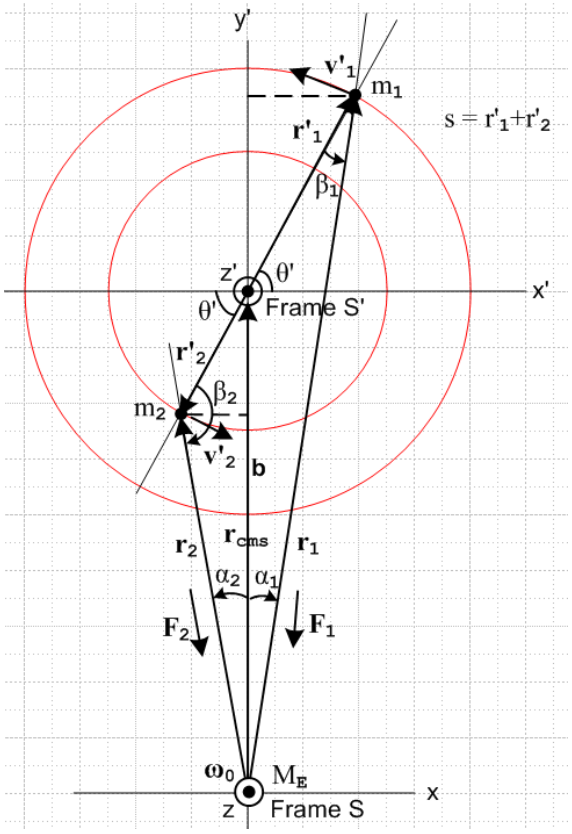
Appendix F v1 PhL

D.6 Equation of Motion for the Dumbbell Satellite in Frame S'

Here again is the satellite where we have added angles β_1 and β_2 along with velocities $\mathbf{v'}_1$ and $\mathbf{v'}_2$.



(D.6.1)

Recalling (D.2.8),

$$m_1\mathbf{r'}_1 = -\, m_2\mathbf{r'}_2 \quad \Rightarrow \quad m_1 r'_1 = m_2 r'_2\,, \quad r'_2/r'_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}'_2 = -\ \hat{\mathbf{r}}'_1 \qquad \textit{(D.2.8)}$$

we apply time derivative $\partial_{S'}$ once and then again to obtain two new sets of facts,

$$m_1\mathbf{v'}_1 = -\, m_2\mathbf{v'}_2 \quad \Rightarrow \quad m_1 v'_1 = m_2 v'_2\,, \quad v'_2/v'_1 = m_1/m_2\,, \quad \hat{\mathbf{v}}'_2 = -\ \hat{\mathbf{v}}'_1 \qquad (D.6.2)$$

$$m_1\mathbf{a'}_1 = -\, m_2\mathbf{a'}_2 \quad \Rightarrow \quad m_1 a'_1 = m_2 a'_2\,, \quad a'_2/a'_1 = m_1/m_2\,, \quad \hat{\mathbf{a}}'_2 = -\ \hat{\mathbf{a}}'_1 \qquad (D.6.3)$$

Then (D.6.2) shows that if $\mathbf{v'}_1$ lies in the plane of paper, so does $\mathbf{v'}_2$ and then mass m_2 is doing circular motion around the inner red circle.

For angular momentum we take our reference point to be $\mathbf{c}' = 0$ in Frame S' and thus $\mathbf{c} = \mathbf{b}$ in Frame S. Then

$$\mathbf{L}'^{(0)} = m_1 \mathbf{r}'_1 \times \mathbf{v}'_1 + m_2 \mathbf{r}'_2 \times \mathbf{v}'_2 = \{m_1 r'_1 v'_1 + m_2 r'_2 v'_2\}\ \hat{\mathbf{z}}' \qquad \text{(D.6.4)}$$

Taking a $\partial_{S'}$ time derivative gives,

$$\dot{\mathbf{L}}'^{(0)} = m_1 [\ \mathbf{r}'_1 \times \mathbf{a}'_1 + \mathbf{v}'_1 \times \mathbf{v}'_1\ ] + m_2 [\ \mathbf{r}'_2 \times \mathbf{a}'_2 + \mathbf{v}'_2 \times \mathbf{v}'_2\ ]$$

$$= m_1\ \mathbf{r}'_1 \times \mathbf{a}'_1 + m_2\ \mathbf{r}'_2 \times \mathbf{a}'_2\ . \qquad \text{(D.6.5)}$$

END

For circular motion we shall make use of polar coordinate unit vectors $\hat{\mathbf{r}}'_1$ and $\hat{\boldsymbol{\theta}}'_1$. Due to the stick constraint on the masses, $\omega'_1 = \omega'_2 = \omega' \equiv (d\theta'/dt)_{S'}$. It is also true that $\hat{\mathbf{r}}'_2 = -\hat{\mathbf{r}}'_1$ and $\hat{\boldsymbol{\theta}}'_2 = -\hat{\boldsymbol{\theta}}'_1$.

For the circular motion of mass m_1 we know that

$$\mathbf{v}'_1 = \omega' r'_1\ \hat{\boldsymbol{\theta}}'_1$$

so

$$\mathbf{a}'_1 = \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1 + \omega' r'_1\ (\partial_{S'} \hat{\boldsymbol{\theta}}'_1) = \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1 + \omega' r'_1\ (-\omega'\ \hat{\mathbf{r}}'_1) = \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1 - \omega'^2 r'_1 \hat{\mathbf{r}}'_1\ .$$

Therefore,

$$\mathbf{r}'_1 \times \mathbf{a}'_1 = \mathbf{r}'_1 \times [\ \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1 - \omega'^2 r'_1 \hat{\mathbf{r}}'_1\ ] = \mathbf{r}'_1 \times [\ \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1] = \dot{\omega}'\ r'^2_1\ \hat{\mathbf{r}}'_1 \times \hat{\boldsymbol{\theta}}'_1 = \dot{\omega}'\ r'^2_1\ \hat{\mathbf{z}}'$$

These same equations apply to m_2, so we summarize the above as

$$\begin{aligned} &\mathbf{v}'_1 = \omega' r'_1\ \hat{\boldsymbol{\theta}}'_1 \qquad && \mathbf{a}'_1 = \dot{\omega}' r'_1\ \hat{\boldsymbol{\theta}}'_1 - \omega'^2 r'_1 \hat{\mathbf{r}}'_2 \qquad && \mathbf{r}'_1 \times \mathbf{a}'_1 = \dot{\omega}'\ r'^2_1\ \hat{\mathbf{z}}' \\ &\mathbf{v}'_2 = \omega' r'_2\ \hat{\boldsymbol{\theta}}'_2 && \mathbf{a}'_2 = \dot{\omega}' r'_2\ \hat{\boldsymbol{\theta}}'_2 - \omega'^2 r'_2 \hat{\mathbf{r}}'_2 && \mathbf{r}'_2 \times \mathbf{a}'_2 = \dot{\omega}'\ r'^2_2\ \hat{\mathbf{z}}' \ . \end{aligned} \qquad \text{(D.6.6)}$$

Therefore from (D.6.5),

$$\dot{\mathbf{L}}'^{(0)} = m_1\ \dot{\omega}'\ r'^2_1\ \hat{\mathbf{z}}' + m_2\ \dot{\omega}'\ r'^2_2\ \hat{\mathbf{z}}' = \dot{\omega}' [m_1\ r'^2_1 + m_2\ r'^2_2\ ]\ \hat{\mathbf{z}}'$$

$$= \ddot{\theta}' [m_1\ r'^2_1 + m_2\ r'^2_2\ ]\ \hat{\mathbf{z}}' \ . \qquad \text{(D.6.7)}$$

The equation of motion for the satellite within Frame S' is given by (11.3.4),

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(0)} = \ddot{\theta}' [m_1\ r'^2_1 + m_2\ r'^2_2\ ]\ \hat{\mathbf{z}}' \ . \qquad \textit{(11.3.4)} \qquad \text{(D.6.8)}$$

Our next task then is to compute the total effective torque on the satellite in Frame S' which from (11.3.5)

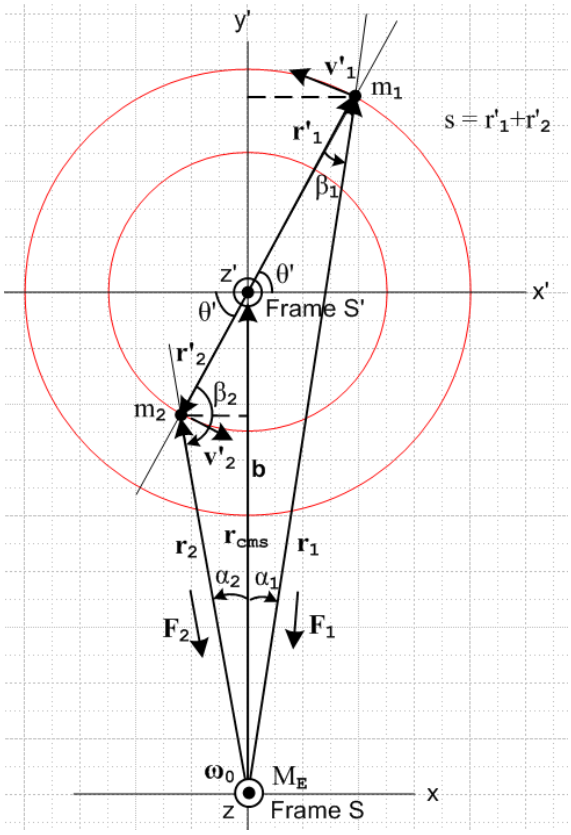
is

$$\mathbf{N'^{(0)}_{eff}} = \mathbf{N^{(b)}} + \mathbf{N'^{(0)}_{fict}} \qquad (11.3.5) \qquad (D.6.9)$$

Here $\mathbf{N^{(b)}}$ is the Frame S torque on the satellite relative to point **b** in Frame S, and $\mathbf{N'^{(0)}_{fict}}$ is the fictitious torque that arises because Frame S' is a rotating frame of reference.

D.7 The true torque on the Dumbbell Satellite in Frame S

In this section we assume that Fig ** is correct at t = 0 so $\mathbf{r'_1}$ is in the plane of paper, but we make no assumption about the direction of $\mathbf{v'_1}$. We repeat the previous drawing.



(D.6.1)

The total torque in Frame S on the satellite is given by

$$\mathbf{N^{(b)}} = \mathbf{r'_1} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2} \qquad (D.7.1)$$

where all four of these vectors are shown in the figure. Recall from (D.2.13) that

$$\mathbf{F_1} = -(GM_E m_1/r_1^2)\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r_1^3)\,\mathbf{r_1}$$
$$\mathbf{F_2} = -(GM_E m_2/r_2^2)\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r_2^3)\,\mathbf{r_2} \quad . \qquad (D.2.13)$$

We really should include the stick tension/compression **T** since these are supposed to be *total* forces, but since $\mathbf{r'}_1 \times \mathbf{T} = \mathbf{r'}_2 \times \mathbf{T} = \mathbf{0}$ so we can ignore **T**. Therefore

$$\mathbf{r'}_1 \times \mathbf{F}_1 = \mathbf{r'}_1 \times [\, -(GM_E m_1/r_1^3)\, \mathbf{r}_1] = -(GM_E m_1/r_1^3)\, \mathbf{r'}_1 \times \mathbf{r}_1$$

$$\mathbf{r'}_2 \times \mathbf{F}_2 = \mathbf{r'}_2 \times [\, -(GM_E m_2/r_2^3)\, \mathbf{r}_2] = -(GM_E m_2/r_2^3)\, \mathbf{r'}_2 \times \mathbf{r}_2 \qquad \text{(D.7.2)}$$

Looking at the drawing we see that the two cross products may be written.

$$\mathbf{r'}_1 \times \mathbf{r}_1 = r'_1 r_1 \sin\beta_1\ \hat{\mathbf{z}}'$$
$$\mathbf{r'}_2 \times \mathbf{r}_2 = r'_2 r_2 \sin\beta_2\ (-\hat{\mathbf{z}}') \qquad \text{(D.7.3)}$$

The Law of Sines applied to the right and left triangles says

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r'_1} = \frac{\sin(\pi/2+\theta')}{r_1} = \frac{\cos\theta'}{r_1}$$
$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r'_2} = \frac{\sin(\pi/2-\theta')}{r_2} = \frac{\cos\theta'}{r_2} \qquad \text{(D.7.4)}$$

from which we conclude that

$$r_1 \sin\beta_1 = b\cos\theta'$$
$$r_2 \sin\beta_2 = b\cos\theta' \qquad \text{(D.7.5)}$$

so then

$$\mathbf{r'}_1 \times \mathbf{r}_1 = r'_1 r_1 \sin\beta_1\ \hat{\mathbf{z}}' = r'_1\, b\cos\theta'\, \hat{\mathbf{z}}'$$
$$\mathbf{r'}_2 \times \mathbf{r}_2 = r'_2 r_2 \sin\beta_2\ (-\hat{\mathbf{z}}') = -r'_2\, b\cos\theta'\, \hat{\mathbf{z}}' \,. \qquad \text{(D.7.6)}$$

The total torque is then

$$\mathbf{N}^{(b)} = \mathbf{r'}_1 \times \mathbf{F}_1 + \mathbf{r'}_2 \times \mathbf{F}_2 = -(GM_E m_1/r_1^3)\, \mathbf{r'}_1 \times \mathbf{r}_1 - (GM_E m_2/r_2^3)\, \mathbf{r'}_2 \times \mathbf{r}_2$$

$$= -(GM_E m_1/r_1^3)\, r'_1\, b\cos\theta'\, \hat{\mathbf{z}}' + (GM_E m_2/r_2^3)\ r'_2\, b\cos\theta'\, \hat{\mathbf{z}}'$$

$$= GM_E b\cos\theta' \,[\, (r'_2 m_2/r_2^3) - (r'_1 m_1/r_1^3)\ ]\, \hat{\mathbf{z}}'$$

$$= GM_E b\cos\theta' \,[\, (r'_1 m_1/r_2^3) - (r'_1 m_1/r_1^3)\ ]\, \hat{\mathbf{z}}' \qquad \text{// (D.2.8)}$$

$$= GM_E b\cos\theta' m_1 r'_1 \,[\, (1/r_2^3) - (1/r_1^3)\ ]\, \hat{\mathbf{z}}'$$

this then is the total torque on the satellite due to the gravitational force of the Earth evaluated in the inertial fixed Frame S. We restate this result as

$$\mathbf{N}^{(b)} = (GM_E m_1 r'_1 b)\cos\theta' \left[(1/r_2^3) - (1/r_1^3) \right] \hat{\mathbf{z}}' \qquad \text{(D.7.7)}$$

where

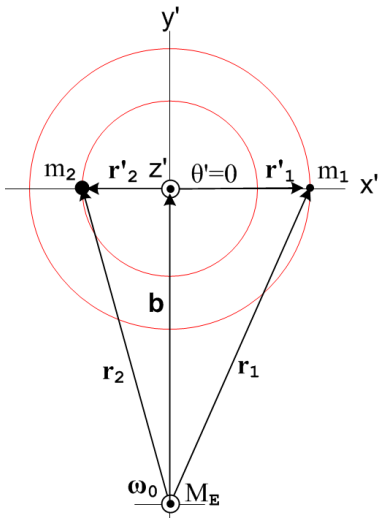
$$r_1^2 = r'^2_1 + b^2 + 2r'_1 b \sin\theta'$$
$$r_2^2 = r'^2_2 + b^2 - 2r'_2 b \sin\theta' \quad . \qquad \textit{(D.2.4)}$$

Quick checks:

• If the satellite is vertically aligned, $\theta' = \pm\pi/2$, $\cos\theta'=0$ and $\mathbf{N}^{(b)} = 0$ as expected (no moment arms).

• If the satellite is horizontally aligned *and* $m_1 = m_2$, then $r_1 = r_2$ so $\mathbf{N}^{(b)} = 0$ (balanced moment arms).

• If $m_2 > m_1$, then $r'_2 < r'_1$ so for horizontal alignment one has $r_2 < r_1$ so $[(1/r_2^3) - (1/r_1^3)] > 0$. Then if m_1 is on the right, we have $\theta' = 1$ and $\cos\theta' = 1$ and then (D.7.7) has $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$. Here is a picture,



(D.7.8)

In this case the lever arms balance in the sense that $m_2 r'_2 = m_1 r'_1$, but m_2 is closer to Earth center so it feels the stronger force and so we expect $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$

If we now assume as before that $r_1, r_2, b >> r'_1, r'_2$ we can approximate $[(1/r_2^3) - (1/r_1^3)]$ by adding on to the Maple code shown in (D.3.5) to get

```
f := (1/r2)^3 - (1/r1)^3:
series(f,e1=0,3): simplify(%);
```

$$-3\frac{\sin(\theta)}{b^3(-1+\mu 1)}\,el - \frac{3}{2}\frac{4 - 8\,\mu 1 - 5\cos(\theta)^2 + 10\,\mu 1\cos(\theta)^2}{b^3(1-2\,\mu 1+\mu 1^2)}\,el^2 + O(el^3)$$

where recall $e1 = \varepsilon_1 \equiv (r'_1/b)$. This time there is a leading linear term in ε_1 so then

$$(1/r_2^3) - (1/r_1^3) \approx -3\,\frac{\sin\theta'}{b^3(-\mu_2)}\,(r'_1/b) = 3r'_1\sin\theta'/(\mu_2 b^4) \qquad \text{(D.7.9)}$$

Then

$$\mathbf{N}^{(b)} = (GM_E m_1 r'_1 b)\cos\theta'\,[\,(1/r_2^3) - (1/r_1^3)\,]\,\hat{\mathbf{z}}' \approx (GM_E m_1 r'_1 b)\cos\theta'\;3r'_1\sin\theta'/(\mu_2 b^4)\;\hat{\mathbf{z}}'$$

$$= (GM_E m_1 r'^2_1 b^{-3}\mu_2^{-1})\,3\sin\theta'\cos\theta'\;\hat{\mathbf{z}}' \qquad \text{(D.7.10)}$$

To this order of approximation, the torque $\mathbf{N}^{(b)}$ vanishes when the satellite is horizontally aligned as well as when it is vertically aligned, and this is due to $r_1 \approx r_2$ in the horizontal case.

D.8 The fictitious torque on the Dumbbell Satellite in Frame S'

At t = 0 the two masses are in the plane of paper in Fig (D.6.1) so $\mathbf{r}'_1$ (and thus $\mathbf{r}'_2$) are in the plane of paper and are thus normal to $\boldsymbol{\omega}$ and $\hat{\mathbf{z}}$. Nothing is assumed about the direction of $\mathbf{v}'_1$ (and $\mathbf{v}'_2$) other than the fact that these velocities must be tangent to the spheres of radius r'_1 and r'_2 to which their masses are constrained in Frame S'.

Recall the general fictitious torque expression given in (11.3.10), acting on a single particle of mass m,

$$\mathbf{N}'^{(c')}_{fict} = -(\mathbf{r}'-\mathbf{c}')\times[\,m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') + 2m\,\boldsymbol{\omega}\times\mathbf{v}' + m\dot{\boldsymbol{\omega}}\times\mathbf{r}']$$
$$+ m(\ddot{\mathbf{c}}' + \boldsymbol{\omega}\times\mathbf{c}' + \ddot{\mathbf{b}}_S)\times(\,\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}' + \dot{\mathbf{b}}_S) - m\ddot{\mathbf{c}}'\times\mathbf{v}' \quad . \qquad \textit{(11.3.10)} \qquad \text{(D.8.1)}$$

Here ω is the angular rotation rate of the satellite about the Earth.

Our application has the torque center at $\mathbf{c}' = 0$ (and $\mathbf{c} = \mathbf{b}$) so this simplifies somewhat to

$$\mathbf{N}'^{(0)}_{fict} = -m\mathbf{r}'\times[\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') + 2\boldsymbol{\omega}\times\mathbf{v}' + \dot{\boldsymbol{\omega}}\times\mathbf{r}'] + \dot{\mathbf{b}}_S\times(\,m\mathbf{v}' + \boldsymbol{\omega}\times[m\mathbf{r}']) \quad . \qquad \text{(D.8.2)}$$
frame centrifugal Coriolis Euler

Recall from (8.1.8) that the square bracket in the above is $-\mathbf{F}'_{fict}/m$ so we trace the origin of the terms.

Now write this separately for each of the two masses of our satellite,

$$\mathbf{N}'^{(0)}_{f,1} = -m_1\mathbf{r}'_1\times[\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_1) + 2\boldsymbol{\omega}\times\mathbf{v}_1' + \dot{\boldsymbol{\omega}}\times\mathbf{r}'_1] + \dot{\mathbf{b}}_S\times(\,m_1\mathbf{v}'_1 + \boldsymbol{\omega}\times[\,m_1\mathbf{r}'_1])$$
$$\mathbf{N}'^{(0)}_{f,2} = -m_2\mathbf{r}'_2\times[\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_2) + 2\boldsymbol{\omega}\times\mathbf{v}'_2 + \dot{\boldsymbol{\omega}}\times\mathbf{r}'_2] + \dot{\mathbf{b}}_S\times(\,m_2\mathbf{v}'_2 + \boldsymbol{\omega}\times[m_2\mathbf{r}'_2]) \,. \qquad \text{(D.8.3)}$$

In the second line, use (D.2.8) to replace $m_2\mathbf{r}'_2 = -m_1\mathbf{r}'_1$ and (D.6.2) to replace $m_2\mathbf{v}'_2 = -m_1\mathbf{v}'_1$:

$$\mathbf{N'^{(0)}_{f,2}} = +\, m_1 \mathbf{r'_1} \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (-m_1 \mathbf{v'_1} + \boldsymbol{\omega} \times [-m_1 \mathbf{r'_1}]) \,. \tag{D.8.4}$$

Next, add the two torques to get the total fictitious torque on the satellite,

$$\mathbf{N'^{(0)}_{fict}} = \mathbf{N'^{(0)}_{f,1}} + \mathbf{N'^{(0)}_{f,2}}$$

$$= -\, m_1 \mathbf{r'_1} \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}] + \dot{\mathbf{b}}_S \times (m_1 \mathbf{v'_1} + \boldsymbol{\omega} \times [\, m_1 \mathbf{r'_1}])$$
$$+\, m_1 \mathbf{r'_1} \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (-m_1 \mathbf{v'_1} + \boldsymbol{\omega} \times [-m_1 \mathbf{r'_1}])$$

$$= -\, m_1 \mathbf{r'_1} \times \{\, \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r'_1} - \mathbf{r'_2}])}_{\text{centrifugal}} + \underbrace{2\boldsymbol{\omega} \times [\mathbf{v_1'} - \mathbf{v'_2}]}_{\text{Coriolis}} + \underbrace{\dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} - \mathbf{r'_2}]}_{\text{Euler}} \,\} \,. \tag{D.8.5}$$

All terms involving $\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$ have cancelled out. Since $[\mathbf{r'_1} - \mathbf{r'_2}]$ lies in the plane of paper, we know from

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{A} - A^2 \mathbf{C}$$

that

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r'_1} - \mathbf{r'_2}]) = (\boldsymbol{\omega} \bullet [\mathbf{r'_1} - \mathbf{r'_2}])\boldsymbol{\omega} - \omega^2 [\mathbf{r'_1} - \mathbf{r'_2}] = -\,\omega^2 [\mathbf{r'_1} - \mathbf{r'_2}] = -\,\omega^2 [1 + \frac{m_1}{m_2}] \mathbf{r'_1} \,. \tag{D.8.6}$$

Since $\mathbf{r'_1} \times \mathbf{r'_1} = 0$ the first term in (D.8.5) vanishes, leaving

$$\mathbf{N'^{(0)}_{fict}} = -\, m_1 \mathbf{r'_1} \times \{\, \underbrace{2\boldsymbol{\omega} \times [\mathbf{v_1'} - \mathbf{v'_2}]}_{\text{Coriolis}} + \underbrace{\dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} - \mathbf{r'_2}]}_{\text{Euler}} \,\} \tag{D.8.7}$$

The Euler term can be written

$$\dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} - \mathbf{r'_2}] = \dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} + \frac{m_1}{m_2} \mathbf{r'_1}] = \dot{\boldsymbol{\omega}} \times \mathbf{r'_1} \,(1 + \frac{m_1}{m_2})$$

Then $\quad \mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{A} - A^2 \mathbf{C}$

$$-\,\mathbf{r'_1} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r'_1}) = \mathbf{r'_1} \times (\mathbf{r'_1} \times \dot{\boldsymbol{\omega}}) = (\mathbf{r'_1} \bullet \dot{\boldsymbol{\omega}})\, \mathbf{r'_1} - r'^2_1 \dot{\boldsymbol{\omega}}$$

Since we only deal with a circular satellite orbit, we know that $\dot{\boldsymbol{\omega}} = 0$, but for an elliptical orbit this would not be the case ("equal areas in equal times") and a

We now set $\dot{\boldsymbol{\omega}} = 0$ for the satellite orbit so only the Coriolis-induced term survives,

$$\mathbf{N'^{(0)}}_{fict} = -2m_1\mathbf{r'}_1 \times [\,\boldsymbol{\omega} \times (\mathbf{v}_1' - \mathbf{v}_2')\,] = -2m_1\mathbf{r'}_1 \times [\,\boldsymbol{\omega} \times (\mathbf{v}_1' + \frac{m_1}{m_2}\mathbf{v}_1')\,]$$

$$= -2m_1\mathbf{r'}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1')\,(1 + \frac{m_1}{m_2}) = -2m_1\mathbf{r'}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1')\,(1 + \frac{\mu_1}{\mu_2})$$

$$= -2(m_1/\mu_2)\,\mathbf{r'}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1') \quad . \qquad \text{(D.8.8)}$$

But now the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

says

$$\mathbf{r'}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1') = (\mathbf{r'}_1\bullet\mathbf{v}_1')\boldsymbol{\omega} - (\mathbf{r'}_1\bullet\boldsymbol{\omega})\mathbf{v}_1' = (\mathbf{r'}_1\bullet\mathbf{v}_1')\boldsymbol{\omega} = (\mathbf{r'}_1\bullet\mathbf{v}_1')\omega\,\hat{\mathbf{z}}' \,. \qquad \text{(D.8.9)}$$

where we used the fact that $\mathbf{r'}_1\bullet\boldsymbol{\omega} = 0$, so at this point

$$\mathbf{N'^{(0)}}_{fict} = -2(m_1/\mu_2)\,(\mathbf{r'}_1\bullet\mathbf{v}_1')\omega\,\hat{\mathbf{z}}' \,.$$

But since $\mathbf{r'}_1$ is constrained by the stick to lie on a sphere of radius r'_1 (a cross section of which is shown in red in Fig D.6.1), the velocity $\mathbf{v}_1'$ must be tangent to that sphere, so $(\mathbf{r'}_1\bullet\mathbf{v}_1') = 0$. We then end up our final elaborate result :

$$\mathbf{N'^{(0)}}_{fict} = 0 \quad . \qquad \text{(D.8.10)}$$

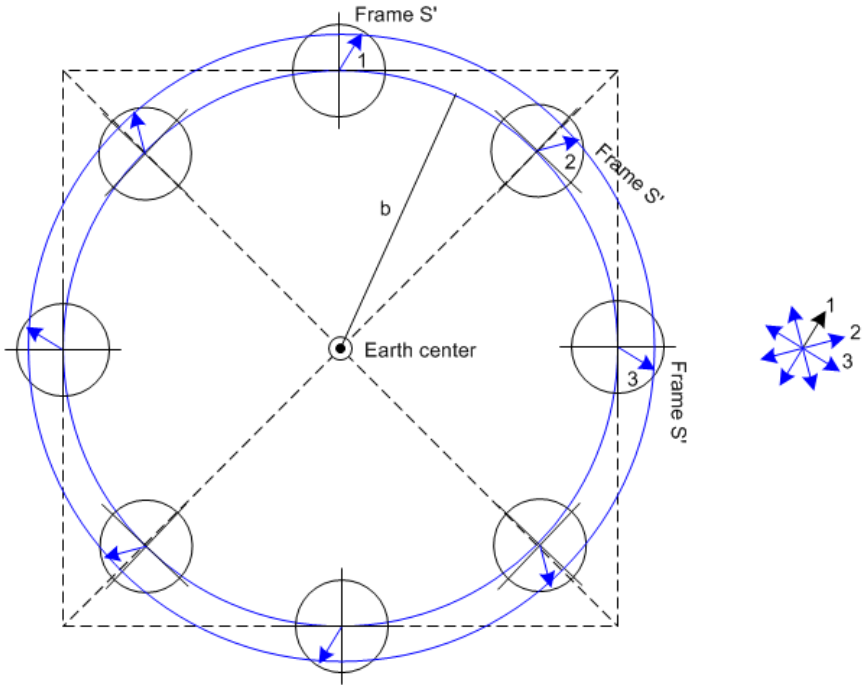
How might one interpret this simple result?

We examine the pieces of $\mathbf{N'^{(0)}}_{fict} = 0$:

(a) the "frame effects" due to the motion of $\mathbf{b}$ ($\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$) are equal and opposite for the two masses because the origin of Frame S' is at the center of mass causing $m_2\mathbf{r'}_2 = -m_1\mathbf{r'}_1$, as shown in (D.8.5).

(b) within Frame S', the centrifugal acceleration term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) = -\omega^2\mathbf{r'}_1$ tries to push m_1 to a larger radius. But $\mathbf{r'}_1$ is constrained to lie on a sphere of radius r'_1 so m_1 cannot *go* to a larger radius. This centrifugal acceleration is neutralized by part of the tension in the stick which we avoided talking about.

This centrifugal term, by the way, involves "the short vector" $\mathbf{r'}_1$ and not the long vector $\mathbf{r}_1$. We discussed this situation in Section 8.2 for an Earth-based Frame S'. In our current context, the picture that corresponds to Fig (8..2.8) is the following



(D.8.11)

The arrow in each location during the orbit represents the vector $\mathbf{r'}_1$ where we assume that other effects are turned off so $\mathbf{r'}_1$ stays fixed in Frame S'. When these arrows are transferred to the picture on the right with common tails, we see that $\mathbf{r'}_1$ does in fact go around in a circle of radius r'_1 and that is why the corresponding centrifugal force acting on m_1 is $-\omega^2\mathbf{r'}_1$.

(c) Within Frame S', the Coriolis force - $2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1'$ tries to deflect mass m_1 "to the right" in Fig (D.6.1). But again, $\mathbf{r'}_1$ is constrained to lie on a sphere of radius r'_1 so m_1 cannot *deflect* to a different radius. This Coriolis force is neutralized by the rest of the tension in the stick.

D.9 The total torque on the Dumbbell Satellite in Frame S'

After much effort, we have arrived at this set of results for the satellite :

$\mathbf{N}^{(b)} = (GM_E m_1 r'_1 b)\cos\theta' \left[(1/r_2^3) - (1/r_1^3) \right] \hat{\mathbf{z}}'$ // true torque *(D.7.7)*

$\mathbf{N}^{(b)} \approx (GM_E m_1 r'^2_1 b^{-3} \mu_2^{-1})\, 3\sin\theta'\cos\theta'\, \hat{\mathbf{z}}'$ // far approximation *(D.7.10)*

$\mathbf{N'}^{(0)}_{fict} = 0$. // fictitious torque *(D.8.10)*

$\dot{\mathbf{L}}'^{(0)} = m_1\ \mathbf{r'}_1 \times \mathbf{a'}_1 + m_2\ \mathbf{r'}_2 \times \mathbf{a'}_2$. *(D.6.5)*

$\mathbf{N'}^{(0)}_{eff} = \mathbf{N}^{(b)} + \mathbf{N'}^{(0)}_{fict} = \dot{\mathbf{L}}'^{(0)}$. *(11.3.6)*

So far we have assumed nothing about directions of $\mathbf{v}_1'$ and $\mathbf{a}'_1$.

Here is a possible equation:

$$\mathbf{N}^{(b)} \bullet \hat{\mathbf{z}}' = [m_1\ \mathbf{r}'_1 \times \mathbf{a}'_1 + m_2\ \mathbf{r}'_2 \times \mathbf{a}'_2\ ] \bullet \hat{\mathbf{z}}' = (GM_E m_1 r'^2_1 b^{-3} \mu_2^{-1})\ 3\sin\theta'\cos\theta'$$

Now

$$(\mathbf{r}'_1 \times \mathbf{a}'_1) \bullet \hat{\mathbf{z}}' = \text{cyclic} = (\hat{\mathbf{z}}' \times \mathbf{r}'_1) \bullet \mathbf{a}'_1$$

I know that $\mathbf{v}'_1$ must be tangent to its sphere. I think this means that $d\mathbf{v}'_1$ must point to sphere center. But how do you do this at some general point on a sphere.

Think totally independent spherical coordinates r,θ,φ having nothing to do with Frame S or Frame S'. I have the point $\mathbf{r}'_1$ lying somewhere on a sphere and I refer to this as **r**. Let **v** be the velocity of a particle at that point, and let **a** be its acceleration. We can write

$$\mathbf{r} = r\,\hat{\mathbf{r}}$$

$$\mathbf{v} = v_\theta \hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// a most general form for } \mathbf{v} \text{ which is tangent to the sphere}$$

What then can we say about acceleration **a** ?

$$\mathbf{a} = \dot{v}_\theta \hat{\boldsymbol{\theta}} + \mathbf{v}_\theta\ \dot{\hat{\boldsymbol{\theta}}} + \dot{v}_\varphi \hat{\boldsymbol{\varphi}} + \mathbf{v}_\varphi\ \dot{\hat{\boldsymbol{\varphi}}}$$

$$\dot{\hat{\boldsymbol{\theta}}} = d\hat{\boldsymbol{\theta}}/dt = (d\hat{\boldsymbol{\theta}}/d\theta)(d\theta/dt) + (d\hat{\boldsymbol{\theta}}/d\varphi)(d\varphi/dt)$$

$$= -\,\hat{\mathbf{r}}\,\dot{\theta} + \cos\theta\,\hat{\boldsymbol{\varphi}}\,\dot{\varphi}$$

$\theta \quad \varphi$