf{r'}_1 \times [-(GM_E m_1/r_1^3)\,\mathbf{r}_1] = -(GM_E m_1/r_1^3)\,\mathbf{r'}_1 \times \mathbf{r}_1$$

$$\mathbf{r'}_2 \times \mathbf{F}_2 = \mathbf{r'}_2 \times [-(GM_E m_2/r_2^3)\,\mathbf{r}_2] = -(GM_E m_2/r_2^3)\,\mathbf{r'}_2 \times \mathbf{r}_2$$

so we are going to need these two cross products! This is then going to require more angle information, so I will rebuild my more complicated original figure. I need to add the two angles β_1 and β_2 which I try to define in a compatible manner, so



Then having done this, I can say

$\mathbf{r'}_1 \times \mathbf{r}_1 = r'_1 r_1 \sin\beta_1 \ \hat{\mathbf{z}}'$

$\mathbf{r'}_2 \times \mathbf{r}_2 = r'_2 r_2 \sin\beta_2 \ (-\hat{\mathbf{z}}')$

I added some small light guidelines to help one to confirm the above two equations.

Now we get to the angle stuff which I did not display early-on:

$\beta_1 + (\theta'+\pi/2) + \alpha_1 = \pi$
$\beta_2 + (\pi/2 - \theta') + \alpha_2 = \pi$

These are very easy to read off the picture and then we have

$\beta_1 + \theta' + \alpha_1 = \pi/2$ $\qquad$ $\alpha_1+\theta' = \pi/2 - \beta_1$
$\beta_2 - \theta' + \alpha_2 = \pi/2$ $\qquad$ $\alpha_2 - \theta' = \pi/2 - \beta_2$

and

$\beta_1 = \pi/2 - [\alpha_1+\theta']$
$\beta_1 = \pi/2 - [\alpha_1-\theta']$

$\sin\beta_1 = \sin\{\pi/2 - [\alpha_1+\theta']\} = \cos(\alpha_1+\theta')$
$\sin\beta_2 = \sin\{\pi/2 - [\alpha_2 - \theta']\} = \cos(\alpha_2 - \theta')$

Thus we may write

$\mathbf{r'}_1 \times \mathbf{r}_1 = r'_1 r_1 \cos(\alpha_1 + \theta') \ \hat{\mathbf{z}}'$
$\mathbf{r'}_2 \times \mathbf{r}_2 = r'_2 r_2 \ \cos(\alpha_2 - \theta')(-\hat{\mathbf{z}}')$

and then

$\mathbf{r'}_1 \times \mathbf{F}_1 = -(GM_E m_1/r_1^3)\, \mathbf{r'}_1 \times \mathbf{r}_1 = -(GM_E m_1/r_1^3)\ r'_1 r_1 \cos(\alpha_1 + \theta')\ \hat{\mathbf{z}}'$
$\mathbf{r'}_2 \times \mathbf{F}_2 = -(GM_E m_2/r_2^3)\, \mathbf{r'}_2 \times \mathbf{r}_2 = -(GM_E m_2/r_2^3)\ r'_2 r_2 \cos(\alpha_2 - \theta')(-\hat{\mathbf{z}}')$

This fine, but really the α_i are "derived angles" and we already know that

$\sin\alpha_1 = (r'_1 \cos\theta') / r_1 \quad \Rightarrow \quad r_1 \sin\alpha_1 = r'_1 \cos\theta'$
$\sin\alpha_2 = (r'_2 \cos\theta') / r_2 \quad \Rightarrow \quad r_2 \sin\alpha_2 = r'_2 \cos\theta'$. (D.2.9)

$\cos\alpha_1 = (b + r'_1 \sin\theta')/r_1 \quad \Rightarrow \quad r_1 \cos\alpha_1 = b + r'_1 \sin\theta'$
$\cos\alpha_2 = (b - r'_2 \sin\theta')/r_2 \quad \Rightarrow \quad r_2 \cos\alpha_2 = b - r'_2 \sin\theta'$. (D.2.10)

Thus I could write things all out in a very complicated manner since

$$\cos(\alpha_1 + \theta') = \cos\alpha_1\cos\theta' - \sin\alpha_1\sin\theta'$$
$$\cos(\alpha_2 - \theta') = \cos\alpha_2\cos\theta' + \sin\alpha_2\sin\theta'$$

so

$$r_1\cos(\alpha_1 + \theta') = r_1\cos\alpha_1\cos\theta' - r_1\sin\alpha_1\sin\theta'$$
$$r_2\cos(\alpha_2 - \theta') = r_2\cos\alpha_2\cos\theta' + r_2\sin\alpha_2\sin\theta'$$

and then

$$r_1\cos(\alpha_1 + \theta') = (b + r'_1\sin\theta')\cos\theta' - r'_1\cos\theta'\sin\theta'$$
$$r_2\cos(\alpha_2 - \theta') = (b - r'_2\sin\theta')\cos\theta' + r'_2\cos\theta'\sin\theta'$$

or

$$r_1\cos(\alpha_1 + \theta') = b\cos\theta' + r'_1\cos\theta'\sin\theta' - r'_1\cos\theta'\sin\theta' = b\cos\theta'$$
$$r_2\cos(\alpha_2 - \theta') = b\cos\theta' - r'_2\cos\theta'\sin\theta' + r'_2\cos\theta'\sin\theta' = b\cos\theta'$$

Oops, there must be a simpler way! Stare at the drawing. I can see that

$$\mathbf{b} \bullet \mathbf{r}'_1 = br'_1\cos(\pi/2-\theta') = br'_1\sin\theta'$$
$$\mathbf{b} \times \mathbf{r}'_1 = br'_1\sin(\pi/2-\theta')(-\hat{\mathbf{z}}') = br'_1\cos\theta'$$

Not too helpful. I cannot see a simple to get the above, so maybe the above is wrong. But I checked and it seems right. So continue on:

I have now shown that

$$r_1\cos(\alpha_1 + \theta') = b\cos\theta'$$
$$r_2\cos(\alpha_2 - \theta') = b\cos\theta'$$

Therefore

$$\mathbf{r}'_1 \times \mathbf{F}_1 = -(GM_E m_1/r_1^3)\, r'_1 r_1 \cos(\alpha_1+\theta')\,\hat{\mathbf{z}}' = -(GM_E m_1/r_1^3)\, r'_1\, b\cos\theta'\,\hat{\mathbf{z}}'$$
$$\mathbf{r}'_2 \times \mathbf{F}_2 = -(GM_E m_2/r_2^3)\, r'_2 r_2 \cos(\alpha_2-\theta')(-\hat{\mathbf{z}}') = -(GM_E m_2/r_2^3)\, r'_2\, b\cos\theta'\,(-\hat{\mathbf{z}}')$$

Finally we arrive at

$$\mathbf{N}^{(b)} = \mathbf{r}'_1 \times \mathbf{F}_1 + \mathbf{r}'_2 \times \mathbf{F}_2$$

$$= -(GM_E m_1/r_1^3)\, r'_1\, b\cos\theta'\,\hat{\mathbf{z}}' - (GM_E m_2/r_2^3)\, r'_2\, b\cos\theta'\,(-\hat{\mathbf{z}}')$$

$$= -GM_E b\cos\theta' \left[(m_1/r_1^3)\, r'_1 - (m_2/r_2^3)\, r'_2 \right] \hat{\mathbf{z}}'$$

$$= GM_E b\cos\theta' \left[(m_2\, r'_2/r_2^3) - (r'_1 m_1/r_1^3) \right] \hat{\mathbf{z}}'$$

But now use

$$m_1 r'_1 = m_2 r'_2 \qquad \text{or} \qquad r'_2/r'_1 = m_1/m_2 \tag{D.2.8}$$

and then

$$\mathbf{N}^{(b)} = GM_E b\cos\theta' \left[(m_1 r'_1/r_2^3) - (r'_1 m_1/r_1^3) \right] \hat{\mathbf{z}}'$$

$$= GM_E b\cos\theta' \, m_1 r'_1 \left[(1/r_2^3) - (1/r_1^3) \right] \hat{\mathbf{z}}'$$

$$= GM_E b\cos\theta' \, m_1 r'_1 \, \frac{r_1^3 - r_2^3}{r_1^3 r_2^3} \, \hat{\mathbf{z}}'$$

which might be reasonable! If the masses are equal, then $r_1 = r_2$ and we expect 0 torque. If the masses are not equal, then $r_1 \neq r_2$ in the drawing, and then there is not zero torque! Also, if $\theta' = \pm\pi/2$ for vertical positions, then $\cos\theta' = 0$ and there is zero torque, so those positions might be stable.

I think this is a big step forward (if it is correct). I could of course do my Maple expansion on this for small parameter ε_1. It is hard to say just how strong this torque is relative to the fictitious torque that I will have to deal with soon.

Dealing with the Fictitious Torque.

$$\mathbf{N'}^{(0)}_{fict} = -\mathbf{r'} \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\,\boldsymbol{\omega} \times \mathbf{v'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r'}]$$
$$+ m\,\dot{\mathbf{b}}_S \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}) \tag{11.3.10}$$

Now digress to quote some Special Case #1 results, since the satellite is Special Case #1:

Torque Stuff for Special Case #1

$$\dot{\mathbf{b}}_S \equiv (d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} \,. \qquad \text{// Special Case \#1} \tag{4.4.3}$$

or

$$\dot{\mathbf{b}}_S = \boldsymbol{\omega} \times \mathbf{b} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r'}) = \boldsymbol{\omega} \times \mathbf{r} - \boldsymbol{\omega} \times \mathbf{r'} \tag{6.10}$$

And also we have

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \,. \qquad \text{Special Case \#1} \tag{7.13}$$

Aside: Can I replace $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$?

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = (\boldsymbol{\omega} \bullet \mathbf{b})\boldsymbol{\omega} - \omega^2 \mathbf{b} \qquad \text{// vector ident}$$

In general Special Case #1 we don't know that $\boldsymbol{\omega} \bullet \mathbf{b} = 0$ (see Fig 1 in Overview). But for the satellite problem we do know that $\boldsymbol{\omega} \bullet \mathbf{b} = 0$ so we can then say

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 \mathbf{b} \qquad \text{Satellite Case} \qquad (7.12)$$

So for the satellite problem we have

$$\dot{\mathbf{b}}_S = \boldsymbol{\omega} \times \mathbf{b}$$

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 \mathbf{b}$$

So insert this into fictitious torque above to get

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{r'} \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\,\boldsymbol{\omega} \times \mathbf{v'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r'}]$$
$$+ m\,\dot{\mathbf{b}}_S \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}) \qquad (11.3.10)$$

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{r'} \times [\, m\{\dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 \mathbf{b}\} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\,\boldsymbol{\omega} \times \mathbf{v'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r'}]$$
$$+ m\,(\boldsymbol{\omega} \times \mathbf{b}) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}) \qquad (11.3.10)$$

Now drop all Euler terms and restate

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{r'} \times [\, -m\omega^2 \mathbf{b} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\,\boldsymbol{\omega} \times \mathbf{v'}]$$
$$+ m\,(\boldsymbol{\omega} \times \mathbf{b}) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}) \qquad (11.3.10)$$

Try to simplify. Write

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) = (\boldsymbol{\omega} \bullet \mathbf{r'})\boldsymbol{\omega} - \omega^2 \mathbf{r'}$$

For the satellite at t = 0 ω is perp to paper and r' for each particle is in paper, so

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) = -\omega^2 \mathbf{r'} \qquad \text{Satellite}$$

so put that in to get

$$\mathbf{N'^{(0)}_{fict}} = -\mathbf{r'} \times [\, -m\omega^2 \mathbf{b} - m\omega^2 \mathbf{r'} + 2m\,\boldsymbol{\omega} \times \mathbf{v'}] + m\,(\boldsymbol{\omega} \times \mathbf{b}) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'})$$

At least I can now write it on one line. This is just for one of our two particles. How about that last term

$$(\boldsymbol{\omega} \times \mathbf{b}) \times (\boldsymbol{\omega} \times \mathbf{r}') = [\boldsymbol{\omega},\mathbf{b},\mathbf{r}']\,\boldsymbol{\omega}$$

so at least I know its direction. Also I know that $\mathbf{b} + \mathbf{r}' = \mathbf{r}$. So insert these two things

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -\,\mathbf{r}' \times [\,-m\omega^2\mathbf{r} + 2m\,\boldsymbol{\omega} \times \mathbf{v}'] + m\,(\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}' + m\,[\boldsymbol{\omega},\mathbf{b},\mathbf{r}']\,\boldsymbol{\omega}$$

I don't have any idea about the direction of $\mathbf{v}'$, though I could insist it is planar at t = 0. That would be an assumption that might be hard to justify. How would you get it into that state?

What I really need to do is show this whole thing is somehow small perhaps because ω is small. I probably got to this same point last time I tried this. Write this out for the two particles

$$\mathbf{N}'^{(0)}_{\mathtt{fict},1} = -\,m_1\mathbf{r}_1' \times [\,-\omega^2\mathbf{r}_1 + 2\,\boldsymbol{\omega} \times \mathbf{v}'_1] + m_1\,(\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_1 + m_1\,[\boldsymbol{\omega},\mathbf{b},\mathbf{r}'_1]\,\boldsymbol{\omega}$$
$$\mathbf{N}'^{(0)}_{\mathtt{fict},2} = -\,m_2\mathbf{r}_2' \times [\,-\omega^2\mathbf{r}_2 + 2\boldsymbol{\omega} \times \mathbf{v}'_2] + m_2\,(\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_2 + m_2\,[\boldsymbol{\omega},\mathbf{b},\mathbf{r}'_2]\,\boldsymbol{\omega}$$

Now recall that

$$m_1 r'_1 = m_2 r'_2 \qquad \text{or} \qquad r'_2/r'_1 = m_1/m_2 \qquad \text{or} \qquad m_1\mathbf{r}'_1 = -\,m_2\mathbf{r}'_2\,. \tag{D.2.8}$$

If I differentiate this last equation in Frame S' then in terms of naturals: $m_1\mathbf{v}'_1 = -\,m_2\mathbf{v}'_2$

So how we can write

$$\mathbf{N}'^{(0)}_{\mathtt{fict},1} = -\;m_1\mathbf{r}_1' \times [\,-\omega^2\mathbf{r}_1 + 2\,\boldsymbol{\omega} \times \mathbf{v}'_1] + m_1\,(\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_1 + m_1\,[\boldsymbol{\omega},\mathbf{b},\mathbf{r}'_1]\,\boldsymbol{\omega}$$
$$\mathbf{N}'^{(0)}_{\mathtt{fict},2} = +\,m_1\mathbf{r}'_1 \times [\,-\omega^2\mathbf{r}_2 + 2\boldsymbol{\omega} \times \mathbf{v}'_2] - m_1\,(\boldsymbol{\omega} \times \mathbf{b}) \times \mathbf{v}'_1 - m_1\,[\boldsymbol{\omega},\mathbf{b},\mathbf{r}'_1]\,\boldsymbol{\omega}$$

If I add these, the last two terms cancel!!! If correct, that would be a major "break" for me. Then

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = m_1\mathbf{r}'_1 \times [\,-\omega^2\mathbf{r}_2 + 2\boldsymbol{\omega} \times \mathbf{v}'_2 + \omega^2\mathbf{r}_1 - 2\,\boldsymbol{\omega} \times \mathbf{v}'_1\,]$$

$$= m_1\mathbf{r}'_1 \times [\,\omega^2(\mathbf{r}_1 - \mathbf{r}_2) - 2\,\boldsymbol{\omega} \times (\mathbf{v}'_1 - \mathbf{v}'_2)\,]$$

Now I know that

$$\mathbf{r}_1 = \mathbf{b} + \mathbf{r}'_1$$
$$\mathbf{r}_2 = \mathbf{b} + \mathbf{r}'_2$$

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}'_1 - \mathbf{r}'_2 = (r'_1 + r'_2)\,\hat{\mathbf{r}}'_1 = s\,\hat{\mathbf{r}}'_1$$

Also we know that

$$\mathbf{v}'_2 = -\,(m_1/m_2)\,\mathbf{v}'_1$$

Then we have

$$\mathbf{N'^{(0)}_{fict}} = m_1\mathbf{r'_1} \times [\, \omega^2 s\, \hat{\mathbf{r}}'_1 - 2\,\boldsymbol{\omega} \times (\mathbf{v'_1} + (m_1/m_2)\,\mathbf{v'_1})\,]$$

$$= m_1\mathbf{r'_1} \times [\, \omega^2 s\, \hat{\mathbf{r}}'_1 - 2\,\boldsymbol{\omega} \times \mathbf{v'_1}\,(1 + (m_1/m_2))\,]$$

$$= m_1\mathbf{r'_1} \times [\, \omega^2 s\, \hat{\mathbf{r}}'_1 - (2/\mu_2)\,\boldsymbol{\omega} \times \mathbf{v'_1}\,]$$

But now $\mathbf{r'_1} \times \hat{\mathbf{r}}'_1 = 0$, so only the second term survives! (I don't believe any of this) so

$$\mathbf{N'^{(0)}_{fict}} = -2(m_1/\mu_2)\,\mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v'_1})$$

that is an amazing simplification if correct. No approximations have been made!!

$$\mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v'_1}) = (\mathbf{r'_1} \bullet \mathbf{v'_1})\boldsymbol{\omega} - (\mathbf{r'_1} \bullet \boldsymbol{\omega})\mathbf{v'_1}$$

But at t = 0 we have $(\mathbf{r'_1} \bullet \boldsymbol{\omega}) = 0$ so get

$$\mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v'_1}) = (\mathbf{r'_1} \bullet \mathbf{v'_1})\,\boldsymbol{\omega}$$

and then

$$\mathbf{N'^{(0)}_{fict}} = -2(m_1/\mu_2)\,(\mathbf{r'_1} \bullet \mathbf{v'_1})\,\boldsymbol{\omega}$$

or

$$\mathbf{N'^{(0)}_{fict}} = -2(m_1/\mu_2)\,(\mathbf{r'_1} \bullet \mathbf{v'_1})\,\omega\,\hat{\mathbf{z}}'$$

Compare this then with

$$\mathbf{N}^{(b)} = GM_E b\cos\theta'\; m_1 r'_1\; \frac{r_1^3 - r_2^3}{r_1^3 r_2^3}\; \hat{\mathbf{z}}'$$

Could this be all more simply done just using fictitious forces?

Question: can you say that $N'_{eff} = r' \times F'_{eff}$?

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I guess I can do the usual steps here. For usual circular motion in Frame S with radius vector **b** we get

$$\mathbf{b} = \mathbf{b}$$

$$\dot{\mathbf{b}}_S = b\,\dot{\theta}\,\hat{\boldsymbol{\theta}} = b\,\omega\,\hat{\boldsymbol{\theta}}$$

$$\ddot{\mathbf{b}}_S = -b\dot{\theta}^2\,\hat{\mathbf{r}} = -b\omega^2\,\hat{\mathbf{r}}$$

and we usually write

$$m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = -m\omega^2 r' \hat{\mathbf{r}}'$$

where $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{r}}$ are Frame S spherical unit vectors (polar). Then insert to get

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -\mathbf{r}' \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m\,\dot{\mathbf{b}}_S \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \qquad (11.3.10)$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -\mathbf{r}' \times [\, -mb\omega^2 \hat{\mathbf{r}} - m\omega^2 r' \hat{\mathbf{r}}' + 2m\,\boldsymbol{\omega} \times \mathbf{v}' \,]$$
$$+ mb\,\omega\,\hat{\boldsymbol{\theta}} \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}') \qquad (11.3.10)$$

where I dumped the Euler term. The result is still quite messy and hence a little scary.

Now $\hat{\boldsymbol{\theta}}$ is to the left at t = 0 as my picture is drawn. It is $-\hat{x}$ at t = 0, and it is $-\hat{x}'$ all the time. The direction the satellite is going. Meanwhile ω x r' is also in the plane of paper, so cannot kill last term, and **v'** could be anything.

Is this a Special Case #1 for torque? I should have said something about that in frames doc, then I would now have to do that here.