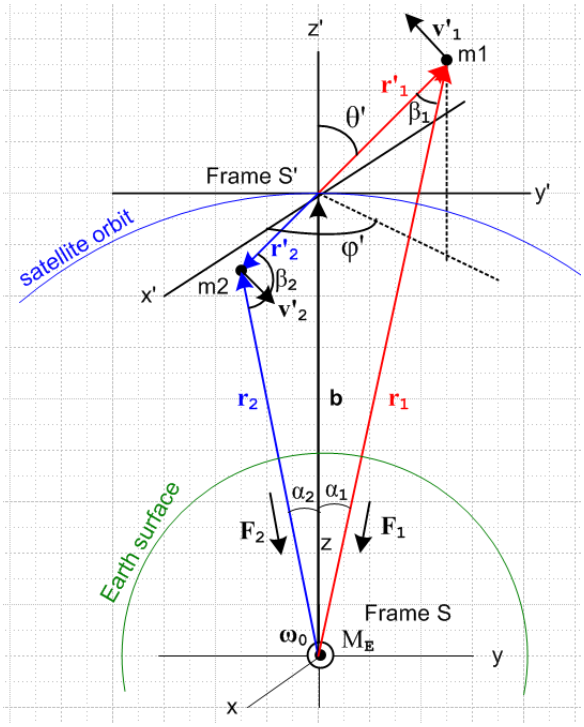
Appendix F

Our purpose here is to exercise our main text expressions for fictitious forces and torques, and to obtain equations of motion for a relatively simple example of a dumbbell satellite. The fictitious forces are reviewed in Section 8.1 and the fictitious torques in Section 11.3. We say "relatively simple" because even with the stick constraint, this is still a 3-body problem and such problems are never simple.

F.1 Kinematics for the Dumbbell Satellite in Frame S'

We now place the dumbbell satellite in a more general orientation than it was in Appendix D :



(F.1.1)

Description of the Figure

Half the battle is having a clear picture of what is going on and we shall expend many words to describe the above picture. It shows the dumbbell satellite in orbit with a completely arbitrary orientation. The two masses m_1 and m_2 are connected by a massless stick of length $r'_1+r'_2 = s$ (not shown). If we find that this stick is always in tension for some situation, we can replace it in that situation with a non-stretching massless tether. If the stick were to go into compression, the replacement tether would lose its linear shape and we don't want to deal with that problem.

Frame S is an inertial frame whose center is the center of the Earth and which is assumed fixed with respect to the stars.

Frame S' is a non-inertial frame whose center is located at the satellite center of mass point. We make the assumption discussed in Section D.* that we can ignore the tiny offset between the center of mass and center of gravity of the satellite, so we then regard the Frame S' origin as travelling in a circular orbit around the Earth.

At time t = 0, the axes of both Frames align with each other. At any time, axes y,z and y',z' are in the plane of paper. The axis z always points to the center of mass point which is the origin of Frame S'. Thus $\mathbf{b} = b\hat{\mathbf{z}} = b\hat{\mathbf{z}}'$ at any time. The axes x,x' always point directly out of the plane of paper.

The orbital rotation vector $\boldsymbol{\omega}_0$ also points out of the plane of paper and recall that $\omega_0 = 2\pi/T_0$ where T_0 is about 88 minutes for a low-Earth orbit, although we have in mind an orbit at any altitude.

Frame S' has spherical coordinates r',θ',φ' define in the usual "physics" convention as shown: angle θ' is the polar angle down from the "vertical" z' axis, φ' is the azimuthal angle measured from the x' axis toward the t' axis. See Appendix E regarding conventions.

Mass m_1 has Frame S' position $\mathbf{r}'_1$ and Frame S position $\mathbf{r}_1$. These two vectors as drawn lie on the viewer's side of the plane of paper and we show them in red. The corresponding vectors $\mathbf{r}'_2$ and $\mathbf{r}_2$ lie to the back of the plane of paper and are shown in blue. So red is near, blue is far.

There are two large non-right triangles in the picture and neither lies in the plane of paper. The triangle on the right side has internal angles α_1 and β_1 as shown, while that on the left side has angles α_2 and β_2.

The particle of mass m_1 has $\mathbf{r}'_1 = (r'_1,\theta',\varphi')$ as its spherical coordinates. Because the Frame S' origin is the center of mass, the particle of mass m_2 has $\mathbf{r}'_1 = (r'_2,\pi-\theta',\varphi'+\pi)$. The vector $\mathbf{r}'_2$ is collinear with $\mathbf{r}'_1$ but points in the opposite direction from $\mathbf{r}'_1$. The relation $m_1 r'_1 = m_2 r'_2$ (see below) shows how the masses and the vector lengths are related.

The particle of mass m_1 is constrained to lie on a sphere of radius r'_1, while the particle of mass m_2 is constrained to lie on a sphere of radius r'_2. The picture assumes $m_1 < m_2$ so $r'_1 > r'_2$. If m_1 lies at a point on its sphere, m_2 lies on the inverse point but on its sphere, as the spherical coordinates above show. If the masses are the same, the two spheres coincide.

Due to these constraints on the vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$, the corresponding velocity vectors $\mathbf{v}'_1$ and $\mathbf{v}'_2$ of the two masses must be tangential to their respective spheres. Thus for example $\mathbf{v}'_1 = v'_{1\theta}\,\hat{\boldsymbol{\theta}}' + v'_{1\varphi}\,\hat{\boldsymbol{\varphi}}'$, whereas $\mathbf{r}'_1 = r'_1\,\hat{\mathbf{r}}'$.

This is an example of a problem where it would probably be better to use the "swap notation" to get rid of the sea of primes we shall encounter, but we shall plod on with all the primes.

The circled dot at the bottom is the center of the Earth (mass M_E) and we have drawn the Earth surface in green. The blue circle is the orbit of the center of mass of the satellite (Frame S' origin) and it lies in the plane of paper. Note that ω_0 is for the orbit and has nothing whatsoever to do with the rotation of the Earth. The above picture is the same whether or not the Earth rotates at its 24 hour ω_1 rate about some obscure $\hat{\boldsymbol{\omega}}_1$ axis (not shown!).

Some Basic Kinematic Facts

The internal angles of the red triangle (not in the plane of paper) are β_1, α_1 and $\pi-\theta'$ at the origin. Thus we know from one of the three laws of cosines that

$$r_1^2 = b^2 + r'^2_1 - 2br'_1\cos(\pi-\theta') = b^2 + r'^2_1 + 2br'_1\cos\theta'$$

The internal angles of the blue triangle (not in the plane of paper) are β_2, α_2 and θ' at the origin. Thus we know that

$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$$

The law of sines for the two triangles tells us that

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r'_1} = \frac{\sin(\pi-\theta')}{r_1} = \frac{\sin\theta'}{r_1}$$

$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r'_2} = \frac{\sin\theta'}{r_2}$$

Another fact: $\mathbf{r}_2 - \mathbf{r}'_2 + \mathbf{r}'_1 = \mathbf{r}_1$ // just looking at the picture

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}'_1 - \mathbf{r}'_2$$

Recall the center of mass condition from (D.2.8) that

$$m_1\mathbf{r}'_1 = -\, m_2\mathbf{r}'_2 \quad \Rightarrow \quad m_1 r'_1 = m_2 r'_2\,, \quad r'_2/r'_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}'_2 = -\,\hat{\mathbf{r}}'_1 \qquad \textit{(D.2.8)}$$

Applying the Frame S' time derivative $\partial_{S'}$ gives similar results for $r \to v$ and $r \to a$, see below.

Summary:

$$r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$$
$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta' \qquad \text{(F.1.2)}$$

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r'_1} = \frac{\sin\theta'}{r_1} \qquad \text{(F.1.3)}$$

$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r'_2} = \frac{\sin\theta'}{r_2} \qquad \text{(F.1.4)}$$

$$\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}'_1 - \mathbf{r}'_2 \qquad \mathbf{r}_1 = \mathbf{b} + \mathbf{r}'_1 \qquad \mathbf{r}_2 = \mathbf{b} + \mathbf{r}'_2 \qquad \text{(F.1.5)}$$

$$m_1\mathbf{r}'_1 = -\, m_2\mathbf{r}'_2 \quad \Rightarrow \quad m_1 r'_1 = m_2 r'_2\,, \quad r'_2/r'_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}'_2 = -\,\hat{\mathbf{r}}'_1 \qquad \text{(F.1.6)}$$

$$m_1\mathbf{v}'_1 = -\, m_2\mathbf{v}'_2 \quad \Rightarrow \quad m_1 v'_1 = m_2 v'_2\,, \quad v'_2/v'_1 = m_1/m_2\,, \quad \hat{\mathbf{v}}'_2 = -\,\hat{\mathbf{v}}'_1 \qquad \text{(F.1.7)}$$

$$m_1\mathbf{a}'_1 = -\, m_2\mathbf{a}'_2 \quad \Rightarrow \quad m_1 a'_1 = m_2 a'_2\,, \quad a'_2/a'_1 = m_1/m_2\,, \quad \hat{\mathbf{a}}'_2 = -\,\hat{\mathbf{a}}'_1\,. \qquad \text{(F.1.8)}$$

F.2 Angular momentum and its time derivative in Frame S'

Comment: I am starting to think that the whole idea of expressing things like angular momentum and torque in terms of spherical unit vectors does not make sense! If so, then everything I have done is a goner. If I want ang mom or torque about the Frame S' origin, if $\hat{\varphi}'$ appears anywhere, it has no meaning.

But wait: when I write $\hat{\varphi}'$ it really means $\hat{\varphi}'_1$ which is the $\hat{\varphi}'$ direction *at the location of* particle 1.

For angular momentum (and later torque) we take our reference point to be $\mathbf{c}' = 0$ in Frame S' (the origin) and thus $\mathbf{c} = \mathbf{b}$ in Frame S. Then

$$\mathbf{L}'^{(0)} = m_1\mathbf{r}'_1 \times \mathbf{v}'_1 + m_2\mathbf{r}'_2 \times \mathbf{v}'_2 = m_1\mathbf{r}'_1 \times \mathbf{v}'_1 - m_1\mathbf{r}'_1 \times \mathbf{v}'_2 \qquad \text{// (F.1.6)}$$

$$= m_1\mathbf{r}'_1 \times (\mathbf{v}'_1 - \mathbf{v}'_2)$$

$$= m_1\mathbf{r}'_1 \times (\mathbf{v}'_1 + (m_1/m_2)\mathbf{v}'_1) \qquad \text{// (F.1.7)}$$

$$= m_1(1 + (m_1/m_2))\ \mathbf{r}'_1 \times \mathbf{v}'_1 \qquad \text{// M L L/T = ML}^2\text{/T}$$

$$= (m_1/m_2)\, M\ \mathbf{r}'_1 \times \mathbf{v}'_1 \ . \qquad M = m_1 + m_2 \qquad \text{(F.2.1)}$$

Identity

$$\mathbf{a} \times \mathbf{b} = (a_i\hat{\mathbf{x}}_i) \times (b_j\hat{\mathbf{x}}_j) = a_ib_j(\hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j) = a_ib_j\varepsilon_{ijk}\,\hat{\mathbf{x}}_k$$

$$= \varepsilon_{ijk}a_ib_j\hat{\mathbf{x}}_k = \varepsilon_{kij}a_ib_j\hat{\mathbf{x}}_k = (\mathbf{a} \times \mathbf{b})_k\,\hat{\mathbf{x}}_k$$

$$= (m_1/m_2)\, M\, r'_1[\cos\varphi' \sin\theta'\ \hat{\mathbf{x}}' + \sin\varphi' \sin\theta'\ \hat{\mathbf{y}}' + \cos\theta'\ \hat{\mathbf{z}}'\,] \times [\, v'_{1x}\hat{\mathbf{x}}' + v'_{1y}\hat{\mathbf{y}}' + v'_{1z}\hat{\mathbf{z}}'\,]$$

$$= (m_1/m_2)\, M\, r'_1\, \{\, \cos\varphi' \sin\theta'$$

From (E.3.5) we know that [this is the velocity at location $\mathbf{r}'_1$]

$$\mathbf{v}'_1 = v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\,\hat{\boldsymbol{\varphi}}' \qquad \text{// velocity is tangent to the sphere : } \mathbf{r}'_1 \bullet \mathbf{v}'_1 = 0$$

$$v'_{\theta 1} = r'_1\,\dot{\theta}'$$

$$v'_{\varphi 1} = r'_1\dot{\varphi}'\sin\theta' \ . \qquad \textit{(E.3.5)} \qquad \text{(F.2.2)}$$

Then using (E.2.12) for the cross products,

$$\mathbf{r}'_1 \times \mathbf{v}'_1 = r'_1\hat{\mathbf{r}}'_1 \times [v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\,\hat{\boldsymbol{\varphi}}'] = r'_1(v'_{\theta 1}\hat{\boldsymbol{\varphi}}' - v'_{\varphi 1}\hat{\boldsymbol{\theta}}') \quad . \qquad \text{(F.2.3)}$$

Thus continuing (F.2.1).

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\ \mathbf{r}'_1 \times \mathbf{v}'_1$$

$$= (m_1/m_2)\, M\, r'_1 (v'_{\theta 1}\hat{\boldsymbol{\varphi}}' - v'_{\varphi 1}\hat{\boldsymbol{\theta}}')$$

$$= (m_1/m_2)\, M\, r'_1 (\, r'_1\, \dot{\theta}'\hat{\boldsymbol{\varphi}}' - r'_1 \dot{\varphi}' \sin\theta'\, \hat{\boldsymbol{\theta}}')$$

$$= (m_1/m_2)\, M\ r'^2_1 (\ \dot{\theta}'\, \hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta'\, \hat{\boldsymbol{\theta}}') \qquad // \quad ML^2/T \qquad (F.2.4)$$

The dumbbell has no angular momentum around the $\hat{\mathbf{r}}'_1$ axis which seems very reasonable since it consists of two point masse aligned with $\hat{\mathbf{r}}'_1$

Taking a $\partial_{S'}$ time derivative gives the rate of change of angular momentum in Frame S' (all variables here are the "natural" ones in Frame S'),

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\ \mathbf{r}'_1 \times \mathbf{v}'_1$$

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, (\, \mathbf{r}'_1 \times \mathbf{a}'_1 + \mathbf{v}'_1 \times \mathbf{v}'_1) = (m_1/m_2)\, M\ \mathbf{r}'_1 \times \mathbf{a}'_1$$

$$= (m_1/m_2)\, M r'_1 \hat{\mathbf{r}}'_1 \times \mathbf{a}'_1\ . \qquad (F.2.5)$$

From (E.3.6) we know that

$$\mathbf{a}'_1 = a'_{r1}\, \hat{\mathbf{r}}'_1 + a'_{\theta 1}\hat{\boldsymbol{\theta}}' + a'_{\varphi 1}\, \hat{\boldsymbol{\varphi}}' \qquad // \text{ acceleration}$$

$$a'_{r1} = -\, r'_1(\dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta')$$
$$a'_{\theta 1} = r'_1(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')$$
$$a'_{\varphi 1} = r'_1(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\ . \qquad (F.2.6)$$

Then again using (E.2.12) for the cross products,

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, Mr'_1 \hat{\mathbf{r}}'_1 \times \mathbf{a}'_1 = (m_1/m_2)\, Mr'_1 \hat{\mathbf{r}}'_1 \times [a'_{r1}\, \hat{\mathbf{r}}'_1 + a'_{\theta 1}\hat{\boldsymbol{\theta}}' + a'_{\varphi 1}\, \hat{\boldsymbol{\varphi}}']$$

$$= (m_1/m_2)\, Mr'_1\, [\ a'_{\theta 1}\hat{\boldsymbol{\varphi}}' - a'_{\varphi 1}\hat{\boldsymbol{\theta}}'\ ]$$

$$= (m_1/m_2)\, Mr'^2_1\, [\ (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}'\ ] \qquad (F.2.7)$$

Here are the conclusions so far:

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 (\, \dot{\theta}'\hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}') \qquad (F.2.4)$$

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \, [\, (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}' \,] \qquad (F.2.7)$$

One obvious statement can be made looking at these equations: there is no angular momentum about the $\hat{\mathbf{r}}'_1$ axis and this vanishing angular momentum never changes.

The equation of motion for the satellite within Frame S' is given by (11.3.4),

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(0)}$$

Our next task then is to compute the total effective torque on the satellite in Frame S' which from (11.3.5) is

$$\mathbf{N}'^{(0)}_{\mathbf{eff}} = \mathbf{N}^{(b)} + \mathbf{N}'^{(0)}_{\mathbf{fict}} \qquad (11.3.5) \qquad (D.6.9)$$

Here $\mathbf{N}^{(b)}$ is the Frame S torque on the satellite relative to point **b** in Frame S, and $\mathbf{N}'^{(0)}_{\mathbf{fict}}$ is the fictitious torque that arises because Frame S' is a rotating frame of reference.

ok to here

F.3 The true torque on the Dumbbell Satellite in Frame S

The total torque in Frame S on the satellite (relative to the Frame S' origin) is given by

$$\mathbf{N}^{(b)} = \mathbf{r}'_1 \times \mathbf{F}_1 + \mathbf{r}'_2 \times \mathbf{F}_2 \qquad // \text{ L M L/T}^2 = \text{ML}^2/\text{T}^2 \qquad (F.3.1)$$

where all four of these vectors are shown in Fig (F.1.1). Recall from (D.2.13) that

$$\mathbf{F}_1 = -(GM_E m_1/r_1^2)\, \hat{\mathbf{r}}_1 = -(GM_E m_1/r_1^3)\, \mathbf{r}_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r_2^2)\, \hat{\mathbf{r}}_2 = -(GM_E m_2/r_2^3)\, \mathbf{r}_2 \quad . \qquad (D.2.13) \qquad (F.3.2)$$

We really should include the stick tension/compression **T** since these are supposed to be *total* forces, but since $\mathbf{r}'_1 \times \mathbf{T} = \mathbf{r}'_2 \times \mathbf{T} = \mathbf{0}$ so we can ignore **T**. Therefore

$$\mathbf{r}'_1 \times \mathbf{F}_1 = \mathbf{r}'_1 \times [\, -(GM_E m_1/r_1^3)\, \mathbf{r}_1] = -(GM_E m_1/r_1^3)\, \mathbf{r}'_1 \times \mathbf{r}_1$$

$$\mathbf{r}'_2 \times \mathbf{F}_2 = \mathbf{r}'_2 \times [\, -(GM_E m_2/r_2^3)\, \mathbf{r}_2] = -(GM_E m_2/r_2^3)\, \mathbf{r}'_2 \times \mathbf{r}_2 \qquad (F.3.3)$$

Using (F.1.5) we evaluate the cross products making use of the first line of (E.2.15),

$$\mathbf{r}'_1 \times \mathbf{r}_1 = \mathbf{r}'_1 \times (\mathbf{b} + \mathbf{r}'_1) = \mathbf{r}'_1 \times \mathbf{b} = r'_1 b(\, \hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}') = r'_1 b\, (\sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' - \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}')$$

$$\mathbf{r'_2} \times \mathbf{r_2} = \mathbf{r'_2} \times (\mathbf{b} + \mathbf{r'_2}) = \mathbf{r'_2} \times \mathbf{b} = r'_2\, b\, \hat{\mathbf{r}}'_2 \times \hat{\mathbf{z}}' = -\, r'_2\, b\, \hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}$$

$$= -r'_2 b\, (\sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' - \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}')$$

Then

$$\mathbf{r'_1} \times \mathbf{F_1} = -\, (GM_E m_1 b\, r'_1/r_1^3)\, (\sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' - \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}')$$

$$\mathbf{r'_2} \times \mathbf{F_2} = +\, (GM_E m_2 b r'_2/r_2^3)\, (\sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' - \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}')$$

$$\mathbf{N}^{(b)} = \mathbf{r'_1} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2}$$

$$= [\, -\, (GM_E m_1 b\, r'_1/r_1^3) + (GM_E m_2 b r'_2/r_2^3)\,] \sin\varphi' \sin\theta'\, \hat{\mathbf{x}}'$$
$$+ [\, (GM_E m_1 b\, r'_1/r_1^3) - (GM_E m_2 b r'_2/r_2^3)\,] \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}'$$

$$= [\, -\, (m_1\, r'_1/r_1^3) + (m_2 r'_2/r_2^3)\,]\, GM_E\, b \sin\varphi' \sin\theta'\, \hat{\mathbf{x}}'$$
$$+ [\, (m_1\, r'_1/r_1^3) - (m_2 r'_2/r_2^3)\,]\, GM_E\, b \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}'$$

Example: If $\varphi' = \pi/2$, masses are in plane of paper, $\cos\varphi' = 0$ and $\sin\varphi' = 1$ so

$$\mathbf{N}^{(b)} = [\, -\, (m_1\, r'_1/r_1^3) + (m_2 r'_2/r_2^3)\,]\, GM_E\, b \sin\theta'\, \hat{\mathbf{x}}'$$

and finally something makes sense to me! The torques are offsetting. It makes no sense to express this result in terms of $\hat{\varphi}'$ because this vector is not defined at the reference point for torque.

So what does this mean: $\hat{\mathbf{x}} = \cos\varphi \sin\theta\, \hat{\mathbf{r}} + \cos\varphi \cos\theta\, \hat{\boldsymbol{\theta}} - \sin\varphi\, \hat{\boldsymbol{\varphi}}$

If you are talking about unit vectors for a point which is at the origin of Frame S, this equation has no meaning!!!

$$\mathbf{N}^{(b)} = \mathbf{r'_1} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2}$$

$$= -\, (GM_E m_1 b\, r'_1/r_1^3) \sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' + (GM_E m_1 b\, r'_1/r_1^3) \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}'$$
$$+ (GM_E m_2 b r'_2/r_2^3) \sin\varphi' \sin\theta'\, \hat{\mathbf{x}}' - (GM_E m_2 b r'_2/r_2^3) \cos\varphi' \sin\theta'\, \hat{\mathbf{y}}'$$

$$= \{\, -\, (GM_E m_1 b\, r'_1/r_1^3) \sin\varphi' \sin\theta' + (GM_E m_2 b r'_2/r_2^3) \sin\varphi' \sin\theta' \}\, \hat{\mathbf{x}}'$$

$$+ \{\, (GM_E m_1 b\, r'_1/r_1^3) \cos\varphi' \sin\theta' - (GM_E m_2 b r'_2/r_2^3) \cos\varphi' \sin\theta'$$

$$\mathbf{r'_1} \times \mathbf{r_1} = \mathbf{r'_1} \times (\mathbf{b} + \mathbf{r'_1}) = \mathbf{r'_1} \times \mathbf{b} = r'_1 b(\hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}') = r'_1 b(-\sin\theta' \hat{\boldsymbol{\varphi}}') = -r'_1 b \sin\theta' \hat{\boldsymbol{\varphi}}'$$

$$\begin{matrix} \textbf{in} & & \textbf{in} & \textbf{in} & \textbf{-} & \textbf{out} \end{matrix}$$

$$\mathbf{r'_2} \times \mathbf{r_2} = \mathbf{r'_2} \times (\mathbf{b} + \mathbf{r'_2}) = \mathbf{r'_2} \times \mathbf{b} = r'_2 b \hat{\mathbf{r}}'_2 \times \hat{\mathbf{z}}' = -r'_2 b \hat{\mathbf{r}}'_1 \times \hat{\mathbf{z}}' = +r'_2 b \sin\theta' \hat{\boldsymbol{\varphi}}' . \qquad \text{(F.3.4)}$$

STOP. Explain direction of $\mathbf{r'_2} \times \mathbf{r_2}$ using right hand rule and the picture. When I do this with the picture, I find that $\mathbf{r'_2} \times \mathbf{r_2}$ is into of plane of paper which is the $-\hat{\boldsymbol{\varphi}}'$ direction since on the left side, but this disagrees with second line above.

Therefore:

$$\mathbf{r'_1} \times \mathbf{F_1} = -(GM_E m_1 / r_1^3)\, \mathbf{r'_1} \times \mathbf{r_1} = -(GM_E m_1 / r_1^3)(-\, r'_1 b \sin\theta' \hat{\boldsymbol{\varphi}}')$$

$$= +GM_E m_1 b\, (r'_1 / r_1^3) \sin\theta' \hat{\boldsymbol{\varphi}}' \qquad \text{(F.3.5)}$$

$$\mathbf{r'_2} \times \mathbf{F_2} = -(GM_E m_2 / r_2^3)\, \mathbf{r'_2} \times \mathbf{r_2} = -(GM_E m_2 / r_2^3)(r'_2 b \sin\theta' \hat{\boldsymbol{\varphi}}')$$

$$= -GM_E m_2 b\, (r'_2 / r_2^3) \sin\theta' \hat{\boldsymbol{\varphi}}' \qquad \text{(F.3.6)}$$

Then the torque about the Frame S' origin (which is at location **b** in Frame S) is,

$$\mathbf{N}^{(b)} = \mathbf{r'_1} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2}$$

$$= GM_E m_1 b\, (r'_1 / r_1^3) \sin\theta' \hat{\boldsymbol{\varphi}}' - GM_E m_1 b\, (r'_2 / r_2^3) \sin\theta' \hat{\boldsymbol{\varphi}}'$$

$$= GM_E m_1 b \sin\theta' \left[(r'_1 / r_1^3) - (r'_2 / r_2^3) \right] \hat{\boldsymbol{\varphi}}' \qquad \text{(F.3.7)}$$

where r_1 and r_2 are given in (F.1.2).

We confirm this torque expression by computing it geometrically. Looking at Fig (F.1.1) one sees that

$$\mathbf{r'_1} \times \mathbf{F_1} = r'_1 F'_1 \sin\beta_1 \, (+\hat{\boldsymbol{\varphi}}') = r'_1 F'_1 (b \sin\theta' / r_1) \hat{\boldsymbol{\varphi}}' \qquad \text{// (F.1.3)}$$

$$= (GM_E m_1 / r_1^2)(r'_1 b / r_1 \sin\theta') \hat{\boldsymbol{\varphi}}' = (GM_E m_1 / r_1^3)\, r'_1 b \sin\theta' \hat{\boldsymbol{\varphi}}' \qquad \text{// same as (F.3.5)}$$

$$\mathbf{r'_2} \times \mathbf{F_2} = r'_2 F'_2 \sin\beta_2 \hat{\boldsymbol{\varphi}}' = r'_2 F'_2 (b \sin\theta' / r_2) \hat{\boldsymbol{\varphi}}' \qquad \text{// (F.1.4)}$$

$$= (GM_E m_2 / r_2^2)(r'_2 b / r_2 \sin\theta') \hat{\boldsymbol{\varphi}}' = (GM_E m_2 / r_2^3)\, r'_1 b \sin\theta' \hat{\boldsymbol{\varphi}}' \qquad \text{// same as (F.3.6)}$$

$(GM_E m_2 / r_2^2)\ (r'_2 b / r_1 \sin\theta') \hat{\boldsymbol{\varphi}}'$

continue here

$$\mathbf{r'}_2 \times \mathbf{F}_2 = r'_1 F'_1 \sin\beta_s \, \hat{\boldsymbol{\varphi}}'$$

$$= b^2 + r'^2_2 - 2br'_2\cos\theta'$$

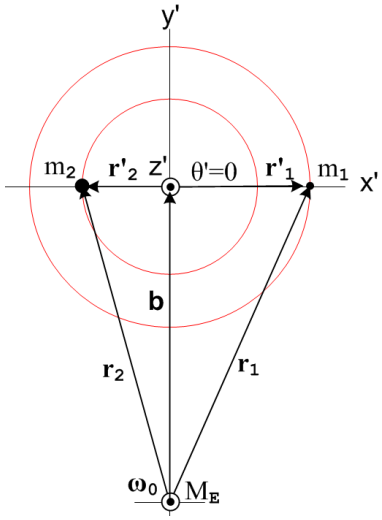
How do I interpret this result that the torque only has one component and it is about the $\hat{\boldsymbol{\varphi}}'$ axis ?? If you were look at $\varphi' = \pi/2$, then we have a planar picture and the torque clearly is about $\hat{\mathbf{z}}'$ and has no components in the other two directions since everything is in a plane. In that case $\hat{\boldsymbol{\varphi}}' = -\hat{\mathbf{z}}'$ so it makes sense.

Quick checks:

• If the satellite is vertically aligned, $\theta' = 0,\pi$, then $\sin\theta'=0$ and $\mathbf{N}^{(b)} = 0$ as expected (no moment arms).

• If the satellite is horizontally aligned *and* $m_1 = m_2$, then $r_1 = r_2$ so $\mathbf{N}^{(b)} = 0$ (balanced moment arms).

• If $m_2 > m_1$, then $r'_2 < r'_1$ so for horizontal alignment one has $r_2 < r_1$ so $[\,(1/r_1^3) - (1/r_2^3)\,] < 0$. Then if m_1 is on the right, we have $\theta' = \pi/2$ and $\sin\theta' = 1$ and then (D.7.7) has $\mathbf{N}^{(b)} = -(\text{positive})\hat{\boldsymbol{\varphi}}'$. But for this orientation $\hat{\boldsymbol{\varphi}}' = -\hat{\mathbf{z}}'$ so we get $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$



(D.7.8)

In this case the lever arms balance in the sense that $m_2 r'_2 = m_1 r'_1$, but m_2 is closer to Earth center so it feels the stronger force and so we expect $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{z}}'$

Following needs repair, will have $\sin\theta \leftrightarrow \cos\theta$, but the result is invariant under this change!

If we now assume as before that $r_1, r_2, b >> r'_1, r'_2$ we can approximate $[(1/r_2^3) - (1/r_1^3)]$ by adding on to the Maple code shown in (D.3.5) to get

```
f := (1/r2)^3 - (1/r1)^3:
series(f,e1=0,3): simplify(%);
```

$$-3\frac{\sin(\theta)}{b^3(-1+\mu 1)}\,el-\frac{3}{2}\frac{4-8\,\mu 1-5\cos(\theta)^2+10\,\mu 1\cos(\theta)^2}{b^3(1-2\,\mu 1+\mu 1^2)}\,el^2+O(el^3)$$

where recall $e1 = \varepsilon_1 \equiv (r'_1/b)$. This time there is a leading linear term in ε_1 so then

$$(1/r_2^3) - (1/r_1^3) \approx -3\frac{\sin\theta'}{b^3(-\mu_2)}(r'_1/b) = 3r'_1\sin\theta'/(\mu_2 b^4) \qquad \text{(D.7.9)}$$

Then

I think the result will be

$$(1/r_2^3) - (1/r_1^3) \approx 3r'_1\cos\theta'/(\mu_2 b^4)$$

so then

$$\mathbf{N}^{(b)} = GM_E m_1 b\, r'_1\sin\theta' [(1/r_1^3) - (1/r_2^3)]\,\hat{\boldsymbol{\varphi}}'$$

$$\approx -GM_E m_1 b\, r'_1\sin\theta' (3r'_1\cos\theta'/(\mu_2 b^4))\,\hat{\boldsymbol{\varphi}}'$$

$$= -3GM_E(m_1/\mu_2)\, b^{-3} r'^2_1 \sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

To this order of approximation, the torque $\mathbf{N}^{(b)}$ vanishes when the satellite is horizontally aligned as well as when it is vertically aligned, and this is due to $r_1 \approx r_2$ in the horizontal case.

D.8 The fictitious torque on the Dumbbell Satellite in Frame S'

Recall the general fictitious torque expression given in (11.3.10), acting on a single particle of mass m,

$$\mathbf{N}'^{(c')}_{fict} = -(\mathbf{r}'-\mathbf{c}') \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$$
$$+ m(\ddot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \ddot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\ddot{\mathbf{c}}' \times \mathbf{v}' \ . \qquad \textit{(11.3.10)} \qquad \text{(D.8.1)}$$

Here ω is the angular rotation rate of the satellite about the Earth.

Our application has the torque center at $\mathbf{c}' = 0$ (and $\mathbf{c} = \mathbf{b}$) so this simplifies somewhat to

$$\mathbf{N'^{(0)}_{fict}} = -\,m\mathbf{r'} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2\boldsymbol{\omega} \times \mathbf{v'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'}] + \dot{\mathbf{b}}_S \times (\,m\mathbf{v'} + \boldsymbol{\omega} \times [m\mathbf{r'}])\ . \qquad \text{(D.8.2)}$$
frame centrifugal Coriolis Euler

Recall from (8.1.8) that the square bracket in the above is $-\,\mathbf{F'_{fict}}/m$ so we can trace the origin of the terms.

Now write this separately for each of the two masses of our satellite,

$$\mathbf{N'^{(0)}_{f,1}} = -\,m_1\mathbf{r'_1} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}] + \dot{\mathbf{b}}_S \times (\,m_1\mathbf{v'_1} + \boldsymbol{\omega} \times [\,m_1\mathbf{r'_1}])$$
$$\mathbf{N'^{(0)}_{f,2}} = -\,m_2\mathbf{r'_2} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (\,m_2\mathbf{v'_2} + \boldsymbol{\omega} \times [m_2\mathbf{r'_2}])\ . \qquad \text{(D.8.3)}$$

In the second line, use (D.2.8) to replace $m_2\mathbf{r'_2} = -\,m_1\mathbf{r'_1}$ and (D.6.2) to replace $m_2\mathbf{v'_2} = -\,m_1\mathbf{v'_1}$:

$$\mathbf{N'^{(0)}_{f,2}} = +\,m_1\mathbf{r'_1} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (\,-m_1\mathbf{v'_1} + \boldsymbol{\omega} \times [-m_1\mathbf{r'_1}])\ . \qquad \text{(D.8.4)}$$

Next, add the two torques to get the total fictitious torque on the satellite,

$$\mathbf{N'^{(0)}_{fict}} = \mathbf{N'^{(0)}_{f,1}} + \mathbf{N'^{(0)}_{f,2}}$$

$$= -\,m_1\mathbf{r'_1} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}] + \dot{\mathbf{b}}_S \times (\,m_1\mathbf{v'_1} + \boldsymbol{\omega} \times [\,m_1\mathbf{r'_1}])$$
$$+\,m_1\mathbf{r'_1} \times [\,\ddot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_2}) + 2\boldsymbol{\omega} \times \mathbf{v'_2} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_2}] + \dot{\mathbf{b}}_S \times (\,-m_1\mathbf{v'_1} + \boldsymbol{\omega} \times [-m_1\mathbf{r'_1}])$$

$$= -\,m_1\mathbf{r'_1} \times \{\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r'_1} - \mathbf{r'_2}]) + 2\boldsymbol{\omega} \times [\mathbf{v_1'} - \mathbf{v'_2}] + \dot{\boldsymbol{\omega}} \times [\mathbf{r'_1} - \mathbf{r'_2}]\ \}\ . \qquad \text{(D.8.5)}$$
centrifugal Coriolis Euler

$$= -\,(m_1/m_2)M\,\mathbf{r'_1} \times [\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1}) + 2\boldsymbol{\omega} \times \mathbf{v_1'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'_1}\ ]$$

All terms involving $\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$ have cancelled out. For the Earth orbit we have $\dot{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}'$.

Now consider this vector identity"

$$\mathbf{C} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = \mathbf{C} \times [\ (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C}\ ] = (\mathbf{A}\bullet\mathbf{C})\ \mathbf{C} \times \mathbf{A} = -\ (\mathbf{A}\bullet\mathbf{C})\ \mathbf{A} \times \mathbf{C}$$

Then

$$\mathbf{r'_1} \times [\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'_1})\ ] = -\ (\boldsymbol{\omega}\bullet\mathbf{r'_1})\ (\boldsymbol{\omega} \times \mathbf{r'_1})$$

$$= -\,\omega^2 r_1'^2\ (\ \hat{\mathbf{x}}'\bullet\ \hat{\mathbf{r}}'_1)(\ \hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1)$$

$$= -\,\omega^2 r_1'^2\ \sin\theta'\cos\varphi'(\ \hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1)$$

Now

$$\hat{\mathbf{x}}' \times \hat{\mathbf{r}}'_1 = [\cos\varphi'\sin\theta'\,\hat{\mathbf{r}}'_1 + \cos\varphi'\cos\theta'\,\hat{\boldsymbol{\theta}}' - \sin\varphi'\,\hat{\boldsymbol{\varphi}}'] \times \hat{\mathbf{r}}'_1$$

$$= [\cos\varphi'\cos\theta'\,\hat{\boldsymbol{\theta}}' - \sin\varphi'\,\hat{\boldsymbol{\varphi}}'] \times \hat{\mathbf{r}}'_1$$

$$= [-\cos\varphi'\cos\theta'\,\hat{\boldsymbol{\varphi}}' - \sin\varphi'\,\hat{\boldsymbol{\theta}}']$$

$$= -\cos\varphi'\cos\theta'\,\hat{\boldsymbol{\varphi}}' - \sin\varphi'\,\hat{\boldsymbol{\theta}}'$$

Therefore

$$\mathbf{r}'_1 \times [\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)\ ] = \omega^2 r'^2_1 \sin\theta'\cos\varphi'\ [\ \cos\varphi'\cos\theta'\,\hat{\boldsymbol{\varphi}}' + \sin\varphi'\,\hat{\boldsymbol{\theta}}'\ ]$$

Next, the Coriolis term involves,

$$\mathbf{r}'_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1') = (\mathbf{r}'_1 \bullet \mathbf{v}_1')\boldsymbol{\omega} - (\mathbf{r}'_1 \bullet \boldsymbol{\omega})\mathbf{v}_1' \qquad // \ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

But $\mathbf{v}_1'$ is tangent to the r'_1 sphere so $(\mathbf{r}'_1 \bullet \mathbf{v}_1') = 0$. Then

$$\mathbf{r}'_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1') = -(\mathbf{r}'_1 \bullet \boldsymbol{\omega})\mathbf{v}_1' = -r'_1\omega(\hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{x}}')\mathbf{v}_1' = -r'_1\omega\sin\theta'\cos\varphi'\mathbf{v}_1'$$

We now have a complicated expression for $\mathbf{N}'^{(0)}_{\mathtt{fict}}$:

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/m_2)M\,\mathbf{r}'_1 \times [\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) + 2\boldsymbol{\omega} \times \mathbf{v}_1'\ ]$$

$$= -(m_1/m_2)M\ \{\ \mathbf{r}'_1 \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)]\ + 2\,\mathbf{r}'_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1')\ \}$$

$$= -(m_1/m_2)M\ \{\ \omega^2 r'^2_1 \sin\theta'\cos\varphi'\ [\ \cos\varphi'\cos\theta'\,\hat{\boldsymbol{\varphi}}' + \sin\varphi'\,\hat{\boldsymbol{\theta}}'\ ]$$
$$- 2r'_1\omega\sin\theta'\cos\varphi'\mathbf{v}_1'\ \}$$

Now use

$$\mathbf{v}_1' = v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\,\hat{\boldsymbol{\varphi}}' = r'_1\,\dot{\theta}'\hat{\boldsymbol{\theta}}' + r'_1\,\dot{\varphi}'\sin\theta'\,\hat{\boldsymbol{\varphi}}'$$

and we then have

$$= -(m_1/m_2)M\ \{\ \omega^2 r'^2_1 \sin\theta'\cos\varphi'\ [\ \cos\varphi'\cos\theta'\,\hat{\boldsymbol{\varphi}}' + \sin\varphi'\,\hat{\boldsymbol{\theta}}'\ ]$$
$$- 2r'^2_1\omega\sin\theta'\cos\varphi'[\ \dot{\theta}'\hat{\boldsymbol{\theta}}' + \dot{\varphi}'\sin\theta'\,\hat{\boldsymbol{\varphi}}'] \ \}$$

$$= -(m_1/m_2)M * \{A\ \hat{\boldsymbol{\theta}}' + B\ \hat{\boldsymbol{\varphi}}'\}$$

where

$$A = \omega^2 r'_1{}^2 \sin\theta'\cos\varphi'\sin\varphi' - 2r'_1{}^2\omega\sin\theta'\cos\varphi'\dot{\theta}'$$

$$= \omega r'_1{}^2 \sin\theta' \cos\varphi' (\omega\sin\varphi' - 2\dot{\theta}')$$

$$B = \omega^2 r'_1{}^2 \sin\theta'\cos\varphi' \cos\varphi'\cos\theta' - 2r'_1{}^2\omega\sin\theta'\cos\varphi' \dot{\varphi}' \sin\theta'$$

$$= \omega r'_1{}^2\sin\theta'\cos\varphi' (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta')$$

Restate the result:

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M \{A\ \hat{\boldsymbol{\theta}}' + B\ \hat{\boldsymbol{\varphi}}'\}$$

where

$$A = \omega r'_1{}^2 \sin\theta' \cos\varphi' (\omega\sin\varphi' - 2\dot{\theta}')$$

$$B = \omega r'_1{}^2\sin\theta'\cos\varphi' (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta')$$

Do it again Sam

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\omega r'_1{}^2\sin\theta'\cos\varphi'$$

$$\{ (\omega\sin\varphi' - 2\dot{\theta}') \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta') \hat{\boldsymbol{\varphi}}' \}$$

and again

$$\mathbf{N'^{(0)}_{fict}} = -(m_1/m_2)M\omega r'_1{}^2\sin\theta'\cos\varphi' [(\omega\sin\varphi' - 2\dot{\theta}') \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta') \hat{\boldsymbol{\varphi}}']$$

Not too bad!

Last time I did this I got the complete result being 0, so what happened? Well set $\varphi' = \pi/2$ to get the planar situation I treated last time. Then $\sin\varphi' = 1$ and $\cos\varphi' = 0$ and you see

$$\mathbf{N'^{(0)}_{fict}} = 0 \ !!$$

So this agrees with what I did previously for my special orientation of the satellite (which therefore missed most of the action!)

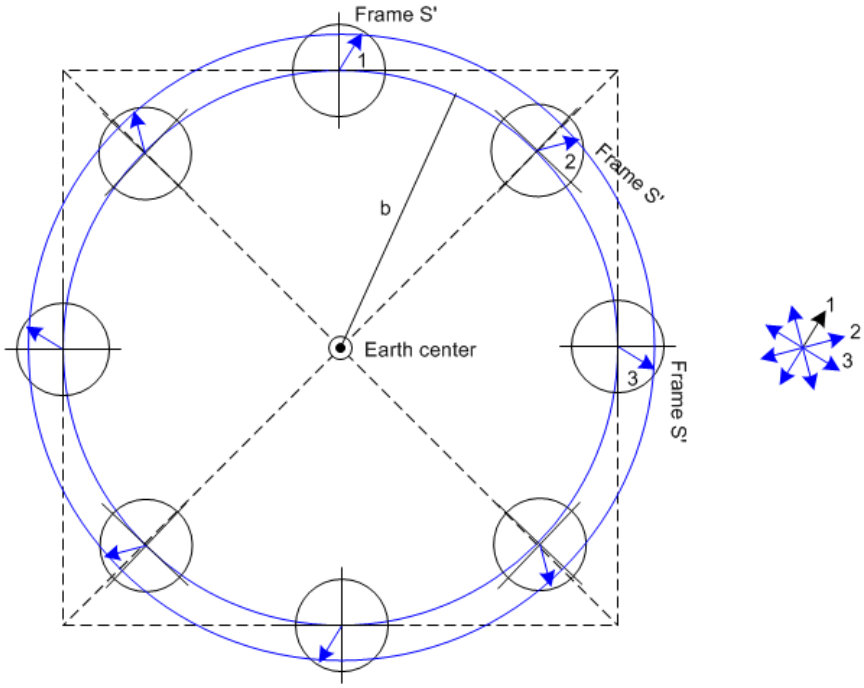
<u>How might one interpret this simple result?</u>

We examine the pieces of $\mathbf{N'}^{(0)}{}_{fict} = 0$:

(a) the "frame effects" due to the motion of **b** ($\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$) are equal and opposite for the two masses because the origin of Frame S' is at the center of mass causing $m_2\mathbf{r'}_2 = - m_1\mathbf{r'}_1$, as shown in (D.8.5).

(b) within Frame S', the centrifugal acceleration term $\boldsymbol{\omega}$ x ($\boldsymbol{\omega}$ x $\mathbf{r'}_1$) = $-\omega^2\mathbf{r'}_1$ tries to push m_1 to a larger radius. But $\mathbf{r'}_1$ is constrained to lie on a sphere of radius r'_1 so m_1 cannot *go* to a larger radius. This centrifugal acceleration is neutralized by part of the tension in the stick which we avoided talking about.

This centrifugal term, by the way, involves "the short vector" $\mathbf{r'}_1$ and not the long vector $\mathbf{r}_1$. We discussed this situation in Section 8.2 for an Earth-based Frame S'. In our current context, the picture that corresponds to Fig (8..2.8) is the following



(D.8.11)

The arrow in each location during the orbit represents the vector $\mathbf{r'}_1$ where we assume that other effects are turned off so $\mathbf{r'}_1$ stays fixed in Frame S'. When these arrows are transferred to the picture on the right with common tails, we see that $\mathbf{r'}_1$ does in fact go around in a circle of radius r'_1 and that is why the corresponding centrifugal force acting on m_1 is $-\omega^2\mathbf{r'}_1$.

(c) Within Frame S', the Coriolis force - $2m_1\,\boldsymbol{\omega}$ x $\mathbf{v}_1'$ tries to deflect mass m_1 "to the right" in Fig (D.6.1). But again, $\mathbf{r'}_1$ is constrained to lie on a sphere of radius r'_1 so m_1 cannot *deflect* to a different radius. This Coriolis force is neutralized by the rest of the tension in the stick.

D.9 The total torque on the Dumbbell Satellite in Frame S'

After much effort, we have arrived at this set of results for the satellite :

$$\mathbf{L}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 (\, \dot{\theta}'\hat{\boldsymbol{\varphi}}' - \dot{\varphi}' \sin\theta' \hat{\boldsymbol{\theta}}')$$

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \, [\, (\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}' \,]$$

$$\mathbf{N}^{(b)} = GM_E m_1 b\, r'_1 \sin\theta' \, [\, (1/r_1^3) - (1/r_2^3)]\, \hat{\boldsymbol{\varphi}}'$$

$$r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$$
$$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/m_2) M\omega r'^2_1 \sin\theta'\cos\varphi' \, [\, (\omega\sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\varphi}}']$$

It has taken a LONG TIME to get this point on 12.16.16,

Now use the orbit equation (8.6.7) applied to the current situation,

$GM_E = \omega^2 b^3$ large b means small ω

Then we have

$$\mathbf{N}^{(b)} = \omega^2 b^4 m_1 r'_1 \sin\theta' \, [\, (1/r_1^3) - (1/r_2^3)]\, \hat{\boldsymbol{\varphi}}' \qquad T^{-2}L^5 M/L^3 = ML^2/T^2 \text{ ok}$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/m_2) M\omega r'^2_1 \sin\theta'\cos\varphi' \, [\, (\omega\sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\varphi}}']$$

$$= MT^{-1}L^2 * T^{-1}$$

What is the relative size of these terms?

Maybe write this way:

$$\mathbf{N}^{(b)} = \omega^2 b^4 m_1 r'_1 \sin\theta' \, [\, (1/r_1^3) - (1/r_2^3)]\, \hat{\boldsymbol{\varphi}}' \qquad T^{-2}L^5 M/L^3 = ML^2/T^2 \text{ ok}$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi' \, [\, (\sin\varphi' - 2(\dot{\theta}'/\omega))\, \hat{\boldsymbol{\theta}}' + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')\, \hat{\boldsymbol{\varphi}}']$$

Magnitudes?

$$\mathbf{N}^{(b)2} = (\omega^2 b^4 m_1 r'_1 \sin\theta')^2 \, [\, (1/r_1^3) - (1/r_2^3)]^2$$

$$\mathbf{N}'^{(0)2}_{fict} = [(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi']^2 \, \{\, (\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')^2 \,\}$$

The ratio is then

$$N'^{(0)}_{fict} / N^{(b)}$$

$$= \frac{(m_1/\mu_2)\omega^2 r'^2_1 \sin\theta'\cos\varphi'}{\omega^2 b^4 m_1 r'_1 \sin\theta'} \sqrt{ (\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega) \sin\theta')^2 } \; / \; [(1/r_1^3) - (1/r_2^3)]$$

$$= \frac{(1/\mu_2) r'_1 \cos\varphi'}{b^4} \sqrt{ (\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega) \sin\theta')^2 } \; / \; [(1/r_1^3) - (1/r_2^3)]$$

So two glaring issues: dealing with the difference in the denom, the with ω ratios in the num. So I don't really have much to say yet about the relative size of these terms.

<u>The Equations of Motion</u>

Start with

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \; [(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') \; \hat{\boldsymbol{\varphi}}' \; - \; (2\dot{\theta}'\dot{\varphi}'\cos\theta' \; + \; \ddot{\varphi}'\sin\theta') \; \hat{\boldsymbol{\theta}}' \;]$$

$$\mathbf{N}^{(b)} \; = \; \omega^2 b^4 m_1 r'_1 \sin\theta' \; [(1/r_1^3) - (1/r_2^3)] \; \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \; [(1/r_1^3) - (1/r_2^3)] \; \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \; [-3 r'_1 \cos\theta' / (\mu_2 b^4)] \; \hat{\boldsymbol{\varphi}}'$$

$$\approx -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta' \; \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{fict} = \; - (m_1/m_2) M \omega r'^2_1 \sin\theta'\cos\varphi' \; [(\omega\sin\varphi' - 2\dot{\theta}') \; \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta') \; \hat{\boldsymbol{\varphi}}']$$

Is there some limit where one or the other torque dominates?

Write again as

$$\mathbf{N}^{(b)} \approx \; -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta' \; \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{fict} = \; - (m_1/\mu_2) \omega r'^2_1 \sin\theta'\cos\varphi' \; [(\omega\sin\varphi' - 2\dot{\theta}') \; \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta') \; \hat{\boldsymbol{\varphi}}']$$

The magnitude ratio factor is

$$[\; -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta' \; / \; - (m_1/\mu_2) \omega r'^2_1 \sin\theta'\cos\varphi' \;]$$

$$= [\; 3\omega\cos\theta' / \cos\varphi' \;]$$

and then

$$N_{fict}/N_{true} = (\cos\varphi'/3\omega\cos\theta')\sqrt{(\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta')^2}$$

$$= (\cos\varphi'/3\cos\theta')\sqrt{(\sin\varphi' - 2[\dot{\theta}'/\omega])^2 + (\cos\varphi'\cos\theta' - 2[\dot{\varphi}'/\omega]\sin\theta')^2}$$

I want an argument that makes this small. Only if $\varphi' = \pi/2$ and it is planar is this small!

Case 1: vertically aligned. Then $\cos\theta' = 1$ and $\sin\theta' = 0$ and the above gives

$$N_{fict}/N_{true} = (\cos\varphi'/3\omega)\sqrt{(\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega\cos\varphi')^2}$$

Not sure what that means.

Limit #1: Let $b \to \infty$ so that $\omega \to 0$. Then both torques vanish and the result is no torque at all.

But in the limit not quite there we might say

$$\mathbf{N}^{(b)} \approx -3\omega^2(m_1/\mu_2)r'_1{}^2\sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,[(-2\dot{\theta}')\,\hat{\boldsymbol{\theta}}' + (-2\dot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\varphi}}']$$

and then the fictional force dominates! We then have in the $\omega \to 0$ limit,

$$\dot{\mathbf{L}}'^{(0)} = (m_1/\mu_2)\,r'_1{}^2\,[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')\,\hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\theta}}']$$

$$\mathbf{N}'_{tot} \approx -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,[(-2\dot{\theta}')\,\hat{\boldsymbol{\theta}}' + (-2\dot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\varphi}}']$$

This gives in this limit two equations of motion

$$(m_1/\mu_2)\,r'_1{}^2\,[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')] = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,(-2\dot{\varphi}'\sin\theta') \qquad \hat{\boldsymbol{\varphi}}'$$

$$-(m_1/\mu_2)\,r'_1{}^2\,(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta') = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,(-2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')] = -\omega\sin\theta'\cos\varphi'\,(-2\dot{\varphi}'\sin\theta') \qquad \hat{\boldsymbol{\varphi}}'$$

$$-(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta') = -\omega\sin\theta'\cos\varphi'\,(-2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta' = +2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' \qquad \hat{\boldsymbol{\varphi}}'$$

$$2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta' = -\omega\sin\theta'\cos\varphi' \quad (2\dot{\theta}') \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 - 2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' = 0$$

$$\sin\theta'\,\ddot{\varphi}' + 2\cos\theta'\dot{\theta}'\dot{\varphi}' + 2\,\omega\sin\theta'\cos\varphi'\dot{\theta}' = 0$$

They are very ugly, but *for the very first time ever* I have something I can write down!! The equations are horribly coupled. Maybe in my limit I can ignore the last terms as $\omega\to 0$, huge high orbit. Then

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 = 0$$

$$\ddot{\varphi}' + 2\cot\theta'\dot{\theta}'\dot{\varphi}' = 0$$

but these equations are inscrutable. Suppose instead I assume close to vertical. Then

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 - 2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' = 0$$
$$\sin\theta'\,\ddot{\varphi}' + 2\cos\theta'\dot{\theta}'\dot{\varphi}' + 2\,\omega\sin\theta'\cos\varphi'\dot{\theta}' = 0$$

$$\ddot{\theta}' - \theta'\,\dot{\varphi}'^2 - 2\omega\theta'^2\cos\varphi'\,\dot{\varphi}' = 0$$
$$\theta'\,\ddot{\varphi}' + 2\,\dot{\theta}'\dot{\varphi}' + 2\,\omega\theta'\cos\varphi'\dot{\theta}' = 0$$

Suppose we try ansatz that φ = constant. Then we get

$$\ddot{\theta}' = 0$$
$$2\,\omega\theta'\cos\varphi'\dot{\theta}' = 0$$

It then just sits there with $\dot{\theta}' = \ddot{\theta}' = 0$, not very exciting.

Let's instead try the large-ω limit and see what happens there. Start over with

$$\mathbf{N}^{(b)} \approx -3\omega^2(m_1/\mu_2)r'_1{}^2\sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi' \left[(\omega\sin\varphi' - 2\dot{\theta}')\,\hat{\boldsymbol{\theta}}' + (\omega\cos\varphi'\cos\theta' - 2\dot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\varphi}}' \right]$$

Now just assume $\mathbf{N}^{(b)}$ dominates because it has ω^2 sitting there. Then we get

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\,M\,r'_1{}^2 \left[(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta')\,\hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\,\hat{\boldsymbol{\theta}}' \right]$$

$$\mathbf{N} \approx -3\omega^2(m_1/\mu_2)r'_1{}^2\sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

This says that we don't change the ang mom about the $\hat{\boldsymbol{\theta}}'$ which is the "dipping" axis.

The two equations of motion now are

$$(m_1/m_2)\,M\,r'_1{}^2\;[\,(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta') = -3\omega^2(m_1/\mu_2)r'_1{}^2\sin\theta'\cos\theta'$$

$$(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta') = 0$$

or

$$(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta') = -3\omega^2\sin\theta'\cos\theta'$$

$$2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta' = 0$$

Now do ansatz where φ = constant. Then 2nd equation is happy and first says

$$\ddot{\theta}' = -3\omega^2\sin\theta'\cos\theta'$$

NOW also assume small angle to get

$$\ddot{\theta}' = -3\omega^2\theta'$$

or

$$\ddot{\theta}' + 3\omega^2\theta' = 0$$

Finally we get some oscillation which involves ω and involves a factor of 3. What did my PDF say about this? Here from p 124,

> Although the vertical orientation of the tether system is a stable one, there are forces which cause it to librate (oscillate) about the vertical. These weak but persistent forces include atmospheric drag due to the different air densities encountered in the northward and southward passes of non-equatorial orbits and due to solar heating and electrodynamic forces (for conducting tethers). Station-keeping and other rocket maneuvers would also contribute to driving (or damping) libration. The natural frequency for in-plane (in the orbit plane) librations is $\sqrt{3}\,\omega_o = 1.732\,\omega_o$, and $2\,\omega_o$ for out-of-plane librations (a detailed derivation is contained in Ref. 2).

So they are saying

$$\omega_{\mathtt{libration}} = \sqrt{3}\,\omega$$

and that agrees with my equation, hurray!!!

So you have to ignore the fictitious force to get this result! What is the condition for doing that? I show above that φ' = π/2 would do it, the planar case only!

What is the Ref 2 above?

2. Beletskii, V. V. and Levin, E. M., "Dynamics of Space Tether Systems," Advances in the Astronautical Sciences, Vol. 83.

D.10 Force analysis of the Dumbbell Satellite in Frame S'

A. COMPUTE TRUE FORCES

The only forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F}_1 = -(GM_E m_1/r_1^3)\,\mathbf{r}_1 - T\,\hat{\mathbf{r}}'_1 = -(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r_2^3)\,\mathbf{r}_2 - T\,\hat{\mathbf{r}}'_2 = -(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2$$

Given that b is the distance to CMS, the above equations are exact no matter where COG lies.

B. COMPUTE FICTITIOUS FORCES

We must now compute the fictitious force on each mass.

$$\mathbf{F}'_{fict,1} \approx -m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$
$$\mathbf{F}'_{fict,2} \approx -m_2\ddot{\mathbf{b}}_S - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_2 - m_2\dot{\boldsymbol{\omega}} \times \mathbf{r}'_2$$

What is the meaning of $\boldsymbol{\omega}$ here? I have assumed that the CMS is in orbit! So the above equations are not exactly true.

C. TOTAL FORCES and EQUATIONS OF MOTION

We then know that,

$$\mathbf{F}'_{eff,1} = m_1\,\ddot{\mathbf{r}}'_1$$
$$\approx -(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$

$$\mathbf{F}'_{eff,2} = m_2\ddot{\mathbf{r}}'_2 = -m_1\ddot{\mathbf{r}}'_1$$
$$\approx -(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2 - m_2\ddot{\mathbf{b}}_S - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_2 - m_2\dot{\boldsymbol{\omega}} \times \mathbf{r}'_2$$

We also know that

$$\mathbf{b} = b\hat{\mathbf{b}} = b\hat{\mathbf{z}} = b\hat{\mathbf{z}}' \qquad // = b\cos\theta'\,\hat{\mathbf{r}}'_1 - b\sin\theta'\,\hat{\boldsymbol{\theta}}'$$

$r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$ $\qquad\qquad$ $\mathbf{r}'_2 = -(m_1/m_2)\mathbf{r}'_1$

$r_2^2 = b^2 + r'^2_2 - 2br'_2\cos\theta'$

$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 b\ \hat{\mathbf{z}}'$ $\qquad\qquad$ $GM_E = \omega^2 b^3$

Now since our setup is a Special Case #1 problem, we can write

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = \dot{\boldsymbol{\omega}} \times \mathbf{b} + (\boldsymbol{\omega}\bullet\mathbf{b})\omega\,\hat{\mathbf{x}}' - \omega^2 b\,\hat{\mathbf{b}}$$

But $(\boldsymbol{\omega}\bullet\mathbf{b}) = 0$ from Fig 1, and $\hat{\mathbf{b}} = \hat{\mathbf{z}} = \hat{\mathbf{z}}'$ so

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 b\ \hat{\mathbf{z}}'$$

The following items are givens: $\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}, m_1, m_2, b, GM_E$

The following items are the unknowns: $\mathbf{r}'_1(t)$, T = 4 unknowns

So we seem to have a system of 6 2nd-order differential equations in 4 unknowns, so the ODE's must not all be linearly independent.

The Sum Equation : What happens if you add the two ODE's (so the right sides cancel out):

$$-(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$
$$-(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2 - m_2\ddot{\mathbf{b}}_S - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_2 - m_2\dot{\boldsymbol{\omega}} \times \mathbf{r}'_2 \approx 0$$

In the second line make the obvious replacements to get $\qquad m_2\mathbf{r}'_2 = -m_1\mathbf{r}'_1$

$$-(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}'_1$$
$$-(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) + T\,\hat{\mathbf{r}}'_1 - m_2\ddot{\mathbf{b}}_S + m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) + 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1 + m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}'_1 \approx 0$$

or

$$-(GM_E m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - m_1\ddot{\mathbf{b}}_S$$
$$-(GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) - m_2\ddot{\mathbf{b}}_S \approx 0$$

or

$$-(\omega^2 b^3 m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - (\omega^2 b^3 m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S$$

or

$$-(m_1/r_1^3)(\mathbf{b} + \mathbf{r}'_1) - (m_2/r_2^3)(\mathbf{b} + \mathbf{r}'_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S/(\omega^2 b^3)$$

or

$$-(1/r_1^3)(m_1\mathbf{b} + m_1\mathbf{r}'_1) - (1/r_2^3)(m_2\mathbf{b} + m_2\mathbf{r}'_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S/(\omega^2 b^3)$$

or

$$(1/r_1^3)(m_1\mathbf{b} + m_1\mathbf{r}'_1) + (1/r_2^3)(m_2\mathbf{b} - m_1\mathbf{r}'_1) \approx -(m_1+m_2)\ddot{\mathbf{b}}_S/(\omega^2 b^3)$$

or

$$\mathbf{b}[(m_1/r_1^3) + (m_2/r_2^3)] + m_1\mathbf{r}'_1[(1/r_1^3) - (1/r_2^3)] \approx -(m_1+m_2)\ddot{\mathbf{b}}_S/(\omega^2 b^3)$$

where $\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 b\,\hat{\mathbf{z}}'$ $\qquad \mathbf{b} = b\hat{\mathbf{z}}'$

Suppose $\dot{\boldsymbol{\omega}} = 0$. Then our sum equation becomes

$$b\hat{\mathbf{z}}'[(m_1/r_1^3) + (m_2/r_2^3)] + m_1\mathbf{r}'_1[(1/r_1^3) - (1/r_2^3)] \approx (m_1+m_2)\omega^2 b\,\hat{\mathbf{z}}'/(\omega^2 b^3)$$

or

$$[b(m_1/r_1^3) + b(m_2/r_2^3)]\hat{\mathbf{z}}' + m_1 r'_1[(1/r_1^3) - (1/r_2^3)]\hat{\mathbf{r}}'_1 \approx M\,\hat{\mathbf{z}}'/b^2$$

or

$$[b(m_1/r_1^3) + b(m_2/r_2^3) - (M/b^2)]\hat{\mathbf{z}}' + m_1 r'_1[(1/r_1^3) - (1/r_2^3)]\hat{\mathbf{r}}'_1 \approx 0$$

But $\hat{\mathbf{z}}'$ and $\hat{\mathbf{r}}'_1$ can be regarded as two linearly independent vectors (if $\theta' \neq 0$) so then

$$b(m_1/r_1^3) + b(m_2/r_2^3) \approx (M/b^2)$$

$$(1/r_1^3) \approx (1/r_2^3)$$

Idea: I have assumed that CMS is COG and maybe this assumption is forcing these results? I think that is in fact the case. Putting $r_1 = r_2 = b$ into the first equation gives an identity.

So does this mean that this whole method is dead in the water? All it really says is that we don't get a contradiction from the sum equation, given that it is the sum of approximate equations.

So I think it is OK to move on. The sum equation is approximately valid, so maybe that means that only one of the two vector equations of motion is linearly independent so there are then 3 equations.

Let's restate the equations with $\mathbf{b} = b\hat{\mathbf{z}}'$ and $\ddot{\mathbf{b}}_S = -\omega^2 b\hat{\mathbf{z}}$ both installed : (no Euler term)

$$\mathbf{F}'_{\mathrm{eff},1} = m_1\ddot{\mathbf{r}}'_1$$
$$\approx -(GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1$$

$$\mathbf{F}'_{\mathrm{eff},2} = m_2\ddot{\mathbf{r}}'_2 = -m_1\ddot{\mathbf{r}}'_1$$
$$\approx -(GM_E m_2/r_2^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2 + m_2\omega^2 b\hat{\mathbf{z}} - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_2$$

So we only need to study the first equation:

$$\mathbf{F'}_{\mathbf{eff},1} = m_1 \ddot{\mathbf{r}}'_1$$

$$\approx \{ - (GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1 \omega^2 b\hat{\mathbf{z}} \} - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1 \boldsymbol{\omega} \times \dot{\mathbf{r}}'_1$$

The following items are givens: $\boldsymbol{\omega}$, b, m_1, m_2, r'_1

The following items are the unknowns: $\hat{\mathbf{r}}'_1(t)$, T = 3 unknowns (so 3 in 3, good).

Consider its first three terms which are in the {...} brackets,

$$- (GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1 \omega^2 b\hat{\mathbf{z}}$$

$$- (\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1 \omega^2 b\hat{\mathbf{z}}$$

$$- (\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}') - (\omega^2 b^3 m_1/r_1^3)\,\mathbf{r}'_1 - T\,\hat{\mathbf{r}}'_1 + m_1 \omega^2 b\hat{\mathbf{z}}$$

$$- (\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}') - (\omega^2 b^3 r'_1 m_1/r_1^3)\,\hat{\mathbf{r}}'_1 - T\,\hat{\mathbf{r}}'_1 + m_1 \omega^2 b\hat{\mathbf{z}}$$

$$[- (\omega^2 b^3 m_1/r_1^3)b + m_1 \omega^2 b]\,\hat{\mathbf{z}} + [- (\omega^2 b^3 r'_1 m_1/r_1^3) - T]\,\hat{\mathbf{r}}'_1$$

$$\omega^2 b m_1 [- (b^3/r_1^3) + 1]\,\hat{\mathbf{z}} - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T]\,\hat{\mathbf{r}}'_1$$

But we know that

$$\hat{\mathbf{z}}' = \cos\theta'\,\hat{\mathbf{r}}'_1 - \sin\theta'\,\hat{\boldsymbol{\theta}}'$$

so the first three terms are

$$\omega^2 b m_1 [- (b^3/r_1^3) + 1]\,[\cos\theta'\,\hat{\mathbf{r}}'_1 - \sin\theta'\,\hat{\boldsymbol{\theta}}'] - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T]\,\hat{\mathbf{r}}'_1$$

$$= \{ \omega^2 b m_1 (- (b^3/r_1^3) + 1) \cos\theta' - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T] \}\,\hat{\mathbf{r}}'_1 + [- \omega^2 b m_1 (-(b^3/r_1^3) + 1) \sin\theta']\,\hat{\boldsymbol{\theta}}'$$

I am wondering whether you should just say $b \approx r_1$ to clear this first term, to our level of approx? Well consider

$$1 - (b^3/r_1^3) = (r_1^3 - b^3)/r_1^3 = (r_1 - b)(r_1^2 + r_1 b + b^2)/r_1^3 \approx (r_1 - b)(3b^2/b^3) = 3(r_1 - b)/b$$

So this factor is linear in $(r_1 - b)$. Now since $r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$ I have shown on scratch that

$$r_1 - b \approx r'_1\cos\theta'$$

So then I claim that

$$1 - (b^3/r_1'^3) \approx 3\, r'_1 \cos\theta'/b$$

So rather than dropping the first term, I will use the above approximation.

Let's next to the centrifugal term:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)$$

$$= (\boldsymbol{\omega}\bullet\mathbf{r}'_1)\boldsymbol{\omega} - \omega^2\, \mathbf{r}'_1$$

$$= (\omega\, \hat{\mathbf{x}}'\bullet\mathbf{r}'_1)\omega\, \hat{\mathbf{x}}' - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1(\hat{\mathbf{x}}'\bullet\hat{\mathbf{r}}'_1)\, \hat{\mathbf{x}}' - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1 \sin\theta'\cos\varphi'\, \hat{\mathbf{x}}' - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1 \sin\theta'\cos\varphi'[\, \cos\varphi'\sin\theta'\, \hat{\mathbf{r}}'_1 + \cos\varphi'\cos\theta'\, \hat{\boldsymbol{\theta}}' - \sin\varphi'\, \hat{\boldsymbol{\varphi}}'] - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1 \sin\theta'\cos\varphi'\cos\varphi'\sin\theta'\, \hat{\mathbf{r}}'_1 + \omega^2 r'_1 \sin\theta'\cos\varphi'\cos\varphi'\cos\theta'\, \hat{\boldsymbol{\theta}}' - \omega^2 r'_1 \sin\theta'\cos\varphi'\sin\varphi'\, \hat{\boldsymbol{\varphi}}' - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1 \sin^2\theta'\cos^2\varphi'\, \hat{\mathbf{r}}'_1 + \omega^2 r'_1 \sin\theta'\cos\theta'\cos^2\varphi'\, \hat{\boldsymbol{\theta}}' - \omega r'^2_1 \sin\theta'\cos\varphi'\sin\varphi'\, \hat{\boldsymbol{\varphi}}' - \omega^2 r'_1\, \hat{\mathbf{r}}'_1$$

$$= [\omega^2 r'_1 \sin^2\theta'\cos^2\varphi' - \omega^2 r'_1]\, \hat{\mathbf{r}}'_1 + [\omega^2 r'_1 \sin\theta'\cos\theta'\cos^2\varphi']\, \hat{\boldsymbol{\theta}}' + [-\omega^2 r'_1 \sin\theta'\cos\varphi'\sin\varphi']\, \hat{\boldsymbol{\varphi}}'$$

$$= \omega^2 r'_1 \{ [\sin^2\theta'\cos^2\varphi' - 1]\, \hat{\mathbf{r}}'_1 + [\sin\theta'\cos\theta'\cos^2\varphi']\, \hat{\boldsymbol{\theta}}' + [-\sin\theta'\cos\varphi'\sin\varphi']\, \hat{\boldsymbol{\varphi}}' \}$$

This certainly is an ugly result. Now move on the Coriolis term factor which is

$$\boldsymbol{\omega} \times \mathbf{v}'_1 = \omega\hat{\mathbf{x}}' \times [v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\, \hat{\boldsymbol{\varphi}}']$$

$$= \omega v'_{\theta 1}\, \hat{\mathbf{x}}' \times \hat{\boldsymbol{\theta}}' + \omega\, v'_{\varphi 1}\, \hat{\mathbf{x}}' \times \hat{\boldsymbol{\varphi}}'$$

$$= \omega v'_{\theta 1}\, [\cos\varphi'\sin\theta'\, \hat{\boldsymbol{\varphi}}' + \sin\varphi'\, \hat{\mathbf{r}}'_1\,] + \omega\, v'_{\varphi 1}\, [-\cos\varphi'\sin\theta'\, \hat{\boldsymbol{\theta}}' + \cos\varphi'\cos\theta'\, \hat{\mathbf{r}}'_1]$$

$$= \omega[\, v'_{\theta 1}\sin\varphi' + v'_{\varphi 1}\cos\varphi'\cos\theta']\, \hat{\mathbf{r}}'_1 + \omega[-v'_{\varphi 1}\cos\varphi'\sin\theta']\, \hat{\boldsymbol{\theta}}' + \omega[\, v'_{\theta 1}\cos\varphi'\sin\theta']\, \hat{\boldsymbol{\varphi}}'$$

where

$$v'_{\theta 1} = \dot{\theta}'$$

$$v'_{\varphi 1} = r'_1 \dot{\varphi}' \sin\theta'$$

so

$$= \omega r'_1 [\, \dot{\theta}' \sin\varphi' + \dot{\varphi}' \sin\theta' \cos\varphi' \cos\theta'] \, \hat{\mathbf{r}}'_1 + \omega r'_1 [- \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + \omega r'_1 [\, \dot{\theta}' \cos\varphi' \sin\theta'] \, \hat{\boldsymbol{\varphi}}'$$

$$= \omega r'_1 \{ [\, \dot{\theta}' \sin\varphi' + \dot{\varphi}' \sin\theta' \cos\varphi' \cos\theta'] \, \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + [\, \dot{\theta}' \cos\varphi' \sin\theta'] \, \hat{\boldsymbol{\varphi}}'$$

$$= \omega r'_1 \{ [\, \dot{\theta}' \sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \, \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + [\, \dot{\theta}' \sin\theta' \cos\varphi'] \, \hat{\boldsymbol{\varphi}}'$$

Here are the conclusions so far

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) =$$

$$= \omega r'_1 \{ [\omega \sin^2\theta' \cos^2\varphi' - \omega] \, \hat{\mathbf{r}}'_1 + [\omega \sin\theta' \cos\theta' \cos^2\varphi'] \, \hat{\boldsymbol{\theta}}' + [- \omega \sin\theta' \cos\varphi' \sin\varphi'] \, \hat{\boldsymbol{\varphi}}' \}$$

$$\boldsymbol{\omega} \times \mathbf{v}'_1 =$$

$$= \omega r'_1 \{ [\, \dot{\theta}' \sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \, \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + [\, \dot{\theta}' \sin\theta' \cos\varphi'] \, \hat{\boldsymbol{\varphi}}'$$

Then

$$-m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)$$

$$= -m_1 \omega r'_1 \{ [\omega \sin^2\theta' \cos^2\varphi' - \omega] \, \hat{\mathbf{r}}'_1 + [\omega \sin\theta' \cos\theta' \cos^2\varphi'] \, \hat{\boldsymbol{\theta}}' + [- \omega \sin\theta' \cos\varphi' \sin\varphi'] \, \hat{\boldsymbol{\varphi}}' \}$$

$$-2m_1 \boldsymbol{\omega} \times \mathbf{v}'_1 =$$

$$= -2m_1 \omega r'_1 \{ [\, \dot{\theta}' \sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \, \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + [\, \dot{\theta}' \sin\theta' \cos\varphi'] \, \hat{\boldsymbol{\varphi}}' \}$$

The sum of these terms will be

$$\text{sum} = -m_1 \omega r'_1 \mathbf{Q}$$

where

$$\mathbf{Q} = [\omega \sin^2\theta' \cos^2\varphi' - \omega] \, \hat{\mathbf{r}}'_1 + [\omega \sin\theta' \cos\theta' \cos^2\varphi'] \, \hat{\boldsymbol{\theta}}' + [- \omega \sin\theta' \cos\varphi' \sin\varphi'] \, \hat{\boldsymbol{\varphi}}'$$

$$+ [\, 2\dot{\theta}' \sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \, \hat{\mathbf{r}}'_1 + [- 2 \dot{\varphi}' \sin^2\theta' \cos\varphi'] \, \hat{\boldsymbol{\theta}}' + [\, 2\dot{\theta}' \sin\theta' \cos\varphi'] \, \hat{\boldsymbol{\varphi}}'$$

$$= [\, \omega \sin^2\theta' \cos^2\varphi' - \omega + 2\dot{\theta}' \sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi' \,] \, \hat{\mathbf{r}}'_1$$

$$+ [\, \omega \sin\theta' \cos\theta' \cos^2\varphi' - 2 \dot{\varphi}' \sin^2\theta' \cos\varphi' \,] \, \hat{\boldsymbol{\theta}}'$$

$$+ [\, - \omega \sin\theta' \cos\varphi' \sin\varphi' + 2\dot{\theta}' \sin\theta' \cos\varphi' \,] \, \hat{\boldsymbol{\varphi}}'$$

$= [\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\ \hat{\mathbf{r}}'_1$

$+ \sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\ \dot{\varphi}'\sin\theta'\ ]\ \hat{\boldsymbol{\theta}}'$

$+ \sin\theta'\cos\varphi'\ [\ - \omega\sin\varphi'\ + 2\dot{\theta}'\ ]\ \hat{\boldsymbol{\varphi}}'$

Finally we can assemble all the terms:

$\mathbf{F}'_{\mathtt{eff},1} = m_1\ \ddot{\mathbf{r}}'_1$

$= \{\ \omega^2 b m_1\ (\ -(b^3/r_1^3) + 1)\cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}\ \hat{\mathbf{r}}'_1 + [-\ \omega^2 b m_1\ (\ -(b^3/r_1^3) + 1)\sin\theta']\ \hat{\boldsymbol{\theta}}'$

$-m_1\omega r'_1\mathbf{Q}$

Take it one component at a time. First the $\hat{\mathbf{r}}'_1$ component:

$\{\ \omega^2 b m_1\ (\ -(b^3/r_1^3) + 1)\cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \{\ \omega^2 b m_1\ (3\ r'_1\cos\theta'/b)\cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$ // approx from above

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ -\ [\ (\omega^2 r'_1 m_1) + T\ ]\ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ - \omega^2 r'_1 m_1 - T\ ]\ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \{\ \omega^2 m_1\ r'_1(3\cos^2\theta'-1) - T\ ]\ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

$\approx\ \omega^2 m_1\ r'_1(3\cos^2\theta'-1)\ -\ T\ \ -m_1\omega r'_1\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$

This is going to get set equal to $m_1 a'_1$ which has this $\hat{\mathbf{r}}'_1$ component

$m_1 r'_1 [\ -\dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta' \ ]$

So here is the equation of motion in the $\hat{\mathbf{r}}'_1$ component:

$$-\ [\ \omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1)\ ]\ -\omega\ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ]\}$$

$$= -\ [\ \dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta'\ ]$$

or

$$\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1) + \omega\ [\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ] - \dot{\theta}'^2 - \dot{\varphi}'^2\sin^2\theta' = 0$$

$$\omega^2(3\cos^2\theta'-2) + T/(m_1 r'_1) + \omega\ [\omega\sin^2\theta'\cos^2\varphi' + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\ ] - \dot{\theta}'^2 - \dot{\varphi}'^2\sin^2\theta' = 0$$

The unknowns here are: $\theta'(t)$, $\varphi'(t)$, $T(t)$

This ODE has the θ' and φ' stuff cross coupled, and it is non-linear because it has $\dot{\theta}'^2$.

Let's get the other two equations at least written down. First here is the $\hat{\boldsymbol{\theta}}'$ component:

$$[-\omega^2 b m_1\ (\ -(b^3/r_1^3) + 1)\ \sin\theta']\ \ -m_1\omega r'_1\ \sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\ \dot{\varphi}'\ \sin\theta'\ ]$$

$$= m_1\ r'_1\ [\ \ddot{\theta}' - \dot{\varphi}'^2\ \sin\theta'\ \cos\theta'\ ]$$

I will approx the first term as before, leaving then

$$-\omega^2 m_1 3\ r'_1 \cos\theta'\ \sin\theta'\ \ -m_1\omega r'_1\ \sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\ \dot{\varphi}'\ \sin\theta'\ ]$$
$$= m_1\ r'_1\ [\ \ddot{\theta}' - \dot{\varphi}'^2\ \sin\theta'\ \cos\theta'\ ]$$

$$-\omega^2 3\ \cos\theta'\ \sin\theta'\ \ -\omega\ \sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\ \dot{\varphi}'\ \sin\theta'\ ]\ =\ \ddot{\theta}' - \dot{\varphi}'^2\ \sin\theta'\ \cos\theta'$$

What happens here if θ' is very small?

$$-3\omega^2\ \theta' - \omega\theta'\cos\varphi'\ [\ \omega\cos\varphi' - 2\ \dot{\varphi}'\ \theta'] = \ddot{\theta}' - \dot{\varphi}'^2\ \theta'$$

$$-3\omega^2\ \theta' - [\ \omega\theta'\cos\varphi'\ \omega\cos\varphi' - \omega\theta'\cos\varphi'\ 2\ \dot{\varphi}'\ \theta'] = \ddot{\theta}' - \dot{\varphi}'^2\ \theta'$$

$$-3\omega^2\ \theta' - [\ \omega^2\theta'\cos^2\varphi'\ \ -\text{order}(\theta')^2] = \ddot{\theta}' - \dot{\varphi}'^2\ \theta'$$

$$-3\omega^2\,\theta' - \omega^2\theta'\cos^2\varphi' \;=\; \ddot{\theta}' - \dot{\varphi}'^2\,\theta'$$

$$0 \;=\; \ddot{\theta}' - \dot{\varphi}'^2\,\theta' + 3\omega^2\,\theta' + \omega^2\theta'\cos^2\varphi'$$

$$\ddot{\theta}' + [\,3\omega^2 - \dot{\varphi}'^2 + \omega^2\cos^2\varphi'\,]\,\theta' = 0$$

Now make the ansatz that $\varphi' = \pi/2$ so we have a fixed planar situation in paper. Then $\cos\varphi' = 0$ so

$$\ddot{\theta}' + [\,3\omega^2\,]\;\theta' = 0$$

and this agrees with what I got in the torque analysis! But there I had to assume ω large I think. But this basic resulting is still lurking in the works !

What about the $\hat{\boldsymbol{\varphi}}'$ equation just to fill this out. Only **Q** has such a component and we get

$$\mathbf{F}'_{\mathtt{eff},1}\bullet\hat{\boldsymbol{\varphi}}' \;=\; -m_1\omega r'_1\mathbf{Q}\bullet\hat{\boldsymbol{\varphi}}' \;=\; -m_1\omega r'_1\sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi'\;\;2\dot{\theta}'\,]$$

and the right side is $m_1 a'_{\varphi 1}$ so we get

$$-m_1\omega r'_1\sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi' + 2\dot{\theta}'\,] \;= m_1 r'_1\,[\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' \;+\; \ddot{\varphi}'\sin\theta']$$

$$-\omega\sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi' + 2\dot{\theta}'\,] \;= [\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' \;+\; \ddot{\varphi}'\sin\theta']$$

$$-\,[\,-\omega\sin\theta'\cos\varphi'\omega\sin\varphi' + \omega\sin\theta'\cos\varphi' 2\dot{\theta}'\,] \;= [\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' \;+\; \ddot{\varphi}'\sin\theta']$$

$$\omega^2\sin\theta'\cos\varphi'\sin\varphi' - \omega\sin\theta'\cos\varphi' 2\dot{\theta}' \;= 2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' \;+\; \ddot{\varphi}'\sin\theta'$$

$$\ddot{\varphi}'\sin\theta' + 2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' \;+\; \omega\sin\theta'\cos\varphi' 2\dot{\theta}' \;-\; \omega^2\sin\theta'\cos\varphi'\sin\varphi' = 0$$

What does this one do for small θ' ?

$$\ddot{\varphi}'\theta' + 2\,\dot{\theta}'\,\dot{\varphi}' + \omega\theta'\cos\varphi' 2\dot{\theta}' \;-\; \omega^2\theta'\cos\varphi'\sin\varphi' = 0$$

$$\ddot{\varphi}' + 2\,(\dot{\theta}'/\theta')\,\dot{\varphi}' + \omega\cos\varphi' 2\dot{\theta}' \;-\; \omega^2\cos\varphi'\sin\varphi' = 0$$

Suppose again that $\varphi' = \pi/2$, then it says

$$\ddot{\varphi}' + 2\,(\dot{\theta}'/\theta')\,\dot{\varphi}' = 0$$

and φ' = constant is a fine solution to this equation. So we have a self-consistent solution. Now recall

$\theta' = A\sin(\omega' t + \alpha)$ $\quad \dot{\theta}' = A\omega\cos(\omega' t+\alpha)$ $\quad (\dot{\theta}'/\theta') = \omega\cos(\omega' t+\alpha)/ \sin(\omega' t + \alpha)$

Then

$$\sin(\omega' t + \alpha)\, \ddot{\varphi}' + 2\ \omega\cos(\omega' t+\alpha)\, \dot{\varphi}' = 0$$

which is some strange but well defined first order ODE for $\dot{\varphi}'$.

Summary of resulting Equations of Motion (the order is r,θ,φ)

$$\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1) +\omega\, [\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' +2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta'$$

$$-\omega^2 3\cos\theta'\sin\theta' \quad -\omega\sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\,\dot{\varphi}'\sin\theta'\ ] = \ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta'$$

$$\omega^2\sin\theta'\cos\varphi'\sin\varphi' - \omega\sin\theta'\cos\varphi' 2\dot{\theta}' \quad = 2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'$$

What does the first equation say for small θ' ?

$$\omega^2(3-2) + T/(m_1 r'_1) +\omega\, [\omega\theta'^2\cos^2\varphi' + 2\dot{\theta}'\sin\varphi' +2\dot{\varphi}'\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2\theta'$$

$$\omega^2 + T/(m_1 r'_1) +\omega\, [2\dot{\theta}'\sin\varphi' +2\dot{\varphi}'\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2\ \theta'$$

$$2\omega^2 + T/(m_1 r'_1) + 2\omega\dot{\theta}'\sin\varphi' +2\omega\dot{\varphi}'\theta'\cos\varphi' = \dot{\theta}'^2 + \dot{\varphi}'^2\ \theta'$$

$$\omega^2 + T/(m_1 r'_1) + 2\omega\dot{\theta}'\sin\varphi' = \dot{\theta}'^2 + [\dot{\varphi}'^2 - 2\omega\dot{\varphi}'\cos\varphi']\ \theta'$$

Suppose it is static. Then equation says

$$\omega^2 + T/(m_1 r'_1) = 0$$

$T = -m_1 r'_1 \omega^2$ what I get here

This is the first time I have had any estimate for T. But what happened to

$$T = 3\,(mMG/r_0^3)(\Delta r) \quad . \qquad (8.6.9)$$

If I set $\Delta r = r'_1$ for mass m_1 and $r_0 = b$, this says

$$T = 3\,(m_1 r'_1 M_E G/b^3) \quad .$$

But as usual

$$GM_E = \omega^2 b^3$$

so my main text says

$$T = 3 (m_1 r'_1 \omega^2)$$ what I get in the main text

and things are not right, though dimensions are correct.

Start again with the r equation

$$\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1) + \omega [\omega \sin^2\theta' \cos^2\varphi' - \omega + 2\dot{\theta}' \sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta'$$

Set $\theta' = 0$ and $\dot{\theta}' = 0$ to get

$$2\omega^2 + T/(m_1 r'_1) + \omega [- \omega] = 0$$

$$\omega^2 + T/(m_1 r'_1) = 0$$

Same bad result. I think T has to be positive, so I have a bug in the r equation which of course is the messiest. I will review this tomorrow and hope it repairs itself.