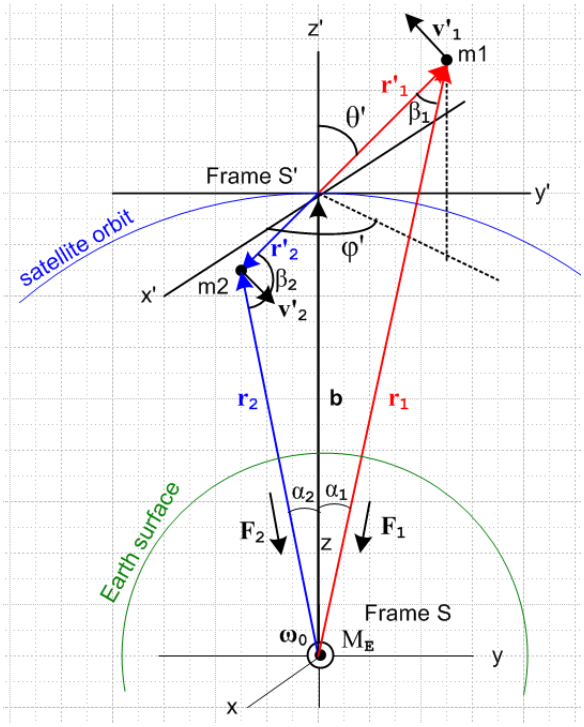
**D.6 Equation of Motion for the Dumbbell Satellite in Frame S'**

We now place the dumbbell satellite in a more general orientation:




Description of the Figure

Half the battle is having a clear picture of what is going on and we shall expend many words to describe the above picture. It shows the dumbbell satellite in orbit with a completely arbitrary orientation. The two masses $m_1$ and $m_2$ are connected by a massless stick of length $r'_1+r'_2 = s$ which we have not drawn. If we find that this stick is always in tension for some situation, we can replace it in that situation with a non-stretching massless tether. If the stick were to go into compression, the replacement tether would lose its linear shape and we don't want to deal with that problem.

As in our earlier pictures, Frame S is an inertial frame whose center is the center of the Earth and which is fixed with respect to the stars. Frame S' rotates as the satellite center of mass (Frame S' origin) moves around its orbit.

At time t = 0, the axes of both Frames align with each other. At any time, axes y,z and y',z' are in the plane of paper. The axis z always points to the center of mass point which is the origin of Frame S'. Thus $\mathbf{b} = b\hat{\mathbf{z}} = b\hat{\mathbf{z}}'$ at any time. The axes x,x' always point direction out of the plane of paper (normal to paper).

The orbital rotation vector $\boldsymbol{\omega}_0$ also points out of the plane of paper and recall that $\omega_0 = 2\pi/T_0$ where $T_0$ is about 88 minutes for a low-Earth orbit, although we have in mind an orbit at any altitude.

Frame S' has spherical coordinates r',θ',φ' define in the usual "physics" convention as shown. Angle θ' is the polar angle down from the "vertical" z' axis, φ' is the azimuthal angle measured from the x' axis toward the t' axis.

Mass $m_1$ has Frame S' position $\mathbf{r'}_1$ and Frame S position $\mathbf{r}_1$.These two vectors as drawn lie on the viewer's side of the plane of paper and we show them in red. The corresponding vectors $\mathbf{r'}_2$ and $\mathbf{r}_2$ lie to the back of the plane of paper and are shown in blue. So red is near, blue is far.

There are two large non-right triangles in the picture and neither lies in the plane of paper. The triangle on the right side has internal angles $\alpha_1$ and $\beta_1$ as shown, while that on the left side has angles $\alpha_2$ and $\beta_2$.

The particle of mass $m_1$ has $\mathbf{r'}_1 = (r'_1,\theta',\varphi')$ as its spherical coordinates. Because the Frame S' origin is the center of mass, the particle of mass $m_2$ has $\mathbf{r'}_1 = (r'_2,\pi-\theta',\varphi'+\pi)$. The vector $\mathbf{r'}_2$ is collinear with $\mathbf{r'}_1$ but points in the opposite direction from $\mathbf{r'}_1$. The relation $m_1 r'_1 = m_2 r'_2$ (see below) shows how the masses and the vector lengths are related.

The particle of mass $m_1$ is constrained to lie on a sphere of radius $r'_1$, while the particle of mass $m_2$ is constrained to lie on a sphere of radius $r'_2$. The picture assumes $m_1 < m_2$ so $r'_1 > r'_2$. If $m_1$ lies at a point on its sphere, $m_2$ lies on the inverse point but on its sphere, as the spherical coordinates above show. If the masses are the same, the two spheres coincide.

Due to these constraints on the vectors $\mathbf{r'}_1$ and $\mathbf{r'}_2$, the corresponding velocity vectors $\mathbf{v'}_1$ and $\mathbf{v'}_2$ of the two masses must be tangential to their respective spheres. Thus for example $\mathbf{v'}_1 = v'_{1\theta}\,\hat{\boldsymbol{\theta}}' + v'_{1\varphi}\,\hat{\boldsymbol{\varphi}}'$, whereas $\mathbf{r'}_1 = r'_1\,\hat{\mathbf{r}}'$ . This is an example of a problem where it would probably be better to use the "swap notation" to get rid of the sea of primes we shall encounter, but we shall plod on with all the primes.

The circled dot at the bottom is the center of the Earth (mass $M_E$) and we have drawn the Earth surface in green. The blue circle is the orbit of the center of mass of the satellite (Frame S' origin) and it lies in the plane of paper. Note that $\omega_0$ is for the orbit and has nothing whatsoever to do with the rotation of the Earth. The above picture is the same whether or not the Earth rotates at its 24 hour $\omega_1$ rate about some obscure $\hat{\boldsymbol{\omega}}_1$ axis (not shown!).

Triangle Trig

The internal angles of the red triangle (not in the plane of paper) are $\beta_1$, $\alpha_1$ and $\pi-\theta'$ at the origin. Thus we know from one of the three laws of cosines that

$$r_1^2 = b^2 + r'^2_1 - 2br'_1\cos(\pi-\theta') = b^2 + r'^2_1 + 2br'_1\cos\theta'$$

The internal angles of the blue triangle (not in the plane of paper) are $\beta_2$, $\alpha_2$ and $\theta'$ at the origin. Thus we know that

$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$$

The law of sines for the two triangles tells us that

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r'_1} = \frac{\sin(\pi-\theta')}{r_1} = -\frac{\sin\theta'}{r_1}$$

$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r'_2} = \frac{\sin\theta'}{r_2}$$

Summary:

$$r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$$
$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$$

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r'_1} = -\frac{\sin\theta'}{r_1}$$
$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r'_2} = +\frac{\sin\theta'}{r_2}$$

Basic Dumbbell Facts and Angular Momentum

Recalling (D.2.8),

$$m_1\mathbf{r}'_1 = -\,m_2\mathbf{r}'_2 \quad \Rightarrow \quad m_1r'_1 = m_2r'_2\,, \quad r'_2/r'_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}'_2 = -\,\hat{\mathbf{r}}'_1 \qquad \textit{(D.2.8)}$$

we apply time derivative $\partial_{S'}$ once and then again to obtain two new sets of facts,

$$m_1\mathbf{v}'_1 = -\,m_2\mathbf{v}'_2 \quad \Rightarrow \quad m_1v'_1 = m_2v'_2\,, \quad v'_2/v'_1 = m_1/m_2\,, \quad \hat{\mathbf{v}}'_2 = -\,\hat{\mathbf{v}}'_1 \qquad \text{(D.6.2)}$$

$$m_1\mathbf{a}'_1 = -\,m_2\mathbf{a}'_2 \quad \Rightarrow \quad m_1a'_1 = m_2a'_2\,, \quad a'_2/a'_1 = m_1/m_2\,, \quad \hat{\mathbf{a}}'_2 = -\,\hat{\mathbf{a}}'_1 \qquad \text{(D.6.3)}$$

For angular momentum we take our reference point to be $\mathbf{c}' = 0$ in Frame S' and thus $\mathbf{c} = \mathbf{b}$ in Frame S. Then

$$m_1\mathbf{v}'_1 = -\,m_2\mathbf{v}'_2$$

$$\mathbf{p}'_1 = -\mathbf{p}'_2$$

$$\mathbf{L}'^{(0)} = m_1\mathbf{r}'_1 \times \mathbf{v}'_1 + m_2\mathbf{r}'_2 \times \mathbf{v}'_2 \qquad \text{(D.6.4)}$$

$$= m_1\mathbf{r}'_1 \times (\mathbf{v}'_1 - \mathbf{v}'_2)$$

$$= m_1\mathbf{r}'_1 \times (\mathbf{v}'_1 + (m_1/m_2)\mathbf{v}'_1)$$

$$= m_1(1 + (m_1/m_2))\ \mathbf{r}'_1 \times \mathbf{v}'_1 \qquad \text{M L L/T} = \text{ML}^2/\text{T}$$

$$= (m_1/m_2)\,M\ \mathbf{r}'_1 \times \mathbf{v}'_1\,.$$

Now we know that

$$\mathbf{v} = v_\theta \hat{\boldsymbol{\theta}} + v_\varphi \hat{\boldsymbol{\varphi}}$$

so

$$\mathbf{v}'_1 = v'_{\theta 1} \hat{\boldsymbol{\theta}}' + v'_{\varphi 1} \hat{\boldsymbol{\varphi}}'$$

Then

$$\mathbf{r}'_1 \times \mathbf{v}'_1 = r'_1 \hat{\mathbf{r}}'_1 \times [v'_{\theta 1} \hat{\boldsymbol{\theta}}' + v'_{\varphi 1} \hat{\boldsymbol{\varphi}}'] = r'_1 (v'_{\theta 1} \hat{\boldsymbol{\varphi}}' - v'_{\varphi 1} \hat{\boldsymbol{\theta}}')$$

Then since

$$v_\theta = r \dot{\theta}$$

$$v_\varphi = r \dot{\varphi} \sin\theta$$

we have

$$v'_{\theta 1} = r'_1 \dot{\theta}'$$

$$v'_{\varphi 1} = r'_1 \dot{\varphi}' \sin\theta'$$

Thus

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, r'_1 (v'_{\theta 1} \hat{\boldsymbol{\varphi}}' - v'_{\varphi 1} \hat{\boldsymbol{\theta}}')$$

$$= (m_1/m_2)\, M\, r'_1 ( r'_1 \dot{\theta}' \hat{\boldsymbol{\varphi}}' - r'_1 \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}')$$

$$= (m_1/m_2)\, M\, r'^2_1 ( \dot{\theta}' \hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}') \qquad \text{// dim} = ML^2/T$$

The dumbbell has no angular momentum around the $\hat{\mathbf{r}}'_1$ axis which seems very reasonable since it consists of two point masse aligned with $\hat{\mathbf{r}}'_1$

I guess I will want to do everything in sphericals and then look at the torque equation? Sound very complicated. But I think these are the natural coordinates to use!!!!

Taking a $\partial_{S'}$ time derivative gives this rate of change of angular momentum in Frame S'

$$\dot{\mathbf{L}}'^{(0)} = m_1 [\, \mathbf{r}'_1 \times \mathbf{a}'_1 + \mathbf{v}'_1 \times \mathbf{v}'_1 \,] + m_2 [\, \mathbf{r}'_2 \times \mathbf{a}'_2 + \mathbf{v}'_2 \times \mathbf{v}'_2 \,]$$

$$= m_1 \mathbf{r}'_1 \times \mathbf{a}'_1 + m_2 \mathbf{r}'_2 \times \mathbf{a}'_2 \ .$$

$$= m_1 \mathbf{r}'_1 \times (\mathbf{a}'_1 - \mathbf{a}'_2) = m_1 ( 1 + (m_1/m_2)) \mathbf{r}'_1 \times \mathbf{a}'_1 = (m_1/m_2)\, M\, \mathbf{r}'_1 \times \mathbf{a}'_1$$

$$\mathbf{a}'_1 = a'_{r1} \hat{\mathbf{r}}'_1 + a'_{\theta 1} \hat{\boldsymbol{\theta}}'_1 + a'_{\varphi 1} \hat{\boldsymbol{\varphi}}'_1 = a'_{r1} \hat{\mathbf{r}}'_1 + a'_{\theta 1} \hat{\boldsymbol{\theta}}' + a'_{\varphi 1} \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{r'_1 x a'_1} = r'_1\ \hat{\mathbf{r}}'_1 x\ [\ a'_{r1}\ \hat{\mathbf{r}}'_1 + a'_{\theta 1}\hat{\boldsymbol{\theta}}'\ + a'_{\varphi 1}\ \hat{\boldsymbol{\varphi}}'\ ]\ =\ r'_1(a'_{\theta 1}\ \hat{\boldsymbol{\varphi}}'\ -\ a'_{\varphi 1})\ \hat{\boldsymbol{\theta}}'$$

So we find that

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'_1\ [\ a'_{\theta 1}\ \hat{\boldsymbol{\varphi}}'\ -\ a'_{\varphi 1}\ \hat{\boldsymbol{\theta}}'\ ]$$

We know that

$$a_\theta = r\ddot{\theta} - r\,\dot{\varphi}^2 \sin\theta\cos\theta = r(\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)$$
$$a_\varphi = 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta = r(2\dot{\theta}\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)$$

Therefore

$$a'_{\theta 1} = r'_1(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta'\cos\theta')$$
$$a'_{\varphi 1} = r'_1(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')$$

And then

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1\ [\ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta'\cos\theta')\ \hat{\boldsymbol{\varphi}}'\ -\ (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\ \hat{\boldsymbol{\theta}}'\ ] \qquad // \ ML^2/T^2$$

Here are the conclusions so far:

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\ r'^2_1(\ \dot{\theta}'\hat{\boldsymbol{\varphi}}' - \dot{\varphi}'\sin\theta'\hat{\boldsymbol{\theta}}')$$

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1\ [\ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta'\cos\theta')\ \hat{\boldsymbol{\varphi}}'\ -\ (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\ \hat{\boldsymbol{\theta}}'\ ]$$

Obviously if L' has no r' component, so $L'_r \equiv 0$, then one must also have $\dot{L}'_r \equiv 0$. So far so good.

The equation of motion for the satellite within Frame S' is given by (11.3.4),

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(0)}$$

Our next task then is to compute the total effective torque on the satellite in Frame S' which from (11.3.5) is

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \mathbf{N}^{(b)} + \mathbf{N}'^{(0)}_{\mathbf{fict}} \qquad \textit{(11.3.5)} \qquad \text{(D.6.9)}$$

Here $\mathbf{N}^{(b)}$ is the Frame S torque on the satellite relative to point **b** in Frame S, and $\mathbf{N}'^{(0)}_{\mathbf{fict}}$ is the fictitious torque that arises because Frame S' is a rotating frame of reference.

ok to here

**D.7 The true torque on the Dumbbell Satellite in Frame S**

The total torque in Frame S on the satellite is given by

$$\mathbf{N}^{(b)} = \mathbf{r'}_1 \times \mathbf{F}_1 + \mathbf{r'}_2 \times \mathbf{F}_2 \qquad \text{L M L/T}^2 = \text{ML}^2/\text{T}^2 \qquad \text{(D.7.1)}$$

where all four of these vectors are shown in the figure. Recall from (D.2.13) that

$$\mathbf{F}_1 = -(GM_E m_1/r_1^2)\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r_1^3)\,\mathbf{r}_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r_2^2)\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r_2^3)\,\mathbf{r}_2 \quad . \qquad \textit{(D.2.13)}$$

We really should include the stick tension/compression **T** since these are supposed to be *total* forces, but since $\mathbf{r'}_1 \times \mathbf{T} = \mathbf{r'}_2 \times \mathbf{T} = \mathbf{0}$ so we can ignore **T**. Therefore

$$\mathbf{r'}_1 \times \mathbf{F}_1 = \mathbf{r'}_1 \times [-(GM_E m_1/r_1^3)\,\mathbf{r}_1] = -(GM_E m_1/r_1^3)\,\mathbf{r'}_1 \times \mathbf{r}_1$$

$$\mathbf{r'}_2 \times \mathbf{F}_2 = \mathbf{r'}_2 \times [-(GM_E m_2/r_2^3)\,\mathbf{r}_2] = -(GM_E m_2/r_2^3)\,\mathbf{r'}_2 \times \mathbf{r}_2 \qquad \text{(D.7.2)}$$

Now we know that

$$\mathbf{r}_1 = \mathbf{b} + \mathbf{r'}_1$$

so that

$$\mathbf{r'}_1 \times \mathbf{r}_1 = \mathbf{r'}_1 \times (\mathbf{b} + \mathbf{r'}_1) = \mathbf{r'}_1 \times \mathbf{b} = r'_1 b(\hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}')$$

Now from App E we know thast

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

so

$$\hat{\mathbf{r}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}} \times (\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = -\sin\theta\,\hat{\boldsymbol{\varphi}}$$

so therefore

$$\hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}' = -\sin\theta'\,\hat{\boldsymbol{\varphi}}'$$

and we then find that

$$\mathbf{r'}_1 \times \mathbf{r}_1 = -\,r'_1 b \sin\theta'\,\hat{\boldsymbol{\varphi}}'$$

Next,

$$\mathbf{r}_2 = \mathbf{b} + \mathbf{r}'_2$$

$$\mathbf{r}'_2 \times \mathbf{r}_2 = \mathbf{r}'_2 \times (\mathbf{b} + \mathbf{r}'_2) = \mathbf{r}'_2 \times \mathbf{b} = r'_2 b(\hat{\mathbf{r}}'_2 \times \hat{\mathbf{z}}') = - r'_2 b(\hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}')$$

$$= (- r'_2 b)( - \sin\theta' \hat{\boldsymbol{\varphi}}') = + r'_2 b \sin\theta' \hat{\boldsymbol{\varphi}}'$$

Therefore:

$$\mathbf{r}'_1 \times \mathbf{F}_1 = - (GM_E m_1 / r_1^3) \mathbf{r}'_1 \times \mathbf{r}_1 = - (GM_E m_1 / r_1^3)(- r'_1 b \sin\theta' \hat{\boldsymbol{\varphi}}') = GM_E m_1 b (r'_1 / r_1^3) \sin\theta' \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{r}'_2 \times \mathbf{F}_2 = - (GM_E m_2 / r_2^3) ( r'_2 b \sin\theta' \hat{\boldsymbol{\varphi}}') = - GM_E m_2 b (r'_2 / r_2^3) \sin\theta' \hat{\boldsymbol{\varphi}}'$$

$$= - GM_E m_1 b (r'_1 / r_2^3) \sin\theta' \hat{\boldsymbol{\varphi}}'$$

Then

$$\mathbf{N}^{(b)} = \mathbf{r}'_1 \times \mathbf{F}_1 + \mathbf{r}'_2 \times \mathbf{F}_2 \qquad LML/T^2 = ML^2/T^2$$

$$= GM_E m_1 b (r'_1 / r_1^3) \sin\theta' \hat{\boldsymbol{\varphi}}' - GM_E m_1 b (r'_1 / r_2^3) \sin\theta' \hat{\boldsymbol{\varphi}}'$$

$$= GM_E m_1 b \, r'_1 \sin\theta' [ (1/r_1^3) - (1/r_2^3)] \hat{\boldsymbol{\varphi}}' \qquad \text{force x L}$$

where

$$r_1^2 = b^2 + r'^2_1 + 2br'_1 \cos\theta'$$
$$r_2^2 = b^2 + r'^2_2 - 2br'_2 \cos\theta'$$

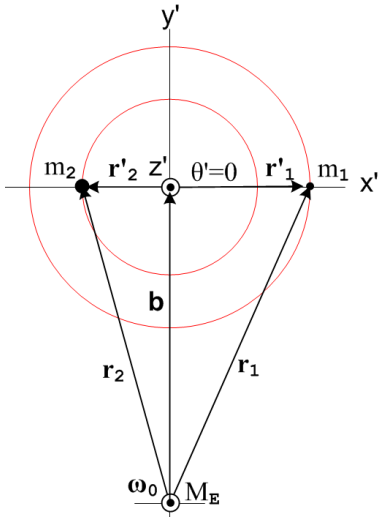
How do I interpret this result that the torque only has one component and it is about the $\hat{\boldsymbol{\varphi}}'$ axis ?? If you were look at $\varphi' = \pi/2$, then we have a planar picture and the torque clearly is about $\hat{\mathbf{z}}'$ and has no components in the other two directions since everything is in a plane. In that case $\hat{\boldsymbol{\varphi}}' = - \hat{\mathbf{z}}'$ so it makes sense.

Quick checks:

- If the satellite is vertically aligned, $\theta' = 0, \pi$, then $\sin\theta' = 0$ and $\mathbf{N}^{(b)} = 0$ as expected (no moment arms).

- If the satellite is horizontally aligned *and* $m_1 = m_2$, then $r_1 = r_2$ so $\mathbf{N}^{(b)} = 0$ (balanced moment arms).

• If $m_2 > m_1$, then $r'_2 < r'_1$ so for horizontal alignment one has $r_2 < r_1$ so $[\,(1/r_1^3) - (1/r_2^3)\,] < 0$. Then if $m_1$ is on the right, we have $\theta' = \pi/2$ and $\sin\theta' = 1$ and then (D.7.7) has $\mathbf{N}^{(b)} = -(\text{positive})\hat{\boldsymbol{\varphi}}'$. But for this orientation $\hat{\boldsymbol{\varphi}}' = -\hat{\mathbf{z}}'$ so we get $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$




(D.7.8)

In this case the lever arms balance in the sense that $m_2 r'_2 = m_1 r'_1$, but $m_2$ is closer to Earth center so it feels the stronger force and so we expect $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$

Following needs repair, will have $\sin\theta \leftrightarrow \cos\theta$, but the result is invariant under this change!

If we now assume as before that $r_1, r_2, b \gg r'_1, r'_2$ we can approximate $[\,(1/r_2^3) - (1/r_1^3)\,]$ by adding on to the Maple code shown in (D.3.5) to get

```
f := (1/r2)^3 - (1/r1)^3:
series(f,e1=0,3): simplify(%);
```

$$-3\frac{\sin(\theta)}{b^3(-1+\mu 1)}\,el - \frac{3}{2}\frac{4 - 8\,\mu 1 - 5\cos(\theta)^2 + 10\,\mu 1\cos(\theta)^2}{b^3(1 - 2\,\mu 1 + \mu 1^2)}\,el^2 + O(el^3)$$

where recall $e1 = \varepsilon_1 \equiv (r'_1/b)$. This time there is a leading linear term in $\varepsilon_1$ so then

$$(1/r_2^3) - (1/r_1^3) \;\approx\; -3\,\frac{\sin\theta'}{b^3(-\mu_2)}\,(r'_1/b) \;=\; 3r'_1\sin\theta'/\,(\mu_2 b^4) \qquad \text{(D.7.9)}$$

Then

I think the result will be

$$(1/r_2^3) - (1/r_1^3) \approx 3r'_1\cos\theta'/\,(\mu_2 b^4)$$

so then

$$\mathbf{N}^{(b)} = GM_E m_1 b\, r'_1 \sin\theta' \left[ (1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx - GM_E m_1 b\, r'_1 \sin\theta' \left( 3r'_1 \cos\theta' / (\mu_2 b^4) \right) \hat{\boldsymbol{\varphi}}'$$

$$= - 3GM_E (m_1/\mu_2)\, b^{-3} r'^2_1 \sin\theta' \cos\theta' \; \hat{\boldsymbol{\varphi}}'$$

To this order of approximation, the torque $\mathbf{N}^{(b)}$ vanishes when the satellite is horizontally aligned as well as when it is vertically aligned, and this is due to $r_1 \approx r_2$ in the horizontal case.

**D.8 The fictitious torque on the Dumbbell Satellite in Frame S'**

Recall the general fictitious torque expression given in (11.3.10), acting on a single particle of mass m,

$$\mathbf{N}'^{(c')}_{fict} = - (\mathbf{r}'-\mathbf{c}') \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\, \boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m(\ddot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \ddot{\mathbf{b}}_S) \times ( \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\ddot{\mathbf{c}}' \times \mathbf{v}' \quad . \qquad \textit{(11.3.10)} \qquad \text{(D.8.1)}$$

Here ω is the angular rotation rate of the satellite about the Earth.

Our application has the torque center at $\mathbf{c}' = 0$ (and $\mathbf{c} = \mathbf{b}$) so this simplifies somewhat to

$$\mathbf{N}'^{(0)}_{fict} = - m\mathbf{r}' \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\boldsymbol{\omega} \times \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}'] + \dot{\mathbf{b}}_S \times ( m\mathbf{v}' + \boldsymbol{\omega} \times [m\mathbf{r}']) \quad . \qquad \text{(D.8.2)}$$
frame centrifugal Coriolis Euler

Recall from (8.1.8) that the square bracket in the above is $- \mathbf{F}'_{fict}/m$ so we can trace the origin of the terms.

Now write this separately for each of the two masses of our satellite,

$$\mathbf{N}'^{(0)}_{f,1} = - m_1\mathbf{r}'_1 \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) + 2\boldsymbol{\omega} \times \mathbf{v}_1' + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1] + \dot{\mathbf{b}}_S \times ( m_1\mathbf{v}'_1 + \boldsymbol{\omega} \times [ m_1\mathbf{r}'_1])$$
$$\mathbf{N}'^{(0)}_{f,2} = - m_2\mathbf{r}'_2 \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) + 2\boldsymbol{\omega} \times \mathbf{v}'_2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_2] + \dot{\mathbf{b}}_S \times ( m_2\mathbf{v}'_2 + \boldsymbol{\omega} \times [m_2\mathbf{r}'_2]) \; . \qquad \text{(D.8.3)}$$

In the second line, use (D.2.8) to replace $m_2\mathbf{r}'_2 = - m_1\mathbf{r}'_1$ and (D.6.2) to replace $m_2\mathbf{v}'_2 = - m_1\mathbf{v}'_1$ :

$$\mathbf{N}'^{(0)}_{f,2} = + m_1\mathbf{r}'_1 \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) + 2\boldsymbol{\omega} \times \mathbf{v}'_2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_2] + \dot{\mathbf{b}}_S \times ( -m_1\mathbf{v}'_1 + \boldsymbol{\omega} \times [-m_1\mathbf{r}'_1]) \; . \qquad \text{(D.8.4)}$$

Next, add the two torques to get the total fictitious torque on the satellite,

$$\mathbf{N'^{(0)}_{fict}} = \mathbf{N'^{(0)}_{f,1}} + \mathbf{N'^{(0)}_{f,2}}$$

$$= -\, m_1 \mathbf{r'_1} \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}] + \dot{\mathbf{b}}_S \times (\, m_1 \mathbf{v'_1} + \boldsymbol{\omega} \times [\, m_1 \mathbf{r'_1}])$$

$$+ m_1 \mathbf{r'_1} \times [\, \ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (\, -m_1 \mathbf{v'_1} + \boldsymbol{\omega} \times [-m_1 \mathbf{r'_1}])$$

$$= -\, m_1 \mathbf{r'_1} \times \{\, \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r'_1} - \mathbf{r'_2}])}_{\text{centrifugal}} + \underbrace{2\boldsymbol{\omega} \times [\mathbf{v_1'} - \mathbf{v'_2}]}_{\text{Coriolis}} + \underbrace{\dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} - \mathbf{r'_2}]}_{\text{Euler}} \} \,. \qquad \text{(D.8.5)}$$

$$= -\, (m_1/m_2) M \, \mathbf{r'_1} \times [\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}\,]$$

All terms involving $\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$ have cancelled out. For the Earth orbit we have $\dot{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\omega} = \omega \hat{\mathbf{x}}'$.

Now consider this vector identity"

$$\mathbf{C} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = \mathbf{C} \times [\, (\mathbf{A} \bullet \mathbf{C})\mathbf{A} - A^2 \mathbf{C}\,] = (\mathbf{A} \bullet \mathbf{C})\, \mathbf{C} \times \mathbf{A} = -\, (\mathbf{A} \bullet \mathbf{C})\, \mathbf{A} \times \mathbf{C}$$

Then

$$\mathbf{r'_1} \times [\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1})\,] = -\, (\boldsymbol{\omega} \bullet \mathbf{r'_1})\, (\boldsymbol{\omega} \times \mathbf{r'_1})$$

$$= -\, \omega^2 r'^2_1 \, (\, \hat{\mathbf{x}}' \bullet \hat{\mathbf{r}}'_1)(\, \hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1)$$

$$= -\, \omega^2 r'^2_1 \sin\theta' \cos\varphi' (\, \hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1)$$

Now

$$\hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1 = [\cos\varphi' \sin\theta' \, \hat{\mathbf{r}}'_1 + \cos\varphi' \cos\theta' \, \hat{\boldsymbol{\theta}}' - \sin\varphi' \, \hat{\boldsymbol{\varphi}}'] \times \hat{\mathbf{r}}'_1$$

$$= [\, \cos\varphi' \cos\theta' \, \hat{\boldsymbol{\theta}}' - \sin\varphi' \, \hat{\boldsymbol{\varphi}}'] \times \hat{\mathbf{r}}'_1$$

$$= [\, -\cos\varphi' \cos\theta' \, \hat{\boldsymbol{\varphi}}' - \sin\varphi' \, \hat{\boldsymbol{\theta}}']$$

$$= -\cos\varphi' \cos\theta' \, \hat{\boldsymbol{\varphi}}' - \sin\varphi' \, \hat{\boldsymbol{\theta}}'$$

Therefore

$$\mathbf{r'_1} \times [\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1})\,] = \omega^2 r'^2_1 \sin\theta' \cos\varphi' \, [\, \cos\varphi' \cos\theta' \, \hat{\boldsymbol{\varphi}}' + \sin\varphi' \, \hat{\boldsymbol{\theta}}'\,]$$

Next, the Coriolis term involves,

$\mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v_1'}) = (\mathbf{r'_1} \bullet \mathbf{v_1'})\boldsymbol{\omega} - (\mathbf{r'_1} \bullet \boldsymbol{\omega})\mathbf{v_1'}$ // $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{B} - (\mathbf{A} \bullet \mathbf{B})\mathbf{C}$

But $\mathbf{v_1'}$ is tangent to the $r'_1$ sphere so $(\mathbf{r'_1} \bullet \mathbf{v_1'}) = 0$. Then

$\mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v_1'}) = -(\mathbf{r'_1} \bullet \boldsymbol{\omega})\mathbf{v_1'} = -r'_1\omega(\hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{x}}')\mathbf{v_1'} = -r'_1\omega\sin\theta'\cos\varphi'\mathbf{v_1'}$

We now have a complicated expression for $\mathbf{N'^{(0)}_{fict}}$ :

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\, \mathbf{r'_1} \times [\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'}\,]$$

$$= -(m_1/m_2)M\, \{\, \mathbf{r'_1} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1})] + 2\, \mathbf{r'_1} \times (\boldsymbol{\omega} \times \mathbf{v_1'})\,\}$$

$$= -(m_1/m_2)M\, \{\, \omega^2 r'^2_1 \sin\theta'\cos\varphi' [\, \cos\varphi'\cos\theta'\, \hat{\boldsymbol{\varphi}}' + \sin\varphi'\, \hat{\boldsymbol{\theta}}'\,] - 2r'_1\omega\sin\theta'\cos\varphi'\mathbf{v_1'}\,\}$$

Now use

$$\mathbf{v_1'} = v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\hat{\boldsymbol{\varphi}}' = r'_1\dot{\theta}'\hat{\boldsymbol{\theta}}' + r'_1\dot{\varphi}'\sin\theta'\,\hat{\boldsymbol{\varphi}}'$$

and we then have

$$= -(m_1/m_2)M\, \{\, \omega^2 r'^2_1 \sin\theta'\cos\varphi' [\, \cos\varphi'\cos\theta'\, \hat{\boldsymbol{\varphi}}' + \sin\varphi'\, \hat{\boldsymbol{\theta}}'\,] - 2r'^2_1\omega\sin\theta'\cos\varphi'[\, \dot{\theta}'\hat{\boldsymbol{\theta}}' + \dot{\varphi}'\sin\theta'\,\hat{\boldsymbol{\varphi}}'\,]\,\}$$

$$= -(m_1/m_2)M\, *\, \{A\, \hat{\boldsymbol{\theta}}' + B\, \hat{\boldsymbol{\varphi}}'\}$$

where

$$A = \omega^2 r'^2_1 \sin\theta'\cos\varphi'\sin\varphi' - 2r'^2_1\omega\sin\theta'\cos\varphi'\dot{\theta}'$$

$$= \omega r'^2_1 \sin\theta'\cos\varphi' (\, \omega\sin\varphi' - 2\dot{\theta}'\,)$$

$$B = \omega^2 r'^2_1 \sin\theta'\cos\varphi'\cos\varphi'\cos\theta' - 2r'^2_1\omega\sin\theta'\cos\varphi'\dot{\varphi}'\sin\theta'$$

$$= \omega r'^2_1 \sin\theta'\cos\varphi' (\, \omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta'\,)$$

Restate the result:

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\, \{A\, \hat{\boldsymbol{\theta}}' + B\, \hat{\boldsymbol{\varphi}}'\}$$

where

$A = \omega r'_1{}^2 \sin\theta' \cos\varphi' \, ( \omega \sin\varphi' - 2\dot{\theta}' )$

$B = \omega r'_1{}^2 \sin\theta' \cos\varphi' \, ( \omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta' )$

Do it again Sam

$$\mathbf{N'}^{(0)}_{fict} = - (m_1/m_2) M \omega r'_1{}^2 \sin\theta' \cos\varphi'$$

$$\{ \, ( \omega \sin\varphi' - 2\dot{\theta}' ) \, \hat{\boldsymbol{\theta}}' + ( \omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta' ) \, \hat{\boldsymbol{\varphi}}' \, \}$$

and again

$$\mathbf{N'}^{(0)}_{fict} = - (m_1/m_2) M \omega r'_1{}^2 \sin\theta' \cos\varphi' \; [ \, ( \omega \sin\varphi' - 2\dot{\theta}' ) \, \hat{\boldsymbol{\theta}}' + ( \omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta' ) \, \hat{\boldsymbol{\varphi}}' ]$$

Not too bad!

Last time I did this I got the complete result being 0, so what happened? Well set $\varphi' = \pi/2$ to get the planar situation I treated last time. Then $\sin\varphi' = 1$ and $\cos\varphi' = 0$ and you see

$$\mathbf{N'}^{(0)}_{fict} = 0 \;\; !!$$

So this agrees with what I did previously for my special orientation of the satellite (which therefore missed most of the action!)

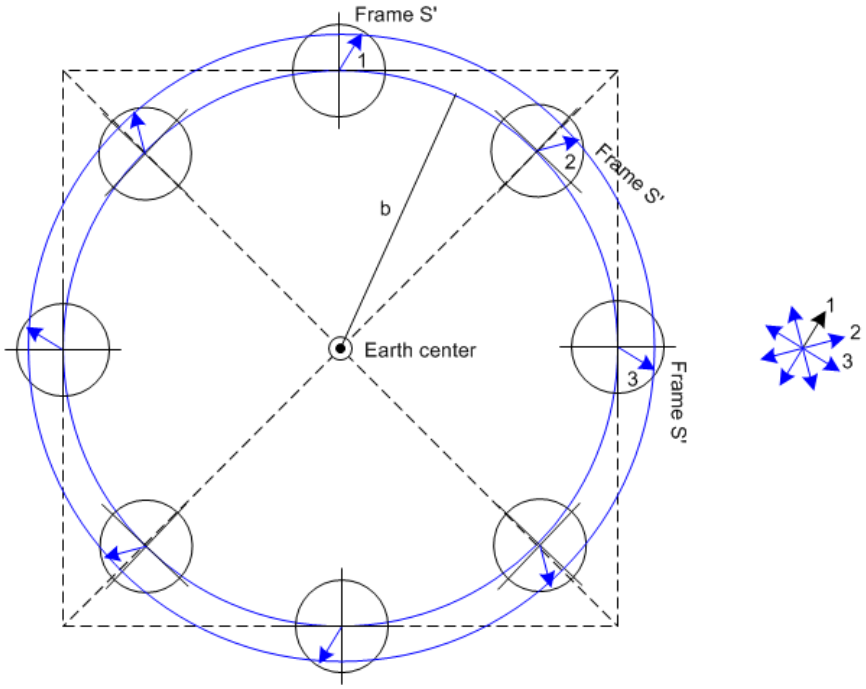
<u>How might one interpret this simple result?</u>

We examine the pieces of $\mathbf{N'}^{(0)}_{fict} = 0$:

(a) the "frame effects" due to the motion of **b** ($\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$) are equal and opposite for the two masses because the origin of Frame S' is at the center of mass causing $m_2 \mathbf{r'}_2 = - m_1 \mathbf{r'}_1$, as shown in (D.8.5).

(b) within Frame S', the centrifugal acceleration term $\boldsymbol{\omega}$ x ($\boldsymbol{\omega}$ x $\mathbf{r'}_1$) $= -\omega^2 \mathbf{r'}_1$ tries to push $m_1$ to a larger radius. But $\mathbf{r'}_1$ is constrained to lie on a sphere of radius $r'_1$ so $m_1$ cannot *go* to a larger radius. This centrifugal acceleration is neutralized by part of the tension in the stick which we avoided talking about.

This centrifugal term, by the way, involves "the short vector" $\mathbf{r'}_1$ and not the long vector $\mathbf{r}_1$. We discussed this situation in Section 8.2 for an Earth-based Frame S'. In our current context, the picture that corresponds to Fig (8..2.8) is the following




(D.8.11)

The arrow in each location during the orbit represents the vector $\mathbf{r'}_1$ where we assume that other effects are turned off so $\mathbf{r'}_1$ stays fixed in Frame S'. When these arrows are transferred to the picture on the right with common tails, we see that $\mathbf{r'}_1$ does in fact go around in a circle of radius $r'_1$ and that is why the corresponding centrifugal force acting on $m_1$ is $-\omega^2\mathbf{r'}_1$ .

(c) Within Frame S', the Coriolis force - $2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1'$ tries to deflect mass $m_1$ "to the right" in Fig (D.6.1). But again, $\mathbf{r'}_1$ is constrained to lie on a sphere of radius $r'_1$ so $m_1$ cannot *deflect* to a different radius. This Coriolis force is neutralized by the rest of the tension in the stick.

### D.9 The total torque on the Dumbbell Satellite in Frame S'

After much effort, we have arrived at this set of results for the satellite :

$$\mathbf{L'}^{(0)} = (m_1/m_2)\, M\, r'^2_1 (\, \dot{\theta}'\hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}')$$

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \, [\, (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}' \,]$$

$$\mathbf{N}^{(b)} = GM_E m_1 b\, r'_1 \sin\theta' \, [\, (1/r_1^3) - (1/r_2^3)]\, \hat{\boldsymbol{\varphi}}'$$

$$r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$$
$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$$

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\omega r'^2_1 \sin\theta'\cos\varphi' \left[ \left( \omega\sin\varphi' - 2\dot{\theta}' \right) \hat{\boldsymbol{\theta}}' + \left( \omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta' \right) \hat{\boldsymbol{\varphi}}' \right]$$

It has taken a LONG TIME to get this point on 12.16.16,

Now use the orbit equation (8.6.7) applied to the current situation,

$GM_E = \omega^2 b^3$ large b means small ω

Then we have

$$\mathbf{N^{(b)}} = \omega^2 b^4 m_1 r'_1 \sin\theta' \left[ (1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}' \qquad T^{-2}L^5 M/L^3 = ML^2/T^2 \text{ ok}$$

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\omega r'^2_1 \sin\theta'\cos\varphi' \left[ \left( \omega\sin\varphi' - 2\dot{\theta}' \right) \hat{\boldsymbol{\theta}}' + \left( \omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta' \right) \hat{\boldsymbol{\varphi}}' \right]$$

$$= MT^{-1}L^2 * T^{-1}$$

<u>What is the relative size of these terms?</u>

Maybe write this way:

$$\mathbf{N^{(b)}} = \omega^2 b^4 m_1 r'_1 \sin\theta' \left[ (1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}' \qquad T^{-2}L^5 M/L^3 = ML^2/T^2 \text{ ok}$$

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi' \left[ \left( \sin\varphi' - 2(\dot{\theta}'/\omega) \right) \hat{\boldsymbol{\theta}}' + \left( \cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta' \right) \hat{\boldsymbol{\varphi}}' \right]$$

Magnitudes?

$$\mathbf{N^{(b)2}} = (\omega^2 b^4 m_1 r'_1 \sin\theta')^2 \left[ (1/r_1^3) - (1/r_2^3) \right]^2$$

$$\mathbf{N'^{(0)2}_{fict}} = \left[(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi'\right]^2 \left\{ \left( \sin\varphi' - 2(\dot{\theta}'/\omega) \right)^2 + \left( \cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta' \right)^2 \right\}$$

The ratio is then

$$\mathbf{N'^{(0)}_{fict}} / \mathbf{N^{(b)}}$$

$$= \frac{(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi'}{\omega^2 b^4 m_1 r'_1 \sin\theta'} \sqrt{\left( \sin\varphi' - 2(\dot{\theta}'/\omega) \right)^2 + \left( \cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta' \right)^2} \Big/ \left[ (1/r_1^3) - (1/r_2^3) \right]$$

$$= \frac{(1/\mu_2) r'_1 \cos\varphi'}{b^4} \sqrt{\left( \sin\varphi' - 2(\dot{\theta}'/\omega) \right)^2 + \left( \cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta' \right)^2} \Big/ \left[ (1/r_1^3) - (1/r_2^3) \right]$$

So two glaring issues: dealing with the difference in the denom, the with ω ratios in the num. So I don't really have much to say yet about the relative size of these terms.

The Equations of Motion

Start with

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \left[ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}' \right]$$

$$\mathbf{N}^{(b)} = \omega^2 b^4 m_1 r'_1 \sin\theta' \left[ (1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \left[ (1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \left[ -3r'_1 \cos\theta' / (\mu_2 b^4) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta'\, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/m_2) M \omega r'^2_1 \sin\theta' \cos\varphi' \left[ (\omega \sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta')\, \hat{\boldsymbol{\varphi}}' \right]$$

Is there some limit where one or the other torque dominates?

Write again as

$$\mathbf{N}^{(b)} \approx -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta'\, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/\mu_2) \omega r'^2_1 \sin\theta' \cos\varphi' \left[ (\omega \sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta')\, \hat{\boldsymbol{\varphi}}' \right]$$

The magnitude ratio factor is

$$\left[ -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta' \,/\, -(m_1/\mu_2) \omega r'^2_1 \sin\theta' \cos\varphi' \right]$$

$$= \left[ 3\omega \cos\theta' / \cos\varphi' \right]$$

and then

$$N_{\mathtt{fict}}/N_{\mathtt{true}} = (\cos\varphi'/3\omega\cos\theta') \sqrt{(\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta')^2}$$

$$= (\cos\varphi'/3\cos\theta') \sqrt{(\sin\varphi' - 2[\dot{\theta}'/\omega])^2 + (\cos\varphi'\cos\theta' - 2[\dot{\varphi}'/\omega]\sin\theta')^2}$$

I want an argument that makes this small. Only if $\varphi' = \pi/2$ and it is planar is this small!

Case 1: vertically aligned. Then $\cos\theta' = 1$ and $\sin\theta' = 0$ and the above gives

$$N_{fict}/N_{true} = (\cos\varphi'/3\omega)\sqrt{(\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega\cos\varphi')^2}$$

Not sure what that means.

Limit #1: Let $b \to \infty$ so that $\omega \to 0$. Then both torques vanish and the result is no torque at all.

But in the limit not quite there we might say

$$\mathbf{N}^{(b)} \approx -3\omega^2(m_1/\mu_2)r'^2_1\sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/\mu_2)\omega r'^2_1\sin\theta'\cos\varphi'\,[(-2\dot{\theta}')\,\hat{\boldsymbol{\theta}}' + (-2\dot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\varphi}}']$$

and then the fictional force dominates! We then have in the $\omega\to 0$ limit,

$$\dot{\mathbf{L}}'^{(0)} = (m_1/\mu_2)\,r'^2_1\,[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')\,\hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\theta}}']$$

$$\mathbf{N}'_{tot} \approx -(m_1/\mu_2)\omega r'^2_1\sin\theta'\cos\varphi'\,[(-2\dot{\theta}')\,\hat{\boldsymbol{\theta}}' + (-2\dot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\varphi}}']$$

This gives in this limit two equations of motion

$$(m_1/\mu_2)\,r'^2_1\,[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')] = -(m_1/\mu_2)\omega r'^2_1\sin\theta'\cos\varphi'\,(-2\dot{\varphi}'\sin\theta') \qquad \hat{\boldsymbol{\varphi}}'$$

$$-(m_1/\mu_2)\,r'^2_1\,(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta') = -(m_1/\mu_2)\omega r'^2_1\sin\theta'\cos\varphi'\,(-2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')] = -\omega\sin\theta'\cos\varphi'\,(-2\dot{\varphi}'\sin\theta') \qquad \hat{\boldsymbol{\varphi}}'$$

$$-(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta') = -\omega\sin\theta'\cos\varphi'\,(-2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta' = +2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' \qquad \hat{\boldsymbol{\varphi}}'$$

$$2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta' = -\omega\sin\theta'\cos\varphi'\,(2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 - 2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' = 0$$

$$\sin\theta'\,\ddot{\varphi}' + 2\cos\theta'\dot{\theta}'\dot{\varphi}' + 2\,\omega\sin\theta'\cos\varphi'\dot{\theta}' = 0$$

They are very ugly, but *for the very first time ever* I have something I can write down!! The equations are horribly coupled. Maybe in my limit I can ignore the last terms as $\omega\to 0$, huge high orbit. Then

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 = 0$$

$$\ddot{\varphi}' + 2\cot\theta'\dot{\theta}'\dot{\varphi}' = 0$$

but these equations are inscrutable. Suppose instead I assume close to vertical. Then

$$\ddot{\theta}' - \sin\theta' \cos\theta' \dot{\varphi}'^2 - 2\omega \sin^2\theta' \cos\varphi' \dot{\varphi}' = 0$$
$$\sin\theta' \ddot{\varphi}' + 2\cos\theta' \dot{\theta}' \dot{\varphi}' + 2\omega \sin\theta' \cos\varphi' \dot{\theta}' = 0$$

$$\ddot{\theta}' - \theta' \dot{\varphi}'^2 - 2\omega\theta'^2 \cos\varphi' \dot{\varphi}' = 0$$
$$\theta' \ddot{\varphi}' + 2\dot{\theta}' \dot{\varphi}' + 2\omega\theta' \cos\varphi' \dot{\theta}' = 0$$

Suppose we try ansatz that $\varphi$ = constant. Then we get

$$\ddot{\theta}' = 0$$
$$2\omega\theta' \cos\varphi' \dot{\theta}' = 0$$

It then just sits there with $\dot{\theta}' = \ddot{\theta}' = 0$, not very exciting.

Let's instead try the large-$\omega$ limit and see what happens there. Start over with

$$\mathbf{N}^{(b)} \approx -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\texttt{fict}} = -(m_1/\mu_2)\omega r'_1{}^2 \sin\theta' \cos\varphi' \left[ (\omega \sin\varphi' - 2\dot{\theta}') \, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta') \, \hat{\boldsymbol{\varphi}}' \right]$$

Now just assume $\mathbf{N}^{(b)}$ dominates because it has $\omega^2$ sitting there. Then we get

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2) M r'_1{}^2 \left[ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') \, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta') \, \hat{\boldsymbol{\theta}}' \right]$$

$$\mathbf{N} \approx -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \, \hat{\boldsymbol{\varphi}}'$$

This says that we don't change the ang mom about the $\hat{\boldsymbol{\theta}}'$ which is the "dipping" axis.

The two equations of motion now are

$$(m_1/m_2) M r'_1{}^2 \left[ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') = -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \right.$$
$$\left. (2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta') \right] = 0$$

or

$$(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') = -3\omega^2 \sin\theta' \cos\theta'$$
$$2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta' = 0$$

Now do ansatz where φ = constant. Then 2nd equation is happy and first says

$$\ddot{\theta}' = -3\omega^2 \sin\theta' \cos\theta'$$

NOW also assume small angle to get

$$\ddot{\theta}' = -3\omega^2 \theta'$$

or

$$\ddot{\theta}' + 3\omega^2 \theta' = 0$$

Finally we get some oscillation which involves ω and involves a factor of 3. What did my PDF say about this? Here from p 124,

> Although the vertical orientation of the tether system is a stable one, there are forces which cause it to librate (oscillate) about the vertical. These weak but persistent forces include atmospheric drag due to the different air densities encountered in the northward and southward passes of non-equatorial orbits and due to solar heating and electrodynamic forces (for conducting tethers). Station-keeping and other rocket maneuvers would also contribute to driving (or damping) libration. The natural frequency for in-plane (in the orbit plane) librations is $\sqrt{3}\ \omega_o = 1.732\ \omega_o$, and $2\ \omega_o$ for out-of-plane librations (a detailed derivation is contained in Ref. 2).

So they are saying

$$\omega_{\mathtt{libration}} = \sqrt{3}\ \omega$$

and that agrees with my equation, hurray!!!

So you have to ignore the fictitious force to get this result! What is the condition for doing that? I show above that φ' = π/2 would do it, the planar case only!

What is the Ref 2 above?

2. Beletskii, V. V. and Levin, E. M., "Dynamics of Space Tether Systems," Advances in the Astronautical Sciences, Vol. 83.

**D.10 Force analysis of the Dumbbell Satellite in Frame S'**

We use the same picture and picture description of Section D.6 and trig relations stated there.

The only forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F}_1 = -(GM_E m_1/r_1^3)\,\mathbf{r}_1 - T\,\hat{\mathbf{r}}'_1 = -(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1$$

$$\mathbf{F}_2 = -(GM_E m_2/r_2^3)\,\mathbf{r}_2 - T\,\hat{\mathbf{r}}'_2 = -(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_1) + T\,\hat{\mathbf{r}}'_1$$

These are forces in Frame S. Without going to Frame S' we could write $\mathbf{F}_i = m_i\mathbf{a}_i$ to get

$$-(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 = m_1\mathbf{a}_1$$
$$-(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_1) + T\,\hat{\mathbf{r}}'_1 = m_2\mathbf{a}_2$$

or

$$[-(GM_E m_1/r_1^3) - T]\,\hat{\mathbf{r}}'_1 + [-(GM_E m_1/r_1^3)]\,\mathbf{b} = m_1\mathbf{a}_1$$
$$[-(GM_E m_2/r_2^3) + T]\,\hat{\mathbf{r}}'_1 + [-(GM_E m_2/r_2^3)]\,\mathbf{b} = m_2\mathbf{a}_2 = -m_1\mathbf{a}_1$$

But we know from Appendix E that

$$\mathbf{b} = b\hat{\mathbf{b}} = b\hat{\mathbf{z}} = b\hat{\mathbf{z}}' = b\cos\theta'\,\hat{\mathbf{r}}'_1 - b\sin\theta'\,\hat{\boldsymbol{\theta}}'$$

so the above equations are then

$$[-(GM_E m_1/r_1^3) - T]\,\hat{\mathbf{r}}'_1 + [-(GM_E m_1/r_1^3)]\,(b\cos\theta'\,\hat{\mathbf{r}}'_1 - b\sin\theta'\,\hat{\boldsymbol{\theta}}') = m_1\mathbf{a}_1$$
$$[-(GM_E m_2/r_2^3) + T]\,\hat{\mathbf{r}}'_1 + [-(GM_E m_2/r_2^3)]\,(b\cos\theta'\,\hat{\mathbf{r}}'_1 - b\sin\theta'\,\hat{\boldsymbol{\theta}}') = m_2\mathbf{a}_2$$

or

$$[-(GM_E m_1/r_1^3) - T - (GM_E m_1/r_1^3)]\,b\cos\theta']\,\hat{\mathbf{r}}'_1 + [(GM_E m_1/r_1^3)\,b\sin\theta']\,\hat{\boldsymbol{\theta}}' = m_1\mathbf{a}_1$$
$$[-(GM_E m_2/r_2^3) + T - (GM_E m_2/r_2^3)\,b\cos\theta']\,\hat{\mathbf{r}}'_1 + [(GM_E m_2/r_2^3)\,b\sin\theta']\,\hat{\boldsymbol{\theta}}' = m_2\mathbf{a}_2$$

or

$$[-(GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T]\,\hat{\mathbf{r}}'_1 + [(GM_E m_1/r_1^3)\,b\sin\theta']\,\hat{\boldsymbol{\theta}}' = m_1\mathbf{a}_1$$
$$[-(GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T]\,\hat{\mathbf{r}}'_1 + [(GM_E m_2/r_2^3)\,b\sin\theta']\,\hat{\boldsymbol{\theta}}' = m_2\mathbf{a}_2$$

These equations are fine, but I don't know really what to do next. They are a correct statement of Newton's Law in Frame S, so no fictitious forces are required. Well we could say

$$\mathbf{b} + \mathbf{r}'_1 = \mathbf{r}_1$$

Now take time derivatives in Frame S

$$\dot{\mathbf{b}}_S + \partial_S\mathbf{r}'_1 = \dot{\mathbf{r}}_1 = \mathbf{v}_1 \qquad \text{// naturals on the right}$$

$$\ddot{\mathbf{b}}_S + \partial^2_S\mathbf{r}'_1 = \ddot{\mathbf{r}}_1 = \dot{\mathbf{v}}_1 = \mathbf{a}_1 \qquad \text{// naturals on the right}$$

We have here some "cross derivatives" to worry about. But let's use as is to get

$$[- (GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_1/r_1^3)\ b\sin\theta'] \ \hat{\boldsymbol{\theta}}' \ = \ m_1 \ddot{\mathbf{b}}_S + m_1(\partial^2_S \mathbf{r}'_1)$$

$$[- (GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_2/r_2^3)\ b\sin\theta'] \ \hat{\boldsymbol{\theta}}' \ = \ m_2 \ddot{\mathbf{b}}_S + m_2(\partial^2_S \mathbf{r}'_2)$$

Basically I am just **bumbling along** the Frame S' pathway ignoring what I already know.

So let's now do things in Frame S' and use are earlier results to get a solution! We have done much of the work already. We know that

$$\mathbf{F}_1 = [- (GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_1/r_1^3)\ b\sin\theta'] \ \hat{\boldsymbol{\theta}}'$$

$$\mathbf{F}_2 = [- (GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_2/r_2^3)\ b\sin\theta'] \ \hat{\boldsymbol{\theta}}'$$

We must now compute the fictitious force on each mass.

$$\mathbf{F}'_{fict,1} = - m_1 \ddot{\mathbf{b}}_S - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) \ - 2m_1 \ \boldsymbol{\omega} \times \mathbf{v}'_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$

$$\mathbf{F}'_{fict,2} = - m_2 \ddot{\mathbf{b}}_S - m_2 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) \ - 2m_2 \ \boldsymbol{\omega} \times \mathbf{v}'_2 - m_2 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_2$$

Use the simple CMS relations to rewrite the second line $\qquad m_2 \mathbf{v}'_2 = - m_1 \mathbf{v}'_1$

$$\mathbf{F}'_{fict,1} = - m_1 \ddot{\mathbf{b}}_S - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) \ - 2m_1 \ \boldsymbol{\omega} \times \mathbf{v}'_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$

$$\mathbf{F}'_{fict,2} = - m_2 \ddot{\mathbf{b}}_S + m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) \ + 2m_1 \ \boldsymbol{\omega} \times \mathbf{v}'_1 + m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$

Now compute

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = \ (\mathbf{A} \bullet \mathbf{C})\mathbf{A} - A^2 \mathbf{C}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) \ = (\boldsymbol{\omega} \bullet \mathbf{r}'_1)\boldsymbol{\omega} - \omega^2 \ \mathbf{r}'_1 \quad = \ (\boldsymbol{\omega} \bullet \mathbf{r}'_1)\omega \ \hat{\mathbf{x}}' - \omega^2 r'_1 \ \hat{\mathbf{r}}'_1$$

We want to vectors expressed in Frame S' unit vectors and the above is a start. Now since our setup is a Special Case #1 problem, we can write

$$\ddot{\mathbf{b}}_S \ = \ \dot{\boldsymbol{\omega}} \times \mathbf{b} + \ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \ = \ \dot{\boldsymbol{\omega}} \times \mathbf{b} + \ (\boldsymbol{\omega} \bullet \mathbf{b})\omega \ \hat{\mathbf{x}}' - \omega^2 b \ \hat{\mathbf{b}}$$

But $(\boldsymbol{\omega} \bullet \mathbf{b}) = 0$ from Fig 1, and $\hat{\mathbf{b}} = \hat{\mathbf{z}} = \hat{\mathbf{z}}'$ so

$$\ddot{\mathbf{b}}_S \ = \ \dot{\boldsymbol{\omega}} \times \mathbf{b} \ - \omega^2 b \ \hat{\mathbf{z}}'$$

At this point let's set $\dot{\boldsymbol{\omega}} = 0$ because the satellite orbit is assumed unperturbed. Then

$$\mathbf{F}'_{fict,1} = + m_1 \omega^2 b \ \hat{\mathbf{z}}' - m_1 [(\boldsymbol{\omega} \bullet \mathbf{r}'_1)\omega \ \hat{\mathbf{x}}' - \omega^2 r'_1 \ \hat{\mathbf{r}}'_1] \ - 2m_1 \ \boldsymbol{\omega} \times \mathbf{v}'_1$$

$$\mathbf{F}'_{fict,2} = + m_2 \omega^2 b \ \hat{\mathbf{z}}' + m_1 [(\boldsymbol{\omega} \bullet \mathbf{r}'_1)\omega \ \hat{\mathbf{x}}' - \omega^2 r'_1 \ \hat{\mathbf{r}}'_1] \ + 2m_1 \ \boldsymbol{\omega} \times \mathbf{v}'_1$$

or

$$\mathbf{F'}_{\mathtt{fict,1}} = +\, m_1\omega^2 b\ \hat{\mathbf{z}}' - m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\omega\ \hat{\mathbf{x}}' + m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ - 2m_1\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1$$
$$\mathbf{F'}_{\mathtt{fict,2}} = +\, m_2\omega^2 b\ \hat{\mathbf{z}}' + m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\omega\ \hat{\mathbf{x}}' + m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + 2m_1\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1$$

or

$$\mathbf{F'}_{\mathtt{fict,1}} = m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + m_1\omega^2 b\ \hat{\mathbf{z}}' - 2m_1\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1 - m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\omega\ \hat{\mathbf{x}}'$$
$$\mathbf{F'}_{\mathtt{fict,2}} = m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + m_2\omega^2 b\ \hat{\mathbf{z}}' + 2m_1\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1 + m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\omega\ \hat{\mathbf{x}}'$$

Next replace

$$\hat{\mathbf{x}}' = \cos\varphi'\sin\theta'\ \hat{\mathbf{r}}'_1\ + \cos\varphi'\cos\theta'\ \hat{\boldsymbol{\theta}}'\ - \sin\varphi'\ \hat{\boldsymbol{\varphi}}'$$
$$\hat{\mathbf{z}}' = \cos\theta'\ \hat{\mathbf{r}}'_1 - \sin\theta'\ \hat{\boldsymbol{\theta}}' \quad .$$
$$\mathbf{v'}_1 = v'_{\theta 1}\hat{\boldsymbol{\theta}}'\ + v'_{\varphi 1}\ \hat{\boldsymbol{\varphi}}'$$

Notice that

$$\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1 = \omega\hat{\mathbf{x}}'\ \mathrm{x}\ [v'_{\theta 1}\hat{\boldsymbol{\theta}}'\ + v'_{\varphi 1}\ \hat{\boldsymbol{\varphi}}']$$

$$= \omega v'_{\theta 1}\ \hat{\mathbf{x}}'\ \mathrm{x}\ \hat{\boldsymbol{\theta}}' + \omega\ v'_{\varphi 1}\ \hat{\mathbf{x}}'\ \mathrm{x}\ \hat{\boldsymbol{\varphi}}'$$

$$= \omega v'_{\theta 1}\ [\cos\varphi'\sin\theta'\ \hat{\boldsymbol{\varphi}}' + \sin\varphi'\ \hat{\mathbf{r}}'_1\ ] + \omega\ v'_{\varphi 1}\ [-\cos\varphi'\sin\theta'\ \hat{\boldsymbol{\theta}}' + \cos\varphi'\cos\theta'\ \hat{\mathbf{r}}'_1]$$

$$= \omega[\ v'_{\theta 1}\sin\varphi' + \ v'_{\varphi 1}\cos\varphi'\cos\theta']\ \hat{\mathbf{r}}'_1 + \omega[-\ v'_{\varphi 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\theta}}' + \omega[\ v'_{\theta 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\varphi}}'$$

Things are now a bit messy. Start with $\mathbf{F'}_{\mathtt{fict,1}}$ :

$$\mathbf{F'}_{\mathtt{fict,1}} = m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + m_1\omega^2 b\ \hat{\mathbf{z}}' - 2m_1\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{v'}_1 - m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\omega\ \hat{\mathbf{x}}'$$

$$= \ m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + m_1\omega^2 b\ [\ \cos\theta'\ \hat{\mathbf{r}}'_1 - \sin\theta'\ \hat{\boldsymbol{\theta}}']$$

$$- 2m_1\omega\ \{\ [\ v'_{\theta 1}\sin\varphi' + \ v'_{\varphi 1}\cos\varphi'\cos\theta']\ \hat{\mathbf{r}}'_1 + [-\ v'_{\varphi 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\theta}}' + [\ v'_{\theta 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\varphi}}'\}$$

$$- m_1(\boldsymbol{\omega}\bullet\mathbf{r'}_1)\ [\cos\varphi'\sin\theta'\ \hat{\mathbf{r}}'_1\ + \cos\varphi'\cos\theta'\ \hat{\boldsymbol{\theta}}'\ - \sin\varphi'\ \hat{\boldsymbol{\varphi}}']$$

$$= \ m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1\ + \ m_1\omega^2 b\cos\theta'\ \hat{\mathbf{r}}'_1 - \ m_1\omega^2 b\sin\theta'\ \hat{\boldsymbol{\theta}}'$$

$$[ - 2m_1\omega v'_{\theta 1}\sin\varphi' - 2m_1\omega\ v'_{\varphi 1}\cos\varphi'\cos\theta']\ \hat{\mathbf{r}}'_1 + [\ 2m_1\omega v'_{\varphi 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\theta}}' + [ - 2m_1\omega v'_{\theta 1}\cos\varphi'\sin\theta']\ \hat{\boldsymbol{\varphi}}'$$

$$- m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\cos\varphi'\sin\theta'\ \hat{\mathbf{r}}'_1 - m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\cos\varphi'\cos\theta'\ \hat{\boldsymbol{\theta}}' + m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\sin\varphi'\ \hat{\boldsymbol{\varphi}}'$$

$$= [\ m_1\omega^2 r'_1 + m_1\omega^2 b\cos\theta' - 2m_1\omega v'_{\theta 1}\sin\varphi' - 2m_1\omega\ v'_{\varphi 1}\cos\varphi'\cos\theta' - m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\cos\varphi'\sin\theta']\ \hat{\mathbf{r}}'_1$$

$$+ [-\ m_1\omega^2 b\sin\theta' + 2m_1\omega v'_{\varphi 1}\cos\varphi'\sin\theta' - m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\cos\varphi'\cos\theta']\ \hat{\boldsymbol{\theta}}'$$

$$+ [- 2m_1\omega v'_{\theta 1}\cos\varphi'\sin\theta' + m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\sin\varphi']\ \hat{\boldsymbol{\varphi}}'$$

$$= m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2\ v'_{\varphi 1}\cos\varphi'\cos\theta' - (\hat{\boldsymbol{\omega}}\bullet\mathbf{r}'_1)\cos\varphi'\sin\theta']\ \hat{\mathbf{r}}'_1$$

$$+ m_1\omega[-\ \omega b\sin\theta' + 2v'_{\varphi 1}\cos\varphi'\sin\theta' - (\hat{\boldsymbol{\omega}}\bullet\mathbf{r}'_1)\cos\varphi'\cos\theta']\ \hat{\boldsymbol{\theta}}'$$

$$+ m_1\omega[- 2v'_{\theta 1}\cos\varphi'\sin\theta' + (\hat{\boldsymbol{\omega}}\bullet\mathbf{r}'_1)\sin\varphi']\ \hat{\boldsymbol{\varphi}}'$$

Meanwhile

$$(\hat{\boldsymbol{\omega}}\bullet\mathbf{r}'_1) = r'_1(\hat{\mathbf{x}}'\bullet\hat{\mathbf{r}}'_1) = r'_1\sin\theta'\cos\varphi'$$

So far then we know that

$$\mathbf{F}'_{\mathrm{fict},1} = m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2\ v'_{\varphi 1}\cos\varphi'\cos\theta' - r'_1\sin\theta'\cos\varphi'\cos\varphi'\sin\theta']\ \hat{\mathbf{r}}'_1$$

$$+ m_1\omega[-\ \omega b\sin\theta' + 2v'_{\varphi 1}\cos\varphi'\sin\theta' - r'_1\sin\theta'\cos\varphi'\cos\varphi'\cos\theta']\ \hat{\boldsymbol{\theta}}'$$

$$+ m_1\omega[- 2v'_{\theta 1}\cos\varphi'\sin\theta' + r'_1\sin\theta'\cos\varphi'\sin\varphi']\ \hat{\boldsymbol{\varphi}}'$$

or

$$\mathbf{F}'_{\mathrm{fict},1} = m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2\ v'_{\varphi 1}\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi']\ \hat{\mathbf{r}}'_1$$

$$+ m_1\omega\sin\theta'[-\ \omega b + 2v'_{\varphi 1}\cos\varphi' - r'_1\cos\theta'\cos^2\varphi']\ \hat{\boldsymbol{\theta}}'$$

$$+ m_1\omega\sin\theta'\cos\varphi'[- 2v'_{\theta 1} + r'_1\sin\varphi']\ \hat{\boldsymbol{\varphi}}'$$

Now recall that

$$\mathbf{F_1} = [- (GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T ]\, \hat{\mathbf{r}}'_1 + [ (GM_E m_1/r_1^3)\, b\sin\theta']\, \hat{\boldsymbol{\theta}}'$$

We combine this with the above then to get

$$\mathbf{F}_{\mathbf{eff},1} = \mathbf{F_1} + \mathbf{F}'_{\mathbf{fict},1}$$

$$= m_1\omega[\, \omega r'_1 + \omega b\cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2\, v'_{\varphi 1}\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T]\, \hat{\mathbf{r}}'_1$$

$$+ m_1\omega\sin\theta'[-\ \omega b\ + 2v'_{\varphi 1}\cos\varphi' - r'_1\cos\theta'\cos^2\varphi' +\ (GM_E m_1/r_1^3)\, b\sin\theta']\ \hat{\boldsymbol{\theta}}'$$

$$+ m_1\omega\sin\theta'\cos\varphi'[- 2v'_{\theta 1} + r'_1\sin\varphi']\, \hat{\boldsymbol{\varphi}}'$$

Now set this to $m_1 a'_1$ which is this

$$m_1\mathbf{a}'_1 = m_1 a'_{r1}\, \hat{\mathbf{r}}'_1 + m_1 a'_{\theta 1}\hat{\boldsymbol{\theta}}' + m_1 a'_{\varphi 1}\, \hat{\boldsymbol{\varphi}}'$$

where

$$a'_{r1} = \ - r'_1\dot{\theta}'^2 - \ r'_1\, \dot{\varphi}'^2\sin^2\theta'$$

$$a'_{\theta 1} = \ r'_1\ddot{\theta}' - \ r'_1\, \dot{\varphi}'^2\sin\theta'\cos\theta'$$

$$a'_{\varphi 1} = 2\, r'_1\, \dot{\theta}'\, \dot{\varphi}'\cos\theta' \ + \ r'_1\ddot{\varphi}'\sin\theta'$$

We then end up with some mind-boggling equations of motion

$$m_1\omega[\, \omega r'_1 + \omega b\cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2\, v'_{\varphi 1}\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + T]$$
$$= \quad m_1 r'_1(\ - \dot{\theta}'^2 - \ \dot{\varphi}'^2\sin^2\theta')$$

$$m_1\omega\sin\theta'[-\ \omega b\ + 2v'_{\varphi 1}\cos\varphi' - r'_1\cos\theta'\cos^2\varphi' +\ (GM_E m_1/r_1^3)\, b\sin\theta']$$
$$= m_1 r'_1(\ \ddot{\theta}' - \ \dot{\varphi}'^2\sin\theta'\cos\theta')$$

$$m_1\omega\sin\theta'\cos\varphi'[- 2v'_{\theta 1} + r'_1\sin\varphi']$$
$$= m_1\, r'_1(\ 2\, \dot{\theta}'\, \dot{\varphi}'\cos\theta' \ + \ \ddot{\varphi}'\sin\theta')$$

Question: What determines the value of T ? Put that on hold.

Question: How to translate this into the $F_2$ situation?

Recall that

$$\mathbf{F_1} = [- (GM_E m_1/r_1^3)\{1+b\cos\theta'\} - T ]\, \hat{\mathbf{r}}'_1 + [ (GM_E m_1/r_1^3)\, b\sin\theta']\, \hat{\boldsymbol{\theta}}'$$

$$\mathbf{F}_2 = [- (GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_2/r_2^3)\ b\sin\theta']\ \hat{\boldsymbol{\theta}}'$$

$$\mathbf{F}'_{fict,1} = m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1 + m_1\omega^2 b\ \hat{\mathbf{z}}' - 2m_1\ \boldsymbol{\omega} \times \mathbf{v}'_1 - m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\omega\ \hat{\mathbf{x}}'$$
$$\mathbf{F}'_{fict,2} = m_1\omega^2 r'_1\ \hat{\mathbf{r}}'_1 + m_2\omega^2 b\ \hat{\mathbf{z}}' + 2m_1\ \boldsymbol{\omega} \times \mathbf{v}'_1 + m_1(\boldsymbol{\omega}\bullet\mathbf{r}'_1)\omega\ \hat{\mathbf{x}}'$$

With regard to the first two lines
- change $m_1/r_1^3$ to $m_2/r_2^3$
- change T to -T

With regard to the last two lines, make these changes
- In the $\omega^2 b$ terms replace $m_1$ by $m_2$ done
- Change sign in the Coriolis terms (any terms including velocity)
- Change sign in the $(\boldsymbol{\omega}\bullet\mathbf{r}'_1)$ terms (these are the last term in each component)

So backing up, we would then have (changes shown in red)

$$\mathbf{F}_2 = [- (GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T ]\ \hat{\mathbf{r}}'_1 + [ (GM_E m_2/r_2^3)\ b\sin\theta']\ \hat{\boldsymbol{\theta}}'$$

$$\mathbf{F}'_{fict,2} = m_1\omega[\ \omega r'_1 - (m_2/m_1)\ \omega b\ \cos\theta' + 2v'_{\theta 1}\sin\varphi' + 2\ v'_{\varphi 1}\ \cos\theta'\cos\varphi' + r'_1\sin^2\theta'\cos^2\varphi']\ \hat{\mathbf{r}}'_1$$
$$+ m_1\omega\sin\theta'[\ +\ (m_2/m_1)\omega b\ - 2v'_{\varphi 1}\cos\varphi' + r'_1\cos\theta'\cos^2\varphi']\ \hat{\boldsymbol{\theta}}'$$
$$+ m_1\omega\sin\theta'\cos\varphi'[+ 2v'_{\theta 1} - r'_1\sin\varphi']\ \hat{\boldsymbol{\varphi}}'$$

Here then is how the three equations of motion will look

With regard to the first two lines
- change $m_1/r_1^3$ to $m_2/r_2^3$ DONE
- change T to -T DONE

With regard to the last two lines, make these changes
- In the $\omega^2 b$ terms replace $m_1$ by $m_2$ DONE
- Change sign in the Coriolis terms (any terms including velocity) DONE
- Change sign in the $(\boldsymbol{\omega}\bullet\mathbf{r}'_1)$ terms (these are the last term in each component) DONE

What are the right sides going to be? Recall that

$\theta'_2 = \pi - \theta'_1$ and $\varphi'_2 = \varphi'_1 + \pi$ $\sin\theta'_2 = +\sin\theta'_1$ $\cos\theta'_2 = -\cos\theta'_1$

so

$\dot{\theta}'_2 = -\dot{\theta}'_1$ and $\dot{\varphi}'_2 = \dot{\varphi}'_1$

So

$$m_1 r'_1( -\dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta') \rightarrow m_2 r'_2( -\dot{\theta}'^2_2 - \dot{\varphi}'^2_2 \sin^2\theta'_2) = m_2 r'_2( -\dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta')$$

$$m_1 r'_1( \ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') \rightarrow m_2 r'_2( \ddot{\theta}'_2 - \dot{\varphi}'^2_2 \sin\theta_2' \cos\theta'_2) = m_2 r'_2( -\ddot{\theta}' + \dot{\varphi}'^2 \sin\theta' \cos\theta')$$

$$m_1 r'_1( 2\dot{\theta}'\dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta') \rightarrow m_2 r'_2( 2\dot{\theta}'_2 \dot{\varphi}'_2 \cos\theta'_2 + \ddot{\varphi}'_2 \sin\theta'_2)$$

$$= m_2 r'_2( +2\dot{\theta}'\dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta')$$

Only the second term picks up a minus sign overall on the right side.

Here then are the equations of motion from the particle 2 forces equation:

$$m_1\omega[ \omega r'_1 - (m_2/m_1)\omega b \cos\theta' + 2v'_{\theta 1}\sin\varphi' + 2 v'_{\varphi 1} \cos\theta'\cos\varphi' + r'_1 \sin^2\theta'\cos^2\varphi' - (GM_E m_2/r_2^3)\{1+b\cos\theta'\} - T] = m_2 r'_2( -\dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta')$$

$$m_1\omega\sin\theta'[+ (m_2/m_1)\omega b - 2v'_{\varphi 1}\cos\varphi' + r'_1\cos\theta'\cos^2\varphi' + (GM_E m_2/r_2^3) b\sin\theta'] = m_2 r'_2(-\ddot{\theta}' + \dot{\varphi}'^2\sin\theta'\cos\theta')$$

$$m_1\omega\sin\theta'\cos\varphi'[+ 2v'_{\theta 1} - r'_1\sin\varphi'] = m_2 r'_2( 2\dot{\theta}'\dot{\varphi}' \cos\theta' + \ddot{\varphi}'\sin\theta')$$

So I end up with SIX equations of motion for the two particles

1. $m_1\omega[ \omega r'_1 + \omega b \cos\theta' - 2v'_{\theta 1}\sin\varphi' - 2 v'_{\varphi 1} \cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + T] = - m_1 r'_1( \dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta')$

2. $m_1\omega\sin\theta'[- \omega b + 2v'_{\varphi 1}\cos\varphi' - r'_1\cos\theta'\cos^2\varphi' + (GM_E m_1/r_1^3) b\sin\theta'] = m_1 r'_1( \ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')$

3. $m_1\omega\sin\theta'\cos\varphi'[- 2v'_{\theta 1} + r'_1\sin\varphi'] = m_1 r'_1( 2\dot{\theta}'\dot{\varphi}' \cos\theta' + \ddot{\varphi}'\sin\theta')$

4. $m_1\omega[ \omega r'_1 - (m_2/m_1)\omega b \cos\theta' + 2v'_{\theta 1}\sin\varphi' + 2 v'_{\varphi 1} \cos\theta'\cos\varphi' + r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_2/r_2^3)\{1+b\cos\theta'\} - T] = -m_2 r'_2( \dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta')$

5. $m_1\omega\sin\theta'[+ (m_2/m_1)\omega b - 2v'_{\varphi 1}\cos\varphi' + r'_1\cos\theta'\cos^2\varphi' + (GM_E m_2/r_2^3) b\sin\theta'] = m_2 r'_2(-\ddot{\theta}' + \dot{\varphi}'^2\sin\theta'\cos\theta')$

6. $m_1\omega\sin\theta'\cos\varphi'[+ 2v'_{\theta 1} - r'_1\sin\varphi'] = m_2 r'_2( +2\dot{\theta}'\dot{\varphi}' \cos\theta' + \ddot{\varphi}'\sin\theta')$

I also know that

$v'_{\theta 1} = r'_1 \dot{\theta}'$

$v'_{\varphi 1} = r'_1 \dot{\varphi}' \sin\theta'$

Install these and rewrite the six equations, but at the same time replace $m_2 r'_2 = m_1 r'_1$ on the right side of the last three equations:

1. $m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2\, r'_1\, \dot{\theta}'\sin\varphi' - 2\, r'_1\, \dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'$
   $- r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + T] = - m_1 r'_1(\ \dot{\theta}'^2 + \dot{\varphi}'^2\sin^2\theta')$

2. $m_1\omega\sin\theta'[-\ \omega b + 2r'_1\, \dot{\varphi}'\sin\theta'\cos\varphi' - r'_1\cos\theta'\cos^2\varphi' + (GM_E m_1/r_1^3)\, b\sin\theta']$
   $= m_1 r'_1(\ \ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')$

3. $m_1\omega\sin\theta'\cos\varphi'[-2\, r'_1\, \dot{\theta}' + r'_1\sin\varphi'] = m_1\, r'_1(\ 2\, \dot{\theta}'\, \dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')$

4. $m_1\omega[\ \omega r'_1 - (m_2/m_1)\omega b\cos\theta' + 2\, r'_1\, \dot{\theta}'\sin\varphi' + 2r'_1\, \dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'$
   $+ r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_2/r_2^3)\{1+b\cos\theta'\} - T] = -m_1 r'_1(\ \dot{\theta}'^2 + \dot{\varphi}'^2\sin^2\theta')$

5. $m_1\omega\sin\theta'[+(m_2/m_1)\omega b - 2r'_1\, \dot{\varphi}'\sin\theta'\cos\varphi' + r'_1\cos\theta'\cos^2\varphi' + (GM_E m_2/r_2^3)\, b\sin\theta']$
   $= m_1 r'_1(-\ddot{\theta}' + \dot{\varphi}'^2\sin\theta'\cos\theta')$

6. $m_1\omega\sin\theta'\cos\varphi'[+2\, r'_1\, \dot{\theta}' - r'_1\sin\varphi'] = m_1\, r'_1(+2\, \dot{\theta}'\, \dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')$

What are we trying to solve for?

$r'_1(t)$, $\theta'(t)$, $\varphi'(t)$, $T(t)$

But $r'_1(t) = r'_1$ = a constant that I know, so only 3 unknowns

$\theta'(t)$, $\varphi'(t)$, $T(t)$

So our F = ma equations result in 6 equations in 3 unknowns, which seems odd.

If we subtract 1-4 the right sides cancel, so

1-4:

$$m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2\,r'_1\,\dot{\theta}'\sin\varphi' - 2\,r'_1\,\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + T]$$

$$-m_1\omega[\ \omega r'_1 - (m_2/m_1)\omega b\cos\theta' + 2\,r'_1\,\dot{\theta}'\sin\varphi' + 2r'_1\,\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi' + r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_2/r_2^3)\{1+b\cos\theta'\} - T] = 0$$

or

$$m_1\omega[\ \omega r'_1 + \omega b\cos\theta' - 2\,r'_1\,\dot{\theta}'\sin\varphi' - 2\,r'_1\,\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + T]$$

$$m_1\omega[\ -\omega r'_1 + (m_2/m_1)\omega b\cos\theta' - 2\,r'_1\,\dot{\theta}'\sin\varphi' - 2r'_1\,\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi' - r'_1\sin^2\theta'\cos^2\varphi' + (GM_E m_2/r_2^3)\{1+b\cos\theta'\} + T] = 0$$

or

$$m_1\omega[+\omega b\cos\theta' - 4\,r'_1\,\dot{\theta}'\sin\varphi' - 4\,r'_1\,\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi' - 2r'_1\sin^2\theta'\cos^2\varphi' - (GM_E m_1/r_1^3)\{1+b\cos\theta'\} + 2T + (m_2/m_1)\omega b\cos\theta' + (GM_E m_2/r_2^3)\{1+b\cos\theta'\}] = 0$$

There is nothing simple about the resulting equation! the 2-4 is similarly ugly.

Next, consider 3-6. They both have the same right side, so we get

3 - 6:

$$m_1\omega\sin\theta'\cos\varphi'[-2\,r'_1\,\dot{\theta}' + r'_1\sin\varphi'] - m_1\omega\sin\theta'\cos\varphi'[+2\,r'_1\,\dot{\theta}' - r'_1\sin\varphi'] = 0$$

or

$$[-2\,r'_1\,\dot{\theta}' + r'_1\sin\varphi'] - [+2\,r'_1\,\dot{\theta}' - r'_1\sin\varphi'] = 0$$

or

$$-4\,r'_1\,\dot{\theta}' + 2r'_1\sin\varphi' = 0$$

or

$$-2\,r'_1\,\dot{\theta}' + r'_1\sin\varphi' = 0$$

or

$$-2\,\dot{\theta}' + \sin\varphi' = 0$$

or

$$2\,\dot{\theta}' = \sin\varphi'$$

How could such a trivial simple equation drop out of the above mess??? More likely I have sign errors.

Question: I know there is no ang mom about the $r'_1$ axis because the masses have no extent. Is there some simple math statement of this fact?

This fact ought to be true in both frames. In Section D.6 I computed in Frame S' that

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, r'_1{}^2 (\, \dot{\theta}'\hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}')$$

and this verifies that there is no ang mom on the $r'_1$ axis. I go on to compute the fictitious torque and I find that it is also 0 in the $r'_1$ component, so everything is consistent.

But this seems to have little bearing on the above mystery. I have shown that in the $\hat{\boldsymbol{\varphi}}'$ component,

$$\mathbf{F}'_{fict,1} = m_1 \omega \sin\theta' \cos\varphi' [-2v'_{\theta 1} + r'_1 \sin\varphi'] \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{F}'_{fict,2} = m_1 \omega \sin\theta' \cos\varphi' [+2v'_{\theta 1} - r'_1 \sin\varphi'] \hat{\boldsymbol{\varphi}}'$$

so they are equal and opposite. Why should this be true? It then says

$$\mathbf{F}'_{fict,\,tot} = 0 \qquad \text{in the } \hat{\boldsymbol{\varphi}}' \text{ component}$$

Go way back to"

$$\mathbf{F}'_{fict,1} = -m_1 \ddot{\mathbf{b}}_S - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\, \boldsymbol{\omega} \times \mathbf{v}'_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$
$$\mathbf{F}'_{fict,2} = -m_2 \ddot{\mathbf{b}}_S - m_2 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\, \boldsymbol{\omega} \times \mathbf{v}'_2 - m_2 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_2$$

and make the trivial replacements so we then have

$$\mathbf{F}'_{fict,1} = -m_1 \ddot{\mathbf{b}}_S - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\, \boldsymbol{\omega} \times \mathbf{v}'_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$
$$\mathbf{F}'_{fict,2} = -m_2 \ddot{\mathbf{b}}_S + m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) + 2m_1\, \boldsymbol{\omega} \times \mathbf{v}'_1 + m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$

Now add to get

$$\mathbf{F}'_{fict,tot} = -M \ddot{\mathbf{b}}_S$$

The centrifugal, Coriolis and Euler terms all cancel out right at the get-go just because of the CMS situation between these two particles. I then show that

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 b\, \hat{\mathbf{z}}' = \dot{\boldsymbol{\omega}} \times \hat{\mathbf{z}}' - \omega^2 b\, \hat{\mathbf{z}}'$$

Since in GENERAL $\dot{\boldsymbol{\omega}}$ could be in any direction, I cannot rule out $\hat{\varphi}'$ component here, but with $\dot{\boldsymbol{\omega}} = 0$ I then have

$$\ddot{\mathbf{b}}_S = -\omega^2 b\, \hat{\mathbf{z}}' = -\omega^2 b [\cos\theta'\, \hat{\mathbf{r}}'_1 - \sin\theta'\, \hat{\boldsymbol{\theta}}']$$

which has no $\hat{\boldsymbol{\varphi}}'$ component. What does that mean? Frame S' is doing circular motion so the origin of Frame S' is accelerating to the Earth which is $-\omega^2 b\ \hat{\mathbf{z}}'$ which is a down vector in my picture, But $\hat{\boldsymbol{\varphi}}'$ is always a horizontal vector in the same picture! So to finish, we get

$$\mathbf{F}'_{\mathtt{fict,tot}} = -M\ddot{\mathbf{b}}_{\mathtt{S}} = M\,\omega^2 b[\ \cos\theta'\ \hat{\mathbf{r}}'_{\mathtt{1}} - \sin\theta'\ \hat{\boldsymbol{\theta}}']$$

and yes, this has no $\hat{\boldsymbol{\varphi}}'$ component.

Gravity also has no component horizontally!

Therefore, the two masses must have equal and opposite $\mathbf{F}'_{\mathtt{fict}}$ in the $\hat{\boldsymbol{\varphi}}'$ direction. But in this direction we also have $\mathbf{F}' = 0$ so then

$$\mathbf{F}'_{\mathtt{eff,tot}} = 0 \qquad \text{in the } \hat{\boldsymbol{\varphi}}' \text{ direction}$$

Now consider

$$\mathbf{F}'_{\mathtt{eff,1}} = m_1\mathbf{a}'_1$$

$$\mathbf{F}'_{\mathtt{eff,2}} = m_2\mathbf{a}'_2$$

Add these to learn that

$$0 = m_1\mathbf{a}'_1 + m_2\mathbf{a}'_2 \qquad \text{in the } \hat{\boldsymbol{\varphi}}' \text{ direction}$$

But the right side is 0 in all directions, so how did I obtain an equation out of this???

Have to stop for now. When resume, find out why equations 3 and 6 have the same right side, when the above equation says they should have opposite sign.

Paradox:

1. We know that (leave off the primes here)

$$m_2\mathbf{a}_2 = -m_1\mathbf{a}_2$$

2. We know that

$$a_{\varphi 1} = 2\,r_1\,\dot{\theta}\,\dot{\varphi}\cos\theta + r_1\ddot{\varphi}\sin\theta$$

$$a_{\varphi 2} = 2\,r_2\,\dot{\theta}_2\,\dot{\varphi}_2\cos\theta_2 + r_2\ddot{\varphi}_2\sin\theta_2$$

$= 2\, r_2\, [\dot{\theta}_2]\, [\dot{\varphi}_2]\, [\cos\theta_2] + r_2[\ddot{\varphi}_2][\sin\theta_2]$

$= 2\, r_2\, [-\dot{\theta}]\, [\dot{\varphi}]\, [-\cos\theta] + r_2[\ddot{\varphi}][\sin\theta]$

$= 2\, r_2\, \dot{\theta}\dot{\varphi}\cos\theta + r_2\ddot{\varphi}\sin\theta$

The $\hat{\varphi}$ component of $m_2\mathbf{a}_2 = -\, m_1\mathbf{a}_1$ then reads

$m_2 a_{\varphi 2} = -\, m_1 a_{\varphi 1}$

or

$m_2(2\, r_2\, \dot{\theta}\dot{\varphi}\cos\theta + r_2\ddot{\varphi}\sin\theta) = -\, m_1(2\, r_1\, \dot{\theta}\, \dot{\varphi}\, \cos\theta + r_1\ddot{\varphi}\sin\theta)$

or

$m_2 r_2(2\, \dot{\theta}\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta) = -\, m_1 r_1(2\, \dot{\theta}\, \dot{\varphi}\, \cos\theta + \ddot{\varphi}\sin\theta)$

or

$m_1 r_1(2\, \dot{\theta}\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta) = -\, m_1 r_1(2\, \dot{\theta}\, \dot{\varphi}\, \cos\theta + \ddot{\varphi}\sin\theta)$

or

$1 = -\, 1$

and there is your paradox.