F.6 Force analysis of the Dumbbell Satellite in Frame S' in Cartesian Coordinates

The only true forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F'}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b}+\mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F'}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2\,. \qquad \text{(F.6.1)}$$

As noted earlier, the center of gravity is not quite at the center of mass in Fig (F.1.1), but the above equations are exact despite this fact. The force are primed because they are forces in the inertial Frame S'. The reader is reminded that we are using the "swap notation" where prime↔noprime relative to the non-swap notation.

In order to use Newton's Law in rotating Frame S, we must include the fictitious forces. We translate the result of (8.1.8) to swap notation to obtain

$$\mathbf{F}_{fict,1} \approx -m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$
$$\mathbf{F}_{fict,2} \approx -m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2 \qquad \text{(F.6.2)}$$
frame centrifugal Coriolis Euler

In these equations, the **b** acceleration is given by the swap version of (7.13) which then states

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}). \qquad \text{// Special Case \#1} \qquad \text{(F.6.3)}$$

and the vectors **b** and **ω** are given by

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$$
$$\mathbf{b} = b\hat{\mathbf{z}}\,. \qquad \text{(F.6.4)}$$

Finally we may state Newton's Law for each mass,

$$\mathbf{F}_{eff,1} = m_1\,\mathbf{a}_1 \qquad \text{(F.6.5)}$$
$$\approx -(GM_E m_1/r'^3_1)\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$

$$\mathbf{F}_{eff,2} = m_2\mathbf{a}_2 \qquad \text{(F.6.6)}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 - m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2$$

where we have now a set of six scalar equations. Using (F.1.9) through (F.1.11), (F.6.6) can be rewritten,

$$\mathbf{F}_{eff,2} = -m_1\mathbf{a}_1 \qquad \text{(F.6.7)}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r'}_2 + T\,\hat{\mathbf{r}}_1 - m_2\ddot{\mathbf{b}}_{S'} + m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) + 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$

Adding (F.6.5) and (F.6.7) gives

$$0 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 + -(GM_E m_2/r'^3_2)\mathbf{r}'_2 - (m_1+m_2)\ddot{\mathbf{b}}_{S'} \qquad \text{(F.6.8)}$$

This equation is just **F** = m**a** in inertial Frame S' for the total satellite where **b** is the center of mass. Ignoring the small offset between center of mass and center of gravity, the three equations (F.6.8) describe the circular orbit of the satellite around the Earth. We may then regard the equation (F.6.5) as a set of three scalar equations for the three unknowns θ,φ and T where recall $\mathbf{r}_1 = (r_1,\theta,\varphi)$ in the spherical coordinates of Fig (F.1.1).

Our next task is to write vector equation (F.6.5) in Cartesian coordinates to obtain the three equations of motion. After expanding the left side, we then consider the right side of (F.6.5) one term at a time:

Left side of (F.6.5): $m_1\mathbf{a}_1 = m_1(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}})$ *(E.3.6)* (F.6.9)

Term 1: $-(GM_E m_1/r'^3_1)\mathbf{r}'_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1)$

$= -(GM_E m_1/r'^3_1)\,[\, b\hat{\mathbf{z}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,]$

$= -(GM_E m_1/r'^3_1)\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (b+z)\hat{\mathbf{z}}\,]$

Term 2: $-T\,\hat{\mathbf{r}}_1 = -(T/r_1)\mathbf{r}_1 = -(T/r_1)\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]$

Term 3: $-m_1\ddot{\mathbf{b}}_{S'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // (F.6.3)

$= -\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // satellite in circular orbit, $\dot{\boldsymbol{\omega}} = 0$

$= -\,m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b}$ //- **A** x (**A** x **C**) = -(**A**•**C**)**A** + A^2**C**

$= m_1\omega^2\mathbf{b} = m_1\omega^2 b\hat{\mathbf{z}}$ (F.6.12)

Term 4: $-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1$ // identity shown above

$= -m_1\omega^2(\hat{\mathbf{x}}\bullet\mathbf{r}_1)\hat{\mathbf{x}} + m_1\omega^2\mathbf{r}_1 = -m_1\omega^2(x)\hat{\mathbf{x}} + m_1\omega^2\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]$

$= m_1\omega^2\,[\, y\hat{\mathbf{y}} + z\hat{\mathbf{z}}]$

Term 5: $-2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 = -2m_1\,[\omega\hat{\mathbf{x}}] \times (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$

$= -2m_1\omega\,[\, v_y\hat{\mathbf{z}} - v_z\hat{\mathbf{y}}]$

(F.6.14)

Term 6: $\quad -m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1 = 0$ because we assume $\dot{\boldsymbol{\omega}} = 0$ (F.6.15)

Having all the bits and pieces, we now assemble the three component equations of (F.6.5). The numbers show the Term above associated with each piece:

$$\mathbf{F}_{\mathbf{eff},1} = m_1 \mathbf{a}_1 \qquad \textit{(F.6.5)}$$

$$\approx \underset{1}{-(GM_E m_1/r'^3_1)\mathbf{r}'_1} \underset{2}{- T\hat{\mathbf{r}}_1} \underset{3}{- m_1\ddot{\mathbf{b}}_S} \underset{4}{- m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)} \underset{5}{- 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1} \underset{6}{- m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1}$$

$\hat{\mathbf{x}}$: $\quad m_1 a_{1x} = \underset{1}{-(GM_E m_1/r'^3_1)\,x} \underset{2}{- (T/r_1)x}$

$\hat{\mathbf{y}}$: $\quad m_1 a_{1y} = \underset{1}{-(GM_E m_1/r'^3_1)\,y} \underset{2}{- (T/r_1)y} + \underset{4}{m_1\omega^2 y} + \underset{5}{2m_1\omega v_z}$

$\hat{\mathbf{z}}$: $\quad m_1 a_{1z} = \underset{1}{-(GM_E m_1/r'^3_1)\,(b+z)} \underset{2}{- (T/r_1)z} + \underset{3}{m_1\omega^2 b} + \underset{4}{m_1\omega^2 z} - \underset{5}{2m_1\omega\, v_y}$

Rewrite these trying to simplify

$$m_1 a_{1x} = -[(GM_E m_1/r'^3_1) + (T/r_1)]x$$

$$m_1 a_{1y} = -[(GM_E m_1/r'^3_1) + (T/r_1) - m_1\omega^2]y + 2m_1\omega v_z$$

$$m_1 a_{1z} = -(GM_E m_1/r'^3_1)(b+z) - (T/r_1)z + m_1\omega^2(b+z) - 2m_1\omega\, v_y$$

We now rewrite the three equations dividing by m_1 and using (F.5.1) that $GM_E = \omega^2 b^3$:

$$a_{1x} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1)]x$$

$$a_{1y} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1) - \omega^2]y + 2\omega v_z$$

$$a_{1z} = -(\omega^2 b^3/r'^3_1)(b+z) - (T/m_1 r_1)z + \omega^2(b+z) - 2\omega\, v_y$$

and now switch to dot notation

$$\ddot{x} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1)]x$$

$$\ddot{y} = -[(\omega^2 b^3/r'^3_1) + (T/m_1 r_1) - \omega^2]y + 2\omega\dot{z}$$

$$\ddot{z} = -(\omega^2 b^3/r'^3_1)(b+z) - (T/m_1 r_1)z + \omega^2(b+z) - 2\omega\,\dot{y}$$

$$x^2+y^2+z^2 = r_1^2$$

where

$$r'_1{}^2 = (r_1+b^2) = r_1{}^2 + b^2 + 2\,\mathbf{r_1}\bullet\,\mathbf{b} = r_1{}^2 + b^2 + 2\,\mathbf{r_1}\bullet\,[b\hat{\mathbf{z}}] = r_1{}^2 + b^2 + 2bz$$

This is a system of 4 equations in 4 unknowns x,y,z,T . Rewrite the last equation

$$\ddot{z} = \omega^2[-(b^3/r'_1{}^3) +1]\,(b+z) - (T/m_1r_1)z - 2\omega\,\dot{y}$$

The plan is to eliminate z and then have 3 equations in just x,y,T. I will do this in "satellite in Cartesians v4.mws". Define

$$Q = (T/m_1r_1)$$

so things are simpler

1 $\ddot{x} = -[(\omega^2b^3/r'_1{}^3) + Q]x$

2 $\ddot{y} = -[(\omega^2b^3/r'_1{}^3) + Q - \omega^2]y + 2\omega\dot{z}$

3 3 $\ddot{z} = \omega^2[-(b^3/r'_1{}^3) +1]\,(b+z) - Qz - 2\omega\,\dot{y}$

4 $x^2+y^2+z^2 = r_1{}^2$

My first shot fails with the usual error! Could it be because Q(t) has no derivatives?? What simple test could I do on this?

$\dot{x} = x+2$

$\dot{x} = y-3$

Here there is no y derivative, will this complain? Yes it does complain.

So I need to **eliminate Q(t)** from my system of equations. How does this work with the above set of equations?

1 $\ddot{x} = -(\omega^2b^3/r'_1{}^3)x - Qx$

2 $\ddot{y} = -(\omega^2b^3/r'_1{}^3)y - Qy + \omega^2y + 2\omega\dot{z}$

3 $\ddot{z} = \omega^2[-(b^3/r'_1{}^3) +1]\,(b+z) - Qz - 2\omega\,\dot{y}$

You could solve equation equation for Q to get

$Q = A$

$Q = B$

Q = C

Then you would have three equations which do not involve Q:

A = B
A = C
B = C

But subjecting the first two equations gives the third, so only two are independent. So I will do this in effect as follows. Start with 1 and 2

1*y $\quad y\ddot{x} = -(\omega^2 b^3/r'_1{}^3)xy - Qxy$

2*x $\quad x\ddot{y} = -(\omega^2 b^3/r'_1{}^3)yx - Qyx + \omega^2 yx + 2\omega\dot{z}x$

Subtract to eliminate Q

$y\ddot{x} - x\ddot{y} = -\omega^2 yx - 2\omega\dot{z}x$

Now repeat using the first and third equation

1*z $\quad z\ddot{x} = -(\omega^2 b^3/r'_1{}^3)xz - Qxz$

3*x $\quad x\ddot{z} = \omega^2[-(b^3/r'_1{}^3) + 1](b+z)x - Qzx - 2\omega\,\dot{y}x$

or

1*z $\quad z\ddot{x} = -(\omega^2 b^3/r'_1{}^3)xz - Qxz$

3*x $\quad x\ddot{z} = [-(\omega^2 b^3/r'_1{}^3) + \omega^2](b+z)x - Qzx - 2\omega\,\dot{y}x$

Temp define $A \equiv (\omega^2 b^3/r'_1{}^3)$ so this then says

1*z $\quad z\ddot{x} = -Axz - Qxz$

3*x $\quad x\ddot{z} = [-A + \omega^2](b+z)x - Qzx - 2\omega\,\dot{y}x$

Subtract to eliminate Q

$z\ddot{x} - x\ddot{z} = -Axz - [-A + \omega^2](b+z)x + 2\omega\,\dot{y}x$

or

$z\ddot{x} - x\ddot{z} = -Axz + [A - \omega^2](b+z)x + 2\omega\,\dot{y}x$

$z\ddot{x} - x\ddot{z} = -Axz + [A - \omega^2](bx+zx) + 2\omega\,\dot{y}x$

$z\ddot{x} - x\ddot{z} = -Axz + Abx + Azx - \omega^2(b+z)x + 2\omega\,\dot{y}x$

$$z\ddot{x} - x\ddot{z} = Abx - \omega^2(b+z)x + 2\omega \dot{y}x$$

$$z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega \dot{y}x$$

So we then end up with these two equations of motion:

$$y\ddot{x} - x\ddot{y} = -\omega^2 yx - 2\omega \dot{z}x$$

$$z\ddot{x} - x\ddot{z} = (\omega^2 b^4/r'_1{}^3)x - \omega^2(b+z)x + 2\omega \dot{y}x$$

$$r'_1{}^2 = r_1{}^2 + b^2 + 2bz$$

Maple Problem: Enter eq1 and eq2, eliminate z as usual, and see if Maple can do a numerical solution. Do this in "satellite in Cartesions v5.mws".

I am able to get it to fly, but the odeplot runs out of memory. Here is my code:

Enter the two equations of motion

```
eq1 := y(t)*diff(x(t),t,t) - x(t)*diff(y(t),t,t) = -w^2*y(t)*x(t)-2*w*zd*x(t);
```

$$eq1 := y(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -w^2\, y(t)\, x(t) - 2\, w\, zd\, x(t)$$

```
eq2 := z(t)* diff(x(t),t,t) - x(t)*zdd =
(w^2*b^4/rp1^3)*x(t)-w^2*(b+z(t))*x(t)+2*w*diff(y(t),t)*x(t);
```

$$eq2 := z(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\, zdd = \frac{w^2\, b^4\, x(t)}{rp1^3} - w^2\,(b + z(t))\, x(t) + 2\, w\left(\frac{\partial}{\partial t}y(t)\right) x(t)$$

Are these correctly entered? Yes they are. Next, here is how to eliminate z:

Equations to eliminate z from the ODE system

```
> z := (t) -> sqrt(r1^2-x(t)^2-y(t)^2);
```

$$z := t \to \sqrt{r1^2 - x(t)^2 - y(t)^2}$$

```
> zd := diff(z(t),t);
```

$$zd := \frac{1}{2}\frac{-2\, x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\, y(t)\left(\frac{\partial}{\partial t}y(t)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}$$

```
> zdd := diff(z(t),t,t);
```

$$zdd := -\frac{1}{4}\frac{\left(-2\, x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\, y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)^2}{\left(r1^2 - x(t)^2 - y(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\frac{-2\left(\frac{\partial}{\partial t}x(t)\right)^2 - 2\, x(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\, y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}$$

Next we have

Enter expression for r1p and enter constants

```
> r1p := r1^2+b^2+2*b*z(t);
```

$$r1p := r1^2 + b^2 + 2\,b\,\sqrt{r1^2 - x(t)^2 - y(t)^2}$$

```
> b := 1000:  r1 := 10:  w := 1:
```

This all looks good. Now we come to the dsolve call:

Setup and make the dsolve call

```
> eqs := {eq1,eq2}:
> funcs := {x(t),y(t)}:
> inits := {x(0)= 1,y(0) = 0, D(x)(0)=0,D(y)(0)=0};
```

$$inits := \{x(0) = 1, y(0) = 0, D(x)(0) = 0, D(y)(0) = 0\}$$

```
> f := dsolve(eqs union inits,funcs,type=numeric,method=classical,output=listprocedure);
```

$$f := \left[t = (\mathbf{proc}(t) \ \ldots \ \mathbf{end}), y(t) = (\mathbf{proc}(t) \ \ldots \ \mathbf{end}), \frac{\partial}{\partial t} y(t) = (\mathbf{proc}(t) \ \ldots \ \mathbf{end}), x(t) = (\mathbf{proc}(t) \ \ldots \ \mathbf{end}), \frac{\partial}{\partial t} x(t) = (\mathbf{proc}(t) \ \ldots \ \mathbf{end})\right]$$

There are only two equations, functions unknown are x(t) and y(t), and there are the inits. I start m_1 at location x = 0 and y = 0 and no velocity.

But then I ask if for f(0.1) and it locks up! So maybe I can try some other methods now. I started with classical and it hangs on f(0.1). I try rfk45 and get this

```
> f(0.1);
Error, (in dsolve/numeric/rkf45) cannot evaluate boolean
```

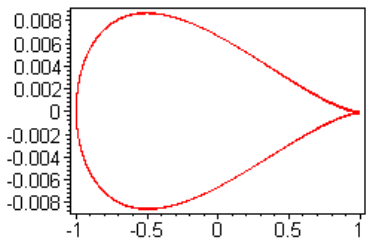
My error was that I had rp1 and r1p both present. Now it works better.

```
> f(0.1);
```

$$t(.1) = .1,\ y(t)(.1) = .002601172186851066,\ \left(\frac{\partial}{\partial t} y(t)\right)(.1) = .06422491379443354,\ x(t)(.1) = .5373794335159814,$$
$$\left(\frac{\partial}{\partial t} x(t)\right)(.1) = -8.476500496449882\Big]$$

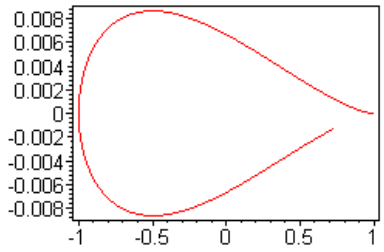
and I get finally some kind of semi-reasonable (perhaps) plot,

```
odeplot(f,[x(t),y(t)],0..2,numpoints=1000,axes = boxed);
```



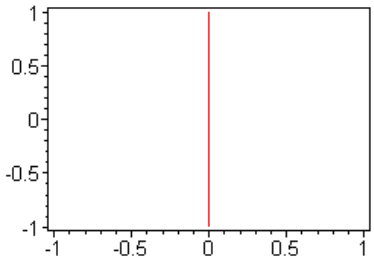
It starts at the right cusp and goes round the loop and repeat Here is just less than one loop

```
odeplot(f,[x(t),y(t)],0..0.55,numpoints=1000,axes = boxed);
```



I think as it starts to move, there is Coriolis pushing it to the right. This is I think a big breakthrough in my Cartesian satellite plotting efforts, even though I don't understand the result and I may have lots of math errors still. But it is a reasonable motion! This would be my "out of plane" motion, and I do expect a deviation.

Let's try x(0) = 0 and y(0) = 1, then I expect to get the "in plane" motion:



Here it stays in the x = 0 plane and swings back and forth. This looks VERY good, I will now take a break, it is 2:30 PM.

Now let's try for a circular path?