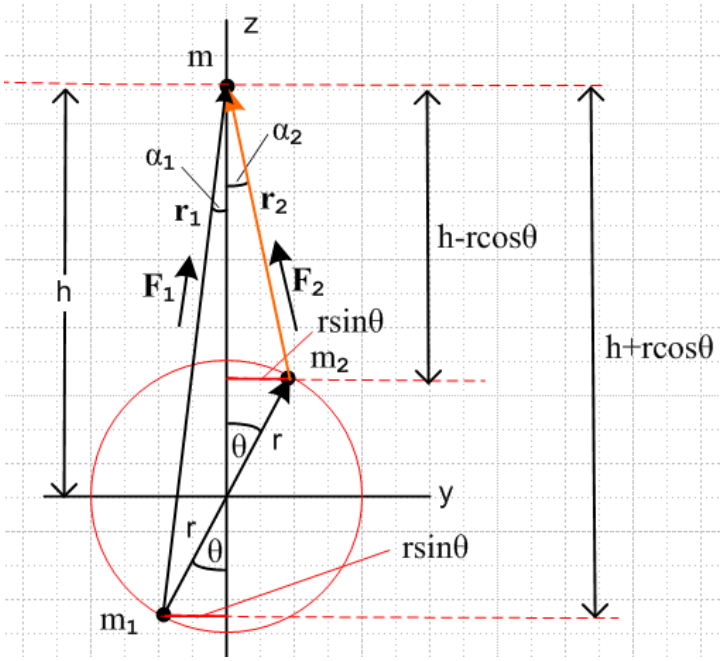
Appendix F v0 PhL 1.11.15

This was an early attempt at doing the torque, and I get the result $\omega_t \approx \sqrt{3}\,\omega$. Still using the planar picture and Earth at the top.

D.6 Gravitational torque on the tethered satellite

As noted in the main text, gravity exerts a torque on a tethered satellite which acts to restore it to the nearest vertically aligned position. Here we derive an expression for that torque.

We start by recalling Fig (D.2.1), where $\hat{\mathbf{x}}$ points toward the viewer,



(D.5.1)

We shall compute the torque about the origin of this figure, but this is the same as the torque about the center of gravity because

I think you have to include the tether tension in this discussion. Maybe it should be computed as a first step. I cannot just say the above picture is "static" because I know it is not static, it is a snapshot of a position while the thing is librating say. The position shown is not stable.

The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ were found in (D.2.6) to be (these forces act *on* m_1 and m_2) ,

$$\mathbf{F}_1 = (Gmm_1/r^3_1)\, r\sin\theta\, \hat{\mathbf{y}} + (Gmm_1/r^3_1)\,(h + r\cos\theta)\hat{\mathbf{z}}$$
$$\mathbf{F}_2 = -(Gmm_2/r^3_2)\, r\sin\theta\, \hat{\mathbf{y}} + (Gmm_2/r^3_2)\,(h - r\cos\theta)\hat{\mathbf{z}} \qquad (D.2.6)$$

where from (D.2.3),

$$r_1^2 = (r\sin\theta)^2 + (h+r\cos\theta)^2$$
$$r_2^2 = (r\sin\theta)^2 + (h-r\cos\theta)^2 \ . \qquad (D.2.3) \qquad (D.5.2)$$

Mass m_2 is at position $\mathbf{r}$ which is

$$r = y\hat{\mathbf{y}} + z\hat{\mathbf{z}} == r\sin\theta\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}} \ . \qquad (D.5.3)$$

With respect to the origin of the drawing we compute these torques on m_1 and m_2

$$\mathbf{N}_1 = \mathbf{(-r)} \times \mathbf{F}_1 = -(Gmm_1/r_1^3)[\ r\sin\theta\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}] \times [\ r\sin\theta\ \hat{\mathbf{y}} + \ (h + r\cos\theta)\hat{\mathbf{z}}]$$

$$= -(Gmm_1/r_1^3)\{\ r\sin\theta\ (h + r\cos\theta)\ \hat{\mathbf{x}} + \ r\cos\theta\ r\sin\theta\ (\ -\hat{\mathbf{x}}\ )\ \}$$

$$= -(Gmm_1 r/r_1^3)\{\ \sin\theta\ (h + r\cos\theta)\ -\ \cos\theta\ r\sin\theta\ \}\hat{\mathbf{x}}$$

$$= -(Gmm_1 r/r_1^3)\ h\ \sin\theta\ \hat{\mathbf{x}} \qquad (D.5.4)$$

$$\mathbf{N}_2 = \mathbf{r} \times \mathbf{F}_2 = (Gmm_2/r^3_2)[\ r\sin\theta\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}] \times [\ -\ r\sin\theta\ \hat{\mathbf{y}} + \ (h\ -\ r\cos\theta)\hat{\mathbf{z}}\ ]$$

$$= (Gmm_2/r^3_2)\ \{\ r\sin\theta\ (h\ -\ r\cos\theta)\hat{\mathbf{x}}\ -\ r\cos\theta\ r\sin\theta(\ -\hat{\mathbf{x}}\ )\ \}$$

$$= (Gmm_2 r/r^3_2)\ \{\ \sin\theta\ (h\ -\ r\cos\theta)\ +\ \cos\theta\ r\sin\theta\ \}\ \hat{\mathbf{x}}$$

$$= (Gmm_2 r/r^3_2)\ \ h\ \sin\theta\ \hat{\mathbf{x}} \qquad (D.5.5)$$

The total torque on the satellite about the origin is then

$$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 = \ \ (Gmr)\ \ (\ m_2/r^3_2\ -\ m_1/r_1^3)\ h\ \sin\theta\ \hat{\mathbf{x}} \qquad \text{dim ok} \qquad (D.5.6)$$

which is in or out of the plane of paper depending on parameter sizes.

In Fig (D.2.1) we take r,θ to be polar coordinates where $-\pi < \theta \leq \pi$. When m_2 is on the right side of the drawing, $\sin\theta > 0$, and when it is on the left, $\sin\theta < 0$. The torque direction is given by,

$$\hat{\mathbf{N}} = \text{sign}(\ m_2/r^3_2\ -\ m_1/r_1^3)\ \text{sign}(\sin\theta)\ \hat{\mathbf{x}} \qquad (D.5.7)$$

or

$$\hat{\mathbf{N}} = \ \text{sign}(\ m_2/r^3_2\ -\ m_1/r_1^3)\ \hat{\mathbf{x}} \qquad \text{for } m_2 \text{ on the right}$$
$$\hat{\mathbf{N}} = -\ \text{sign}(\ m_2/r^3_2\ -\ m_1/r_1^3)\ \hat{\mathbf{x}} \qquad \text{for } m_2 \text{ on the left} \qquad (D.5.8)$$

Suppose m_2 is a large mass so that $(m_2/r^3_2 - m_1/r_1^3) > 0$ and m_2 is on the right. In this case, $\hat{\mathbf{N}} = \hat{\mathbf{x}}$ and the torque acts to move m_2 to the top of the circle. If m_2 were on the left, one gets instead $\hat{\mathbf{N}} = -\hat{\mathbf{x}}$ and again the torque acts to move m_2 to the top of the circle. In this case, the stable position of the satellite has m_2 on the top of the circle, which means it is closest to the Earth.

Suppose m_2 is a *small* mass so that $(m_2/r^3_2 - m_1/r_1^3) < 0$. In this case the torque acts to move m_2 to the bottom of the circle, farthest from Earth.

If the masses are equal, for m_2 on the upper half of the circle one has $r_1 > r_2$ so then

$$\hat{\mathbf{N}} = \text{sign}(1/r^3_2 - 1/r_1^3)\,\hat{\mathbf{x}} = +\hat{\mathbf{x}} \qquad \text{for } m_2 \text{ on the right}$$
$$\hat{\mathbf{N}} = -\text{sign}(1/r^3_2 - 1/r_1^3)\,\hat{\mathbf{x}} = -\hat{\mathbf{x}} \qquad \text{for } m_2 \text{ on the left} \qquad \text{(D.5.8)}$$

and this is the situation of a restoring torque, so either mass is stable in the vertically aligned case.

Tether Oscillation

Suppose m_2 is a large enough mass so m_2 is restored by the torque to the top of the circle. If θ is a small angle, the satellite oscillates in a harmonic fashion about this position. Similar to $\mathbf{F} = m\ddot{\mathbf{x}}$ for linear motion, one has $\mathbf{N} = I\,\ddot{\boldsymbol{\theta}}$ for angular motion where $I = (m_1+m_2)r^2$ is the moment of inertia about the origin. If θ is small then $\sin\theta \approx \theta$ and we then have

$$\mathbf{N} = I\,\ddot{\boldsymbol{\theta}}$$

$$N\hat{\mathbf{x}} = I\,\ddot{\theta}\,(-\hat{\mathbf{x}}) \qquad \text{// because } \theta \text{ is defined clockwise in Fig (D.5.1), the "wrong way"}$$

$$N = -I\,\ddot{\theta}$$

$$(Gmr)\,(m_2/r^3_2 - m_1/r_1^3)\,h\,\theta + (m_1+m_2)r^2\,\ddot{\theta} = 0$$

$$[(Gmh/r)\,(\mu_2/r^3_2 - \mu_1/r_1^3)]\,\theta + \ddot{\theta} = 0 \qquad \text{dim ok}$$

$$\ddot{\theta} + \omega_t^2\theta = 0. \qquad \text{(D.5.9)}$$

The tethered satellite oscillates with motion $\theta = \sin(\omega_t t + C)$ where

$$\omega_t = \sqrt{(Gmh/r)\,(\mu_2/r^3_2 - \mu_1/r_1^3)} \qquad \text{(D.5.10)}$$

But for small θ in this position we have $r_1 \approx h+r$ and $r_2 \approx h-r$ so

$$\omega_t = \sqrt{(Gmh/r)\ (\mu_2/(h-r)^3 - \mu_1/(h+r)^3)} \qquad \text{(D.5.11)}$$

What happens in the limit that $m_1 \rightarrow 0$? Then have

$$\omega_t \approx \sqrt{(Gmh/r)\ (1/(h-r)^3)} \approx \sqrt{(Gmh/r)\ h^{-3}} = (1/h)\sqrt{(Gm/r)}$$

$$\omega_t \approx (1/R_E)\sqrt{(Gm_E/r)} = \sqrt{(Gm_E/R_E^3)(R_E/r)} = \sqrt{\omega^2(R_E/r)} = \omega\sqrt{R_E/r}$$

This says as $r \rightarrow 0$, $\omega_t \rightarrow \infty$. How do I explain that? Just have a single point mass m_2 in orbit, why should there be a libration going on in the first place?

$R_E = 6371$ km $\approx 6.4 \times 10^6$ m $r \approx 10$m $\sqrt{R_E/r} = \sqrt{6.4 \times 10^5} \approx 2500$

$T_t = 88$ min / 2500 = 88 x 60 sec/2500 = 2 second

For h >> r (typically the case!) one has

$$1/(h-r)^3 = (h-r)^{-3} = h^{-3}(1- r/h)^{-3} \approx h^{-3}[1 + (-3)(-r/h)] \qquad // (1+a)^n \approx 1 + nb$$

$$= h^{-3}[1 + 3(r/h)]$$

$$1/(h+r)^3 = h^{-3}[1 - 3(r/h)]\ . \qquad \text{(D.5.12)}$$

Then

$$(\mu_2/r^3_2 - \mu_1/r_1^3) \approx \mu_2/(h-r)^3 - \mu_1/(h+r)^3 \approx \mu_2 h^{-3}[1 + 3(r/h)] - \mu_1 h^{-3}[1 - 3(r/h)]$$

$$= h^{-3}[(\mu_2-\mu_1) + 3(r/h)(\mu_2+\mu_1)]$$

$$= h^{-3}[(\mu_2-\mu_1) + 3(r/h)] \qquad \text{(D.5.13)}$$

and therefore

$$\omega_t \approx \sqrt{(Gmh/r)\ h^{-3}[(\mu_2-\mu_1) + 3(r/h)]}$$

$$= (1/h)\sqrt{(Gm/r)[(\mu_2-\mu_1) + 3(r/h)]} \qquad \text{dim ok} \qquad \text{(D.5.14)}$$

If the masses are exactly equal so $\mu_1=\mu_2$ one gets

$$\omega_t \approx (1/h)\sqrt{(Gm/r)[3(r/h)]} = (1/h)\sqrt{(3Gm/h)} = \sqrt{(3Gm/h^3)}$$

Recall from (8.22) that

$$GM_E/r^3_0 = \omega^2 \ . \qquad (8.22)$$

where ω is the orbital frequency of a satellite and r_0 is the radius of its center of gravity. For h>> r we have $r_0 \approx h$ and then

$$\omega^2 \approx GM_E/h^3$$

and this the natural tether frequency can be written

$$\omega_t \approx \sqrt{3}\,\omega \qquad \text{or} \qquad T_t = T/\sqrt{3}$$

A low earth orbit satellite has a period of about 88 minutes so this tether oscillation period would be about 51 minutes, a little less than an hour, so the oscillation is relatively "slow". Result *** appears on page 126 of Ref [] .

Suppose the masses are *slightly* unequal such that $(\mu_2-\mu_1) >> 3(r/h)$. Then we get

$$\omega_t \approx (1/h)\sqrt{(Gm/r)(\mu_2-\mu_1)} \qquad \text{dim ok}$$

and then

$$\omega_t = \omega\sqrt{(r_E/r)(\mu_2-\mu_1)}$$

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Question: In (D.1.4) I claim that $\mathbf{N}^{(r_{cog})} = 0$. I also say that

$$\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$$

where **F** is the total force on the system. In finding $\mathbf{r}_{cog}$ I assume it is along **F** so then $\mathbf{N}^{(0)} = 0$ as well.

So what the heck is the non-zero torque I am talking about above???