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F.6 Force analysis of the Dumbbell Satellite in Frame S'

The only true forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F'}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b}+\mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F'}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2\,. \qquad \text{(F.6.1)}$$

As noted earlier, the center of gravity is not quite at the center of mass in Fig (F.1.1), but the above equations are exact despite this fact. The force are primed because they are forces in the inertial Frame S'. The reader is reminded that we are using the "swap notation" where prime↔noprime relative to the non-swap notation.

In order to use Newton's Law in rotating Frame S, we must include the fictitious forces. We translate the result of (8.1.8) to swap notation to obtain

$$\mathbf{F}_{fict,1} \approx -m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$
$$\mathbf{F}_{fict,2} \approx -m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2 \qquad \text{(F.6.2)}$$
frame centrifugal Coriolis Euler

In these equations, the **b** acceleration is given by the swap version of (7.13) which then states

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}). \qquad \text{// Special Case \#1} \qquad \text{(F.6.3)}$$

and the vectors **b** and **ω** are given by

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$$
$$\mathbf{b} = b\hat{\mathbf{z}}\,. \qquad \text{(F.6.4)}$$

Finally we may state Newton's Law for each mass,

$$\mathbf{F}_{eff,1} = m_1\,\mathbf{a}_1 \qquad \text{(F.6.5)}$$
$$\approx -(GM_E m_1/r'^3_1)\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$

$$\mathbf{F}_{eff,2} = m_2\mathbf{a}_2 \qquad \text{(F.6.6)}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 - m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2$$

where we have now a set of six scalar equations. Using (F.1.9) through (F.1.11), (F.6.6) can be rewritten,

$$\mathbf{F}_{\mathbf{eff},2} = -m_1\mathbf{a}_1 \qquad \text{(F.6.7)}$$

$$\approx -(GM_E m_2/r'^3_2)\mathbf{r}'_2 + T\,\hat{\mathbf{r}}_1 - m_2\ddot{\mathbf{b}}_{S'} + m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$

Adding (F.6.5) and (F.6.7) gives

$$0 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 - (GM_E m_2/r'^3_2)\mathbf{r}'_2 - (m_1+m_2)\ddot{\mathbf{b}}_{S'} \qquad \text{(F.6.8)}$$

This equation is just $\mathbf{F} = m\mathbf{a}$ in inertial Frame S' for the total satellite where **b** is the center of mass. Ignoring the small offset between center of mass and center of gravity, the three equations (F.6.8) describe the circular orbit of the satellite around the Earth. We may then regard the equation (F.6.5) as a set of three scalar equations for the three unknowns θ,φ and T where recall $\mathbf{r}_1 = (r_1,\theta,\varphi)$ in the spherical coordinates of Fig (F.1.1).

Our next task is to write vector equation (F.6.5) in spherical coordinates to obtain the three equations of motion. After expanding the left side, we then consider the right side of (F.6.5) one term at a time:

Left side of (F.6.5): $m_1\mathbf{a}_1 = m_1(a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}})$ (E.3.6)

$$= m_1 r_1[(-\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta)\,\hat{\mathbf{r}}_1 + (\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta)\,\hat{\boldsymbol{\theta}} + (2\,\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\varphi}}] \qquad \text{(F.6.9)}$$

Term 1: $-(GM_E m_1/r'^3_1)\mathbf{r}'_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1)$ // (F.1.8)

$$= -(GM_E m_1/r'^3_1)(b\hat{\mathbf{z}} + r_1\hat{\mathbf{r}_1}) \qquad \text{// (F.6.4)}$$

$$= -(GM_E m_1/r'^3_1)(b\cos\theta\,\hat{\mathbf{r}}_1 - b\sin\theta\,\hat{\boldsymbol{\theta}} + r_1\hat{\mathbf{r}_1}) \qquad \text{// (E.2.7)}$$

$$= -(GM_E m_1/r'^3_1)[(b\cos\theta + r_1)\,\hat{\mathbf{r}}_1 - b\sin\theta\,\hat{\boldsymbol{\theta}}] \qquad \text{(F.6.10)}$$

Term 2: $-T\,\hat{\mathbf{r}}_1$ as is (F.6.11)

Term 3: $-m_1\ddot{\mathbf{b}}_{S'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // (F.6.3)

$$= -m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{// satellite in circular orbit, } \dot{\boldsymbol{\omega}} = 0$$

$$= -m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b} \qquad \text{//- } \mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = -(\mathbf{A}\bullet\mathbf{C})\mathbf{A} + A^2\mathbf{C}$$

$$= m_1\omega^2\mathbf{b} = m_1\omega^2 b\hat{\mathbf{z}} \qquad \text{// (F.6.4)}$$

$$= m_1\omega^2 b\,[\cos\theta\,\hat{\mathbf{r}}_1 - \sin\theta\,\hat{\boldsymbol{\theta}}] \qquad \text{// (E.2.7)} \qquad \text{(F.6.12)}$$

Term 4: $-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1$ // identity shown above

$= -m_1\omega^2 r_1(\hat{\mathbf{x}}\bullet\hat{\mathbf{r}}_1)\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (F.6.4)

$= -m_1\omega^2 r_1 \sin\theta\cos\varphi\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (E.2.4))

$= -m_1\omega^2 r_1 \sin\theta\cos\varphi[\sin\theta\cos\varphi\,\hat{\mathbf{r}}_1 + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (E.2.7))

$= -m_1\omega^2 r_1 [\,(\sin^2\theta\cos^2\varphi - 1)\,\hat{\mathbf{r}}_1 + (\sin\theta\cos\theta\cos^2\varphi)\,\hat{\boldsymbol{\theta}} + (-\sin\theta\cos\varphi\sin\varphi)\,\hat{\boldsymbol{\varphi}}$ (F.6.13)

Term 5: $-2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 = -2m_1\,[\omega\hat{\mathbf{x}}] \times (v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}})$ // (E.3.5)

$= -2m_1\omega\,[v_\theta\,\hat{\mathbf{x}} \times \hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\mathbf{x}} \times \hat{\boldsymbol{\varphi}}\,]$

$= -2m_1\omega\,[v_\theta\,(\sin\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} + \sin\varphi\,\hat{\mathbf{r}}_1) + v_\varphi(-\sin\theta\cos\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\varphi\,\hat{\mathbf{r}}_1)]$ // (E.2.13)

$= -2m_1\omega\,[\,(v_\theta\sin\varphi + v_\varphi\cos\theta\cos\varphi)\hat{\mathbf{r}}_1 + (-v_\varphi\sin\theta\cos\varphi)\,\hat{\boldsymbol{\theta}} + (v_\theta\sin\theta\cos\varphi)\,\hat{\boldsymbol{\varphi}}\,]$

$= -2m_1\omega r_1\,[\,(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)\hat{\mathbf{r}}_1 + (-\dot{\varphi}\sin^2\theta\cos\varphi)\,\hat{\boldsymbol{\theta}} + (\dot{\theta}\sin\theta\cos\varphi)\,\hat{\boldsymbol{\varphi}}\,]$ // (E.3.2)
(F.6.14)

Term 6: $-m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1 = 0$ because we assume $\dot{\boldsymbol{\omega}} = 0$ (F.6.15)

Having all the bits and pieces, we now assemble the three component equations of (F.6.5). The numbers show the Term above associated with each piece:

$\mathbf{F}_{\mathbf{eff},1} = m_1\,\mathbf{a}_1$ *(F.6.5)*

$\approx \underset{1}{-(GM_E m_1/r'^3_1)\mathbf{r}'_1} \underset{2}{- T\,\hat{\mathbf{r}}_1} \underset{3}{- m_1\ddot{\mathbf{b}}_{S'}} \underset{4}{- m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)} \underset{5}{- 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1} \underset{6}{- m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1}$

$\hat{\mathbf{r}}_1$: $m_1 r_1(-\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta) = \underset{1}{-(GM_E m_1/r'^3_1)(b\cos\theta + r_1)} \underset{2}{- T} + \underset{3}{m_1\omega^2 b\cos\theta} \underset{4}{- m_1\omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)}$

$\underset{5}{- 2m_1\omega r_1(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)}$ (F.6.16)ok

$\hat{\boldsymbol{\theta}}$: $m_1 r_1(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta) = \underset{1}{+(GM_E m_1/r'^3_1)\,b\sin\theta} \underset{3}{- m_1\omega^2 b\sin\theta}$

$\underset{4}{- m_1\omega^2 r_1\sin\theta\cos\theta\cos^2\varphi} \underset{5}{+ 2m_1\omega r_1\dot{\varphi}\sin^2\theta\cos\varphi}$ (F.6.17)ok

$\hat{\boldsymbol{\varphi}}$: $m_1 r_1(2\,\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta) = \underset{4}{+ m_1\omega^2 r_1\sin\theta\cos\varphi\sin\varphi} \underset{5}{- 2m_1\omega r_1(\dot{\theta}\sin\theta\cos\varphi)}$ (F.6.18)ok

We now rewrite the three equations dividing by m_1 and using (F.5.1) that $GM_E = \omega^2 b^3$:

$\hat{\mathbf{r}}_1$: $r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = - (\omega^2 b^3/r'_1{}^3)(b\cos\theta + r_1) - T/m_1 + \omega^2 b \cos\theta - \omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)$

$- 2\omega r_1 (\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$ (F.6.19)ok

$\hat{\boldsymbol{\theta}}$: $r_1 (\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta) = + (\omega^2 b^3/r'_1{}^3) b \sin\theta - \omega^2 b\sin\theta$

$- \omega^2 r_1 \sin\theta\cos\theta\cos^2\varphi + 2\omega r_1 \dot{\varphi} \sin^2\theta\cos\varphi$ (F.6.20)ok

$\hat{\boldsymbol{\varphi}}$: $(2 \dot{\theta} \dot{\varphi} \cos\theta + \ddot{\varphi}\sin\theta) = + \omega^2\sin\theta\cos\varphi\sin\varphi - 2\omega(\dot{\theta} \sin\theta\cos\varphi)$ (F.6.21)ok

If one uses (F.1.4) that $r'_1{}^2 = b^2 + r_1{}^2 + 2br_1\cos\theta$ in (F.6.20), the pair of equations (F.6.20) and (F.6.21) can in theory be solved for $\theta(t)$ and $\varphi(t)$, given appropriate initial conditions. The solutions can then be inserted into (F.6.19) to obtain a result for the stick tension T(t).

We can rewrite(F.6.21) as

$\hat{\boldsymbol{\varphi}}$: $\sin\theta \ddot{\varphi} + 2 \dot{\theta} \dot{\varphi} \cos\theta - \omega\sin\theta\cos\varphi (\omega\sin\varphi - 2\dot{\theta})$ (F.6.23)ok

which exactly matches the $\hat{\boldsymbol{\theta}}$ torque equation (F.5.7),

$\sin\theta \ddot{\varphi} + 2\dot{\theta} \dot{\varphi} \cos\theta - \omega\sin\theta\cos\varphi (\omega\sin\varphi - 2\dot{\theta}) = 0$ // $\hat{\boldsymbol{\theta}}$ *(F.5.7)*

Next, the $\hat{\boldsymbol{\theta}}$ equation (F.6.20) may be rewritten

$r_1 (\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta) = + (\omega^2 b^3/r'_1{}^3) b \sin\theta - \omega^2 b\sin\theta - \omega^2 r_1 \sin\theta\cos\theta\cos^2\varphi + 2\omega r_1 \dot{\varphi} \sin^2\theta\cos\varphi$

or

$r_1 (\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta) = +\omega^2 b\sin\theta [(b/r'_1)^3 - 1] - \omega^2 r_1 \sin\theta\cos\theta\cos^2\varphi + 2\omega r_1 \dot{\varphi} \sin^2\theta\cos\varphi$

or

$\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta = +\omega^2 b\sin\theta [(b/r'_1)^3 - 1]/r_1 - \omega^2\sin\theta\cos\theta\cos^2\varphi + 2\omega\dot{\varphi} \sin^2\theta\cos\varphi$. (F.6.24)

Now for the first time we assume the far approximation where $r'_1, b >> r$. Recall that

$r'_1{}^2 = b^2 + r_1{}^2 + 2br_1\cos\theta$

$(r'_1/b)^2 = 1 + (r_1/b)^2 + 2(r_1/b)\cos\theta$

$(r'_1/b)^3 = [1 + (r_1/b)^2 + 2(r_1/b)\cos\theta]^{3/2}$

$(b/r'_1)^3 = [1 + (r_1/b)^2 + 2(r_1/b)\cos\theta]^{-3/2} \approx 1 + (-3/2) [(r_1/b)^2 + 2(r_1/b)\cos\theta]$

$$\approx 1 + (-3/2)\, 2(r_1/b)\cos\theta = 1 - 3(r_1/b)\cos\theta$$

so

$$[(b/r'_1)^3 - 1] \approx - 3(r_1/b)\cos\theta \ . \qquad \text{(F.6.25)}$$

Then (F.6.24) becomes

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta = +\omega^2 b\sin\theta\,[- 3(r_1/b)\cos\theta]/r_1 - \omega^2\sin\theta\cos\theta\cos^2\varphi + 2\omega\dot{\varphi}\sin^2\theta\cos\varphi$$

or

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta = -3\omega^2\sin\theta\cos\theta - \omega^2\sin\theta\cos\theta\cos^2\varphi + 2\omega\dot{\varphi}\sin^2\theta\cos\varphi$$

or

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta + 3\omega^2\sin\theta\cos\theta + \omega^2\sin\theta\cos\theta\cos^2\varphi - 2\omega\dot{\varphi}\sin^2\theta\cos\varphi = 0$$

or

$$\ddot{\theta} + 3\omega^2\sin\theta\cos\theta - \dot{\varphi}^2 \sin\theta\cos\theta + \omega\sin\theta\cos\varphi(\omega\cos\theta\cos\varphi - 2\dot{\varphi}\sin\theta) = 0 \qquad \text{(F.6.26)}$$

which exactly matches the $\hat{\boldsymbol{\varphi}}$ torque equation (F.5.7),

$$\ddot{\theta} + 3\omega^2\sin\theta\cos\theta - \dot{\varphi}^2 \sin\theta\cos\theta + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \quad // \hat{\boldsymbol{\varphi}} \qquad \textit{(F.5.7)}$$

Finally we come to the $\hat{\mathbf{r}}_1$ equation which we divide by b:

$$\hat{\mathbf{r}}_1\text{:}\ [r_1/b](- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = - (\omega^2 b^3/r'^3_1)(\cos\theta + [r_1/b]) - T/(m_1 b) + \omega^2\cos\theta - \omega^2[r_1/b](\sin^2\theta\cos^2\varphi - 1)$$
$$- 2\omega[r_1/b]\,(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi) \qquad \text{(F.6.27)}$$

If we now drop terms over order $[r_1/b]$ in the far approximation, we then have

$$\hat{\mathbf{r}}_1\text{:} \qquad 0 \approx - (\omega^2 b^3/r'^3_1)(\cos\theta) - T/(m_1 b) + \omega^2\cos\theta \qquad \text{(F.6.28)}$$

From (F.6.25) one has

$$(b/r'_1)^3 \approx 1 - 3(r_1/b)\cos\theta$$

so (F.6.28) becomes

$$0 \approx - (\omega^2[1 - 3(r_1/b)\cos\theta])(\cos\theta) - T/(m_1 b) + \omega^2\cos\theta$$

$$= -\omega^2\cos\theta + 3\,\omega^2(r_1/b)\cos^2\theta - T/(m_1 b) + \omega^2\cos\theta$$

$$= 3\,\omega^2(r_1/b)\cos^2\theta - T/(m_1 b) \ .$$

Thus, in the far approximation, the tension in the stick is given by

$$T = 3m_1 \omega^2 r_1 \cos^2\theta \qquad \text{(F.6.29)}$$

Since this is non-negative for all angles θ, we may conclude that the stick may be replaced with a tether in the dumbbell satellite and the tether will never be under compression.