**F.6 Force analysis of the Dumbbell Satellite in Frame S' in Cartesian Coordinates**

The only true forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F'}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b}+\mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F'}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2\,. \qquad \text{(F.6.1)}$$

As noted earlier, the center of gravity is not quite at the center of mass in Fig (F.1.1), but the above equations are exact despite this fact. The force are primed because they are forces in the inertial Frame S'. The reader is reminded that we are using the "swap notation" where prime↔noprime relative to the non-swap notation.

In order to use Newton's Law in rotating Frame S, we must include the fictitious forces. We translate the result of (8.1.8) to swap notation to obtain

$$\mathbf{F}_{fict,1} \approx -m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$
$$\mathbf{F}_{fict,2} \approx -m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2 \qquad \text{(F.6.2)}$$
frame centrifugal Coriolis Euler

In these equations, the **b** acceleration is given by the swap version of (7.13) which then states

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}). \qquad \text{// Special Case \#1} \qquad \text{(F.6.3)}$$

and the vectors **b** and **ω** are given by

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$$
$$\mathbf{b} = b\hat{\mathbf{z}}\,. \qquad \text{(F.6.4)}$$

Finally we may state Newton's Law for each mass,

$$\mathbf{F}_{eff,1} = m_1\,\mathbf{a}_1 \qquad \text{(F.6.5)}$$
$$\approx -(GM_E m_1/r'^3_1)\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$

$$\mathbf{F}_{eff,2} = m_2\mathbf{a}_2 \qquad \text{(F.6.6)}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 - m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2$$

where we have now a set of six scalar equations. Using (F.1.9) through (F.1.11), (F.6.6) can be rewritten,

$$\mathbf{F}_{eff,2} = -m_1\mathbf{a}_1 \qquad \text{(F.6.7)}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r'}_2 + T\,\hat{\mathbf{r}}_1 - m_2\ddot{\mathbf{b}}_{S'} + m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) + 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$

Adding (F.6.5) and (F.6.7) gives

$$0 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 + -(GM_E m_2/r'^3_2)\mathbf{r}'_2 - (m_1+m_2)\ddot{\mathbf{b}}_{S'} \qquad \text{(F.6.8)}$$

This equation is just **F** = m**a** in inertial Frame S' for the total satellite where **b** is the center of mass. Ignoring the small offset between center of mass and center of gravity, the three equations (F.6.8) describe the circular orbit of the satellite around the Earth. We may then regard the equation (F.6.5) as a set of three scalar equations for the three unknowns θ,φ and T where recall $\mathbf{r}_1 = (r_1,\theta,\varphi)$ in the spherical coordinates of Fig (F.1.1).

Our next task is to write vector equation (F.6.5) in Cartesian coordinates to obtain the three equations of motion. After expanding the left side, we then consider the right side of (F.6.5) one term at a time:

Left side of (F.6.5): $\quad m_1\mathbf{a}_1 = m_1(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}})$ *(E.3.6)* (F.6.9)

Term 1: $-(GM_E m_1/r'^3_1)\mathbf{r}'_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1)$ // (F.1.8)

$$= -(GM_E m_1/r'^3_1)(\,b\hat{\mathbf{z}} + r_1\sin\theta\cos\varphi\hat{\mathbf{x}} + r_1\sin\theta\sin\varphi\hat{\mathbf{y}} + r_1\cos\theta\hat{\mathbf{z}}\,)$$

$$= -(GM_E m_1/r'^3_1)[\,r_1\sin\theta\cos\varphi\hat{\mathbf{x}} + r_1\sin\theta\sin\varphi\hat{\mathbf{y}} + (b+r_1\cos\theta)\hat{\mathbf{z}}] \qquad \text{(F.6.10)}$$

Term 2: $\quad -T\,\hat{\mathbf{r}}_1 = -T\,[r_1\sin\theta\cos\varphi\hat{\mathbf{x}} + r_1\sin\theta\sin\varphi\hat{\mathbf{y}} + r_1\cos\theta\hat{\mathbf{z}}\,]$ (F.6.11)

Term 3: $\quad -m_1\ddot{\mathbf{b}}_{S'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // (F.6.3)

$$= -\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{// satellite in circular orbit, } \dot{\boldsymbol{\omega}} = 0$$

$$= -\,m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b} \qquad \text{//- } \mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = -(\mathbf{A}\bullet\mathbf{C})\mathbf{A} + A^2\mathbf{C}$$

$$= m_1\omega^2\mathbf{b} = m_1\omega^2 b\hat{\mathbf{z}} \qquad \text{(F.6.12)}$$

Term 4: $\quad -m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1$ // identity shown above

$$= -m_1\omega^2 r_1(\hat{\mathbf{x}}\bullet\hat{\mathbf{r}}_1)\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1 \qquad \text{// (F.6.4)}$$

$$= -m_1\omega^2 r_1\sin\theta\cos\varphi\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1 \qquad \text{// (E.2.4))}$$

$$= -m_1\omega^2 r_1\sin\theta\cos\varphi\hat{\mathbf{x}} + m_1\omega^2 r_1[\sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}] \qquad \text{// (E.2.4))}$$

$$= m_1\omega^2 r_1[\,(\sin\theta\cos\varphi - \sin\theta\cos\varphi)\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\,]$$

$= m_1\omega^2 r_1[\ \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}\ ]$ (F.6.13)

Term 5: $-2m_1\ \boldsymbol{\omega}\ x\ \mathbf{v}_1 = -2m_1\ [\omega\hat{\mathbf{x}}]\ x\ (\ v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$

$= -2m_1\omega\ [\ v_y\hat{\mathbf{z}} - v_z\hat{\mathbf{y}}]$ (F.6.14)

Term 6: $-m_1\dot{\boldsymbol{\omega}}\ x\ \mathbf{r}_1 = 0$ because we assume $\dot{\boldsymbol{\omega}} = 0$ (F.6.15)

Having all the bits and pieces, we now assemble the three component equations of (F.6.5). The numbers show the Term above associated with each piece:

$\mathbf{F}_{\mathbf{eff},1} = m_1\ \mathbf{a}_1$ *(F.6.5)*

$\approx -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega}\ x\ (\boldsymbol{\omega}\ x\ \mathbf{r}_1) - 2m_1\ \boldsymbol{\omega}\ x\ \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\ x\ \mathbf{r}_1$

1 2 3 4 5 6

$\hat{\mathbf{x}}$: $m_1 a_{1x} = -(GM_E m_1/r'^3_1)\, r_1\sin\theta\cos\varphi - T r_1\sin\theta\cos\varphi$

1 2

$\hat{\mathbf{y}}$: $m_1 a_{1y} = -(GM_E m_1/r'^3_1)\, r_1\sin\theta\sin\varphi - T\, r_1\sin\theta\sin\varphi + m_1\omega^2 r_1 \sin\theta\sin\varphi + 2m_1\omega v_z$

1 2 4 5

$\hat{\mathbf{z}}$: $m_1 a_{1z} = -(GM_E m_1/r'^3_1)\,(b+r_1\cos\theta) - T\,(b+r_1\cos\theta) + m_1\omega^2 b + m_1\omega^2 r_1\cos\theta - 2m_1\omega\, v_y$

1 2 3 4 5

: $m_1 r_1(2\,\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta) = +m_1\omega^2 r_1\sin\theta\cos\varphi\sin\varphi - 2m_1\omega r_1(\dot{\theta}\sin\theta\cos\varphi)$ (F.6.18)ok

4 5

We now rewrite the three equations dividing by $m_1$ and using (F.5.1) that $GM_E = \omega^2 b^3$ :

$\hat{\mathbf{x}}$: $a_{1x} = -(\omega^2 b^3/r'^3_1)\, r_1\sin\theta\cos\varphi - (T/m_1) r_1\sin\theta\cos\varphi$

$\hat{\mathbf{y}}$: $a_{1y} = -(\omega^2 b^3/r'^3_1)\, r_1\sin\theta\sin\varphi - (T/m_1)\, r_1\sin\theta\sin\varphi + \omega^2 r_1 \sin\theta\sin\varphi + 2\omega v_z$

$\hat{\mathbf{z}}$: $a_{1z} = -(\omega^2 b^3/r'^3_1)\,(b+r_1\cos\theta) - (T/m_1)\,(b+r_1\cos\theta) + \omega^2 b + \omega^2 r_1\cos\theta - 2\omega\, v_y$

or

$\hat{\mathbf{x}}$: $\ddot{x} = -(\omega^2 b^3/r'^3_1)\, r_1\sin\theta\cos\varphi - (T/m_1) r_1\sin\theta\cos\varphi$

$\hat{\mathbf{y}}$: $\ddot{y} = -(\omega^2 b^3/r'^3_1)\, r_1\sin\theta\sin\varphi - (T/m_1)\, r_1\sin\theta\sin\varphi + \omega^2 r_1 \sin\theta\sin\varphi + 2\omega\dot{z}$

$\hat{\mathbf{z}}$: $\ddot{z} = -(\omega^2 b^3/r'^3_1)\,(b+r_1\cos\theta) - (T/m_1)\,(b+r_1\cos\theta) + \omega^2 b + \omega^2 r_1\cos\theta - 2\omega\,\dot{y}$

Comments: These are "not bad" really, could have been much worse. But now we have the following unknown functions x, y,z,θ,φ. I guess I could have done it all simpler and I now rewrite

$\hat{\mathbf{x}}$: $\ddot{x} = -(\omega^2 b^3/r'_1{}^3)\, x - (T/m_1)x$

$\hat{\mathbf{y}}$: $\ddot{y} = -(\omega^2 b^3/r'_1{}^3)\, y - (T/m_1)y + \omega^2 y + 2\omega\dot{z}$

$\hat{\mathbf{z}}$: $\ddot{z} = -(\omega^2 b^3/r'_1{}^3)\,(b+z) - (T/m_1)\,(b+z) + \omega^2(b+z) - 2\omega\,\dot{y}$

Then

$$r'_1{}^2 = (r_1+b^2) = r_1{}^2 + b^2 + 2\,\mathbf{r_1}\bullet\,\mathbf{b} = r_1{}^2 + b^2 + 2b\,\mathbf{r_1}\bullet\,\mathbf{b}$$

$$= r_1{}^2 + b^2 + 2bz$$

Rewrite them again

$\ddot{x} = -[(\omega^2 b^3/r'_1{}^3) + (T/m_1)]\, x$ $\qquad$ $r'_1{}^2 = r_1{}^2 + b^2 + 2bz$

$\ddot{y} = -[(\omega^2 b^3/r'_1{}^3) + (T/m_1) - \omega^2]y + 2\omega\dot{z}$

$\ddot{z} = -[(\omega^2 b^3/r'_1{}^3) + (T/m_1) + \omega^2](b+z) - 2\omega\,\dot{y}$

$x^2+y^2+z^2 = r_1{}^2$

This is a system of 4 equations in 4 unknowns x,y,z,T .

Suppose I try to eliminate x. Then with some Maple help,

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x := sqrt(r1^2-y(t)^2-z(t)^2);
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$$x := \sqrt{r1^2 - y(t)^2 - z(t)^2}$$

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diff(x,t);
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$$\frac{1}{2}\,\frac{-2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right) - 2\,z(t)\left(\frac{\partial}{\partial t}z(t)\right)}{\sqrt{r1^2 - y(t)^2 - z(t)^2}}$$

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diff(x,t,t);
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$$-\frac{1}{4}\,\frac{\left(-2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right) - 2\,z(t)\left(\frac{\partial}{\partial t}z(t)\right)\right)^2}{\left(r1^2 - y(t)^2 - z(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\,\frac{-2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\,y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) - 2\left(\frac{\partial}{\partial t}z(t)\right)^2 - 2\,z(t)\left(\frac{\partial^2}{\partial t^2}z(t)\right)}{\sqrt{r1^2 - y(t)^2 - z(t)^2}}$$

We then end up with 3 equations in three unknowns which are y,z,T.

Maybe one should eliminate T early in the game. Rewrite the above

$$\ddot{y}x = -[(\omega^2 b^3/r'^3_1) + (T/m_1)]\, yx \qquad r'^2_1 = r_1^2 + b^2 + 2bz$$

$$x\ddot{y} = -[(\omega^2 b^3/r'^3_1) + (T/m_1) - \omega^2]yx + 2\omega\dot{z}x$$

$$\ddot{z} = -[(\omega^2 b^3/r'^3_1) + (T/m_1) + \omega^2](b+z) - 2\omega\,\dot{y}$$

Subtract the first two to get

$$y\ddot{x} - x\ddot{y} = -\omega^2 yx - 2\omega\dot{z}x$$

Go back again

$$(b+z)\ddot{x} = -[(\omega^2 b^3/r'^3_1) + (T/m_1)]\, x(b+z) \qquad r'^2_1 = r_1^2 + b^2 + 2bz$$

$$\ddot{y} = -[(\omega^2 b^3/r'^3_1) + (T/m_1) - \omega^2]y + 2\omega\dot{z}$$

$$x\ddot{z} = -[(\omega^2 b^3/r'^3_1) + (T/m_1) + \omega^2](b+z)x - 2\omega x\,\dot{y}$$

Now subtract the third from the first,

$$(b+z)\ddot{x} - x\ddot{z} = +\omega^2(b+z)x + 2\omega x\,\dot{y}$$

Now I have three equations in three unknowns which are x,y,z

$$y\ddot{x} - x\ddot{y} = -\omega^2 yx - 2\omega\dot{z}x$$

$$(b+z)\ddot{x} - x\ddot{z} = +\omega^2(b+z)x + 2\omega x\,\dot{y}$$

$$x^2+y^2+z^2 = r_1^2$$

Once you solve these, then T is determined by any of the equations. Maybe I can assume b >> z to get

$$y\ddot{x} - x\ddot{y} = -\omega^2 yx - 2\omega\dot{z}x$$

$$b\ddot{x} - x\ddot{z} = +\omega^2 bx + 2\omega x\,\dot{y}$$

$$x^2+y^2+z^2 = r_1^2$$

I will pause this effort right here and work on the gas valve which just arrived.