Now use the orbit equation (8.6.7) applied to the current situation,

$GM_E = \omega^2 b^3$ large b means small ω

Then we have

$\mathbf{N'}^{(b)} = (\omega^2 b^4 m_1 r \sin\theta)\,(1/r'^3 - 1/r'_2{}^3)\,\hat{\boldsymbol{\varphi}}$

$\mathbf{N}^{(0)}{}_{fict} = -(m_1/m_2) M\omega r^2 \sin\theta\cos\varphi\,[\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}}\,]$

What is the relative size of these terms?

Maybe write this way:

$\mathbf{N}^{(b)} = \omega^2 b^4 m_1 r'_1 \sin\theta'\,[\,(1/r_1{}^3) - (1/r_2{}^3)]\,\hat{\boldsymbol{\varphi}}'$ $T^{-2}L^5 M/L^3 = ML^2/T^2$ ok

$\mathbf{N'}^{(0)}{}_{fict} = -(m_1/\mu_2)\omega^2 r'_1{}^2 \sin\theta'\cos\varphi'\,[\,(\sin\varphi' - 2(\dot{\theta}'/\omega))\,\hat{\boldsymbol{\theta}}' + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')\,\hat{\boldsymbol{\varphi}}']$

Magnitudes?

$\mathbf{N}^{(b)\,2} = (\omega^2 b^4 m_1 r'_1 \sin\theta')^2\,[\,(1/r_1{}^3) - (1/r_2{}^3)]^2$

$\mathbf{N'}^{(0)}{}_{fict}{}^2 = [(m_1/\mu_2)\omega^2 r'_1{}^2 \sin\theta'\cos\varphi']^2\,\{\,(\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')^2\,\}$

The ratio is then

$\mathbf{N'}^{(0)}{}_{fict} / \mathbf{N}^{(b)}$

$$= \frac{(m_1/\mu_2)\omega^2 r'_1{}^2 \sin\theta'\cos\varphi'}{\omega^2 b^4 m_1 r'_1 \sin\theta'} \sqrt{(\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')^2}\ /\ [(1/r_1{}^3) - (1/r_2{}^3)]$$

$$= \frac{(1/\mu_2) r'_1 \cos\varphi'}{b^4} \sqrt{(\sin\varphi' - 2(\dot{\theta}'/\omega))^2 + (\cos\varphi'\cos\theta' - 2(\dot{\varphi}'/\omega)\sin\theta')^2}\ /\ [(1/r_1{}^3) - (1/r_2{}^3)]$$

So two glaring issues: dealing with the difference in the denom, the with ω ratios in the num. So I don't really have much to say yet about the relative size of these terms.

The Equations of Motion

Start with

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2)\, M\, r'^2_1 \left[(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta')\, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta')\, \hat{\boldsymbol{\theta}}' \right]$$

$$\mathbf{N}^{(b)} = \omega^2 b^4 m_1 r'_1 \sin\theta' \left[(1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \left[(1/r_1^3) - (1/r_2^3) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx \omega^2 b^4 m_1 r'_1 \sin\theta' \left[-3r'_1 \cos\theta' / (\mu_2 b^4) \right] \hat{\boldsymbol{\varphi}}'$$

$$\approx -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta'\, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/m_2) M \omega r'^2_1 \sin\theta' \cos\varphi' \left[(\omega \sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta')\, \hat{\boldsymbol{\varphi}}' \right]$$

Is there some limit where one or the other torque dominates?

Write again as

$$\mathbf{N}^{(b)} \approx -3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta'\, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -(m_1/\mu_2) \omega r'^2_1 \sin\theta' \cos\varphi' \left[(\omega \sin\varphi' - 2\dot{\theta}')\, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta')\, \hat{\boldsymbol{\varphi}}' \right]$$

The magnitude ratio factor is

$$\left[-3\omega^2 (m_1/\mu_2) r'^2_1 \sin\theta' \cos\theta' \,/\, -(m_1/\mu_2) \omega r'^2_1 \sin\theta' \cos\varphi' \right]$$

$$= \left[3\omega \cos\theta' / \cos\varphi' \right]$$

and then

$$N_{\mathtt{fict}}/N_{\mathtt{true}} = (\cos\varphi'/3\omega\cos\theta') \sqrt{ (\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega \cos\varphi'\cos\theta' - 2\dot{\varphi}' \sin\theta')^2 }$$

$$= (\cos\varphi'/3\cos\theta') \sqrt{ (\sin\varphi' - 2[\dot{\theta}'/\omega])^2 + (\cos\varphi'\cos\theta' - 2[\dot{\varphi}'/\omega] \sin\theta')^2 }$$

I want an argument that makes this small. Only if $\varphi' = \pi/2$ and it is planar is this small!

Case 1: vertically aligned. Then $\cos\theta' = 1$ and $\sin\theta' = 0$ and the above gives

$$N_{\mathtt{fict}}/N_{\mathtt{true}} = (\cos\varphi'/3\omega) \sqrt{ (\omega\sin\varphi' - 2\dot{\theta}')^2 + (\omega \cos\varphi')^2 }$$

Not sure what that means.

Limit #1: Let $b \to \infty$ so that $\omega \to 0$. Then both torques vanish and the result is no torque at all.

But in the limit not quite there we might say

$$\mathbf{N}^{(b)} \approx -3\omega^2(m_1/\mu_2)r'_1{}^2\sin\theta'\cos\theta'\,\hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{fict} = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,\left[\left(-2\dot{\theta}'\right)\hat{\boldsymbol{\theta}}' + \left(-2\dot{\varphi}'\sin\theta'\right)\hat{\boldsymbol{\varphi}}'\right]$$

and then the fictional force dominates! We then have in the $\omega \to 0$ limit,

$$\dot{\mathbf{L}}'^{(0)} = (m_1/\mu_2)\, r'_1{}^2\,\left[\left(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta'\right)\hat{\boldsymbol{\varphi}}' - \left(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'\right)\hat{\boldsymbol{\theta}}'\right]$$

$$\mathbf{N}'_{tot} \approx -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,\left[\left(-2\dot{\theta}'\right)\hat{\boldsymbol{\theta}}' + \left(-2\dot{\varphi}'\sin\theta'\right)\hat{\boldsymbol{\varphi}}'\right]$$

This gives in this limit two equations of motion

$$(m_1/\mu_2)\, r'_1{}^2\,\left[\left(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta'\right)\right] = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,\left(-2\dot{\varphi}'\sin\theta'\right) \qquad \hat{\boldsymbol{\varphi}}'$$

$$-(m_1/\mu_2)\, r'_1{}^2\,\left(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'\right) = -(m_1/\mu_2)\omega r'_1{}^2\sin\theta'\cos\varphi'\,\left(-2\dot{\theta}'\right) \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\left[\left(\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta'\right)\right] = -\omega\sin\theta'\cos\varphi'\,\left(-2\dot{\varphi}'\sin\theta'\right) \qquad \hat{\boldsymbol{\varphi}}'$$

$$-\left(2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'\right) = -\omega\sin\theta'\cos\varphi'\,\left(-2\dot{\theta}'\right) \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta' = +2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' \qquad \hat{\boldsymbol{\varphi}}'$$

$$2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta' = -\omega\sin\theta'\cos\varphi'\,\left(2\dot{\theta}'\right) \qquad \hat{\boldsymbol{\theta}}'$$

or

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 - 2\omega\sin^2\theta'\cos\varphi'\,\dot{\varphi}' = 0$$

$$\sin\theta'\,\ddot{\varphi}' + 2\cos\theta'\dot{\theta}'\dot{\varphi}' + 2\,\omega\sin\theta'\cos\varphi'\dot{\theta}' = 0$$

They are very ugly, but *for the very first time ever* I have something I can write down!! The equations are horribly coupled. Maybe in my limit I can ignore the last terms as $\omega \to 0$, huge high orbit. Then

$$\ddot{\theta}' - \sin\theta'\cos\theta'\,\dot{\varphi}'^2 = 0$$

$$\ddot{\varphi}' + 2\cot\theta'\dot{\theta}'\dot{\varphi}' = 0$$

but these equations are inscrutable. Suppose instead I assume close to vertical. Then

$$\ddot{\theta}' - \sin\theta' \cos\theta' \dot{\varphi}'^2 - 2\omega \sin^2\theta' \cos\varphi' \dot{\varphi}' = 0$$
$$\sin\theta' \ddot{\varphi}' + 2\cos\theta' \dot{\theta}' \dot{\varphi}' + 2\omega \sin\theta' \cos\varphi' \dot{\theta}' = 0$$

$$\ddot{\theta}' - \theta' \dot{\varphi}'^2 - 2\omega\theta'^2 \cos\varphi' \dot{\varphi}' = 0$$
$$\theta' \ddot{\varphi}' + 2\dot{\theta}' \dot{\varphi}' + 2\omega\theta' \cos\varphi' \dot{\theta}' = 0$$

Suppose we try ansatz that φ = constant. Then we get

$$\ddot{\theta}' = 0$$
$$2\omega\theta' \cos\varphi' \dot{\theta}' = 0$$

It then just sits there with $\dot{\theta}' = \ddot{\theta}' = 0$, not very exciting.

Let's instead try the large-ω limit and see what happens there. Start over with

$$\mathbf{N}^{(b)} \approx -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \, \hat{\boldsymbol{\varphi}}'$$

$$\mathbf{N}'^{(0)}_{\texttt{fict}} = -(m_1/\mu_2)\omega r'_1{}^2 \sin\theta' \cos\varphi' \left[(\omega \sin\varphi' - 2\dot{\theta}') \, \hat{\boldsymbol{\theta}}' + (\omega \cos\varphi' \cos\theta' - 2\dot{\varphi}' \sin\theta') \, \hat{\boldsymbol{\varphi}}' \right]$$

Now just assume $\mathbf{N}^{(b)}$ dominates because it has ω^2 sitting there. Then we get

$$\dot{\mathbf{L}}'^{(0)} = (m_1/m_2) M r'_1{}^2 \left[(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') \, \hat{\boldsymbol{\varphi}}' - (2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta') \, \hat{\boldsymbol{\theta}}' \right]$$

$$\mathbf{N} \approx -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \, \hat{\boldsymbol{\varphi}}'$$

This says that we don't change the ang mom about the $\hat{\boldsymbol{\theta}}'$ which is the "dipping" axis.

The two equations of motion now are

$$(m_1/m_2) M r'_1{}^2 \left[(\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') = -3\omega^2 (m_1/\mu_2) r'_1{}^2 \sin\theta' \cos\theta' \right.$$
$$\left. (2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta') \right] = 0$$

or

$$\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta') = -3\omega^2 \sin\theta' \cos\theta'$$
$$2\dot{\theta}' \dot{\varphi}' \cos\theta' + \ddot{\varphi}' \sin\theta' = 0$$

Now do ansatz where φ = constant. Then 2nd equation is happy and first says

$$\ddot{\theta}' = -3\omega^2 \sin\theta' \cos\theta'$$

NOW also assume small angle to get

$$\ddot{\theta}' = -3\omega^2 \theta'$$

or

$$\ddot{\theta}' + 3\omega^2 \theta' = 0$$

****

D.10 Force analysis of the Dumbbell Satellite in Frame S'

A. COMPUTE TRUE FORCES

The only forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r}'_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b} + \mathbf{r}_2) - T\,\hat{\mathbf{r}}_2$$

Given that b is the distance to CMS, the above equations are exact no matter where COG lies.

B. COMPUTE FICTITIOUS FORCES

We must now compute the fictitious force on each mass seen in Frame S:

$$\mathbf{F}_{fict,1} \approx -m_1 \ddot{\mathbf{b}}_{S'} - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\, \boldsymbol{\omega} \times \dot{\mathbf{r}}_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$
$$\mathbf{F}_{fict,2} \approx -m_2 \ddot{\mathbf{b}}_{S'} - m_2 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) - 2m_2\, \boldsymbol{\omega} \times \dot{\mathbf{r}}_2 - m_2 \dot{\boldsymbol{\omega}} \times \mathbf{r}_2$$

What is the meaning of **ω** here? I have assumed that the CMS is in orbit! So the above equations are not exactly true.

C. TOTAL FORCES and EQUATIONS OF MOTION

We then know that,

$$\mathbf{F}_{eff,1} = m_1 \ddot{\mathbf{r}}_1$$
$$\approx -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1) - T\,\hat{\mathbf{r}}_1 - m_1 \ddot{\mathbf{b}}_{S'} - m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\, \boldsymbol{\omega} \times \dot{\mathbf{r}}_1 - m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$

$$\mathbf{F}_{\mathbf{eff},2} = m_2\ddot{\mathbf{r}}_2 = -m_1\ddot{\mathbf{r}}_1$$

$$\approx -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2 - m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\dot{\mathbf{r}}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2$$

We also know that

$$\mathbf{b} = b\hat{\mathbf{b}} = b\hat{\mathbf{z}} = b\hat{\mathbf{z}}' \qquad // = b\cos\theta\,\hat{\mathbf{r}}_1 - b\sin\theta\,\hat{\boldsymbol{\theta}}$$

$$r'^2_1 = b^2 + r_1^2 + 2br_1\cos\theta \qquad \mathbf{r}_2 = -(m_1/m_2)\mathbf{r}_1$$
$$r'^2_2 = b^2 + r_2^2 - 2br_2\cos\theta$$

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}}\times\mathbf{b} - \omega^2 b\,\hat{\mathbf{z}}' \qquad GM_E = \omega^2 b^3$$

ok to here

Now since our setup is a Special Case #1 problem, we can write

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}) = \dot{\boldsymbol{\omega}}\times\mathbf{b} + (\boldsymbol{\omega}\bullet\mathbf{b})\omega\,\hat{\mathbf{x}}' - \omega^2 b\,\hat{\mathbf{b}}$$

But $(\boldsymbol{\omega}\bullet\mathbf{b}) = 0$ from Fig 1, and $\hat{\mathbf{b}} = \hat{\mathbf{z}} = \hat{\mathbf{z}}'$ so

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}}\times\mathbf{b} - \omega^2 b\,\hat{\mathbf{z}}'$$

The following items are givens: $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$, m_1, m_2, b, GM_E

The following items are the unknowns: $\mathbf{r}'_1(t)$, T = 4 unknowns

So we seem to have a system of 6 2nd-order differential equations in 4 unknowns, so the ODE's must not all be linearly independent.

The Sum Equation : What happens if you add the two ODE's (so the right sides cancel out):

$$-(GM_E m_1/r_1^3)(\mathbf{b}+\mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega}\times\dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}'_1$$
$$-(GM_E m_2/r_2^3)(\mathbf{b}+\mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2 - m_2\ddot{\mathbf{b}}_S - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega}\times\dot{\mathbf{r}}'_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}'_2 \approx 0$$

In the second line make the obvious replacements to get $m_2\mathbf{r}'_2 = -m_1\mathbf{r}'_1$

$$-(GM_E m_1/r_1^3)(\mathbf{b}+\mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega}\times\dot{\mathbf{r}}'_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}'_1$$
$$-(GM_E m_2/r_2^3)(\mathbf{b}+\mathbf{r}'_2) + T\,\hat{\mathbf{r}}'_1 - m_2\ddot{\mathbf{b}}_S + m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_1) + 2m_1\,\boldsymbol{\omega}\times\dot{\mathbf{r}}'_1 + m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}'_1 \approx 0$$

or

$$-(GM_E m_1/r_1^3)(\mathbf{b}+\mathbf{r}'_1) - m_1\ddot{\mathbf{b}}_S$$

$$- (GM_E m_2/r_2^3)(\mathbf{b} + \mathbf{r'}_2) - m_2 \ddot{\mathbf{b}}_S \approx 0$$

or

$$- (\omega^2 b^3 m_1/r_1^3)(\mathbf{b} + \mathbf{r'}_1) - (\omega^2 b^3 m_2/r_2^3)(\mathbf{b} + \mathbf{r'}_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S$$

or

$$- (m_1/r_1^3)(\mathbf{b} + \mathbf{r'}_1) - (m_2/r_2^3)(\mathbf{b} + \mathbf{r'}_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S /(\omega^2 b^3)$$

or

$$- (1/r_1^3)(m_1\mathbf{b} + m_1\mathbf{r'}_1) - (1/r_2^3)(m_2\mathbf{b} + m_2 \mathbf{r'}_2) \approx (m_1+m_2)\ddot{\mathbf{b}}_S /(\omega^2 b^3)$$

or

$$(1/r_1^3)(m_1\mathbf{b} + m_1\mathbf{r'}_1) + (1/r_2^3)(m_2\mathbf{b} - m_1\mathbf{r'}_1) \approx -(m_1+m_2)\ddot{\mathbf{b}}_S /(\omega^2 b^3)$$

or

$$\mathbf{b}[(m_1/r_1^3) + (m_2/r_2^3)] + m_1\mathbf{r'}_1 [(1/r_1^3) - (1/r_2^3)] \approx -(m_1+m_2)\ddot{\mathbf{b}}_S /(\omega^2 b^3)$$

where $\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} - \omega^2 b\, \hat{\mathbf{z}}'$ $\qquad$ $\mathbf{b} = b\hat{\mathbf{z}}'$

Suppose $\dot{\boldsymbol{\omega}} = 0$. Then our sum equation becomes

$$b\hat{\mathbf{z}}'[(m_1/r_1^3) + (m_2/r_2^3)] + m_1\mathbf{r'}_1 [(1/r_1^3) - (1/r_2^3)] \approx (m_1+m_2)\omega^2 b\, \hat{\mathbf{z}}' /(\omega^2 b^3)$$

or

$$[b(m_1/r_1^3) + b(m_2/r_2^3)]\hat{\mathbf{z}}' + m_1 r'_1 [(1/r_1^3) - (1/r_2^3)]\hat{\mathbf{r}}'_1 \approx M\, \hat{\mathbf{z}}' /b^2$$

or

$$[b(m_1/r_1^3) + b(m_2/r_2^3) - (M/b^2)]\hat{\mathbf{z}}' + m_1 r'_1 [(1/r_1^3) - (1/r_2^3)]\hat{\mathbf{r}}'_1 \approx 0$$

But $\hat{\mathbf{z}}'$ and $\hat{\mathbf{r}}'_1$ can be regarded as two linearly independent vectors (if $\theta' \neq 0$) so then

$$b(m_1/r_1^3) + b(m_2/r_2^3) \approx (M/b^2)$$

$$(1/r_1^3) \approx (1/r_2^3)$$

Idea: I have assumed that CMS is COG and maybe this assumption is forcing these results? I think that is in fact the case. Putting $r_1 = r_2 = b$ into the first equation gives an identity.

So does this mean that this whole method is dead in the water? All it really says is that we don't get a contradiction from the sum equation, given that it is the sum of approximate equations.

So I think it is OK to move on. The sum equation is approximately valid, so maybe that means that only one of the two vector equations of motion is linearly independent so there are then 3 equations.

Let's restate the equations with $\mathbf{b} = b\hat{\mathbf{z}}'$ and $\ddot{\mathbf{b}}_S = -\omega^2 b\hat{\mathbf{z}}$ both installed : (no Euler term)

$$\mathbf{F'}_{\mathtt{eff},1} = m_1 \ddot{\mathbf{r}}'_1$$

$$\approx -(GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1$$

$$\mathbf{F'}_{\mathtt{eff},2} = m_2\ddot{\mathbf{r}}'_2 = -m_1\ddot{\mathbf{r}}'_1$$

$$\approx -(GM_E m_2/r_2^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_2) - T\,\hat{\mathbf{r}}'_2 + m_2\omega^2 b\hat{\mathbf{z}} - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_2) - 2m_2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_2$$

So we only need to study the first equation:

$$\mathbf{F'}_{\mathtt{eff},1} = m_1 \ddot{\mathbf{r}}'_1$$

$$\approx \{ -(GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}} \} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) - 2m_1\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'_1$$

The following items are givens: $\boldsymbol{\omega}$, b, m_1, m_2, r'_1

The following items are the unknowns: $\hat{\mathbf{r}}'_1(t)$, T = 3 unknowns (so 3 in 3, good).

Consider its first three terms which are in the {...} brackets,

$$-(GM_E m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}}$$

$$-(\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}' + \mathbf{r}'_1) - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}}$$

$$-(\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}') - (\omega^2 b^3 m_1/r_1^3)\,\mathbf{r}'_1 - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}}$$

$$-(\omega^2 b^3 m_1/r_1^3)(b\hat{\mathbf{z}}') - (\omega^2 b^3 r'_1 m_1/r_1^3)\,\hat{\mathbf{r}}'_1 - T\,\hat{\mathbf{r}}'_1 + m_1\omega^2 b\hat{\mathbf{z}}$$

$$[-(\omega^2 b^3 m_1/r_1^3)b + m_1\omega^2 b]\,\hat{\mathbf{z}} + [-(\omega^2 b^3 r'_1 m_1/r_1^3) - T]\,\hat{\mathbf{r}}'_1$$

$$\omega^2 b m_1[-(b^3/r_1^3) + 1]\,\hat{\mathbf{z}} - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T]\,\hat{\mathbf{r}}'_1$$

But we know that

$$\hat{\mathbf{z}}' = \cos\theta'\,\hat{\mathbf{r}}'_1 - \sin\theta'\,\hat{\boldsymbol{\theta}}'$$

so the first three terms are

$$\omega^2 b m_1[-(b^3/r_1^3) + 1]\,[\cos\theta'\,\hat{\mathbf{r}}'_1 - \sin\theta'\,\hat{\boldsymbol{\theta}}'] - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T]\,\hat{\mathbf{r}}'_1$$

$$= \{ \omega^2 b m_1 (-(b^3/r_1^3) + 1)\cos\theta' - [(\omega^2 b^3 r'_1 m_1/r_1^3) + T] \}\,\hat{\mathbf{r}}'_1 + [-\omega^2 b m_1 (-(b^3/r_1^3) + 1)\sin\theta']\,\hat{\boldsymbol{\theta}}'$$

I am wondering whether you should just say $b \approx r_1$ to clear this first term, to our level of approx? Well consider

$$1 - (b^3/r_1^3) = (r_1^3 - b^3)/r_1^3 = (r_1 - b)(r_1^2 + r_1 b + b^2)/r_1^3 \approx (r_1 - b)(3b^2/b^3) = 3(r_1 - b)/b$$

So this factor is linear in $(r_1 - b)$. Now since $r_1^2 = b^2 + r'^2_1 + 2br'_1\cos\theta'$ I have shown on scratch that

$$r_1 - b \approx r'_1\cos\theta'$$

So then I claim that

$$1 - (b^3/r_1^3) \approx 3\, r'_1\cos\theta'/b$$

So rather than dropping the first term, I will use the above approximation.

Let's next to the centrifugal term:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)$$

$$= (\boldsymbol{\omega}\bullet\mathbf{r}'_1)\boldsymbol{\omega} - \omega^2\,\mathbf{r}'_1$$

$$= (\omega\,\hat{\mathbf{x}}'\bullet\mathbf{r}'_1)\omega\,\hat{\mathbf{x}}' - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1(\hat{\mathbf{x}}'\bullet\hat{\mathbf{r}}'_1)\,\hat{\mathbf{x}}' - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1\sin\theta'\cos\varphi'\,\hat{\mathbf{x}}' - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1\sin\theta'\cos\varphi'[\cos\varphi'\sin\theta'\,\hat{\mathbf{r}}'_1 + \cos\varphi'\cos\theta'\,\hat{\boldsymbol{\theta}}' - \sin\varphi'\,\hat{\boldsymbol{\varphi}}'] - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1\sin\theta'\cos\varphi'\cos\varphi'\sin\theta'\,\hat{\mathbf{r}}'_1 + \omega^2 r'_1\sin\theta'\cos\varphi'\cos\varphi'\cos\theta'\,\hat{\boldsymbol{\theta}}' - \omega^2 r'_1\sin\theta'\cos\varphi'\sin\varphi'\,\hat{\boldsymbol{\varphi}}' - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= \omega^2 r'_1\sin^2\theta'\cos^2\varphi'\,\hat{\mathbf{r}}'_1 + \omega^2 r'_1\sin\theta'\cos\theta'\cos^2\varphi'\,\hat{\boldsymbol{\theta}}' - \omega r'^2_1\sin\theta'\cos\varphi'\sin\varphi'\,\hat{\boldsymbol{\varphi}}' - \omega^2 r'_1\,\hat{\mathbf{r}}'_1$$

$$= [\omega^2 r'_1\sin^2\theta'\cos^2\varphi' - \omega^2 r'_1]\,\hat{\mathbf{r}}'_1 + [\omega^2 r'_1\sin\theta'\cos\theta'\cos^2\varphi']\,\hat{\boldsymbol{\theta}}' + [-\omega^2 r'_1\sin\theta'\cos\varphi'\sin\varphi']\,\hat{\boldsymbol{\varphi}}'$$

$$= \omega^2 r'_1\{[\sin^2\theta'\cos^2\varphi' - 1]\,\hat{\mathbf{r}}'_1 + [\sin\theta'\cos\theta'\cos^2\varphi']\,\hat{\boldsymbol{\theta}}' + [-\sin\theta'\cos\varphi'\sin\varphi']\,\hat{\boldsymbol{\varphi}}'\}$$

This certainly is an ugly result. Now move on the Coriolis term factor which is

$$\boldsymbol{\omega} \times \mathbf{v}'_1 = \omega\hat{\mathbf{x}}' \times [v'_{\theta 1}\hat{\boldsymbol{\theta}}' + v'_{\varphi 1}\,\hat{\boldsymbol{\varphi}}']$$

$$= \omega v'_{\theta 1}\,\hat{\mathbf{x}}' \times \hat{\boldsymbol{\theta}}' + \omega\, v'_{\varphi 1}\,\hat{\mathbf{x}}' \times \hat{\boldsymbol{\varphi}}'$$

$$= \omega v'_{\theta 1} [\cos\varphi'\sin\theta' \hat{\boldsymbol{\varphi}}' + \sin\varphi' \hat{\mathbf{r}}'_1] + \omega v'_{\varphi 1} [-\cos\varphi'\sin\theta' \hat{\boldsymbol{\theta}}' + \cos\varphi'\cos\theta' \hat{\mathbf{r}}'_1]$$

$$= \omega[v'_{\theta 1}\sin\varphi' + v'_{\varphi 1} \cos\varphi'\cos\theta'] \hat{\mathbf{r}}'_1 + \omega[- v'_{\varphi 1}\cos\varphi'\sin\theta'] \hat{\boldsymbol{\theta}}' + \omega[v'_{\theta 1}\cos\varphi'\sin\theta'] \hat{\boldsymbol{\varphi}}'$$

where

$$v'_{\theta 1} = \dot{\theta}'$$

$$v'_{\varphi 1} = r'_1 \dot{\varphi}' \sin\theta'$$

so

$$= \omega r'_1[\dot{\theta}'\sin\varphi' + \dot{\varphi}' \sin\theta' \cos\varphi'\cos\theta'] \hat{\mathbf{r}}'_1 + \omega r'_1[- \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + \omega r'_1[\dot{\theta}'\cos\varphi'\sin\theta'] \hat{\boldsymbol{\varphi}}'$$

$$= \omega r'_1\{ [\dot{\theta}'\sin\varphi' + \dot{\varphi}' \sin\theta' \cos\varphi'\cos\theta'] \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + [\dot{\theta}'\cos\varphi'\sin\theta'] \hat{\boldsymbol{\varphi}}'$$

$$= \omega r'_1\{ [\dot{\theta}'\sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + [\dot{\theta}'\sin\theta'\cos\varphi'] \hat{\boldsymbol{\varphi}}'$$

Here are the conclusions so far

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1) =$$

$$= \omega r'_1 \{ [\omega\sin^2\theta'\cos^2\varphi' - \omega] \hat{\mathbf{r}}'_1 + [\omega\sin\theta'\cos\theta'\cos^2\varphi'] \hat{\boldsymbol{\theta}}' + [- \omega\sin\theta'\cos\varphi' \sin\varphi'] \hat{\boldsymbol{\varphi}}'\}$$

$$\boldsymbol{\omega} \times \mathbf{v}'_1 =$$

$$= \omega r'_1\{ [\dot{\theta}'\sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + [\dot{\theta}'\sin\theta'\cos\varphi'] \hat{\boldsymbol{\varphi}}'$$

Then

$$-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_1)$$

$$= -m_1\omega r'_1 \{ [\omega\sin^2\theta'\cos^2\varphi' - \omega] \hat{\mathbf{r}}'_1 + [\omega\sin\theta'\cos\theta'\cos^2\varphi'] \hat{\boldsymbol{\theta}}' + [- \omega\sin\theta'\cos\varphi' \sin\varphi'] \hat{\boldsymbol{\varphi}}'\}$$

$$-2m_1\boldsymbol{\omega} \times \mathbf{v}'_1 =$$

$$= -2m_1\omega r'_1\{ [\dot{\theta}'\sin\varphi' + \dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \hat{\mathbf{r}}'_1 + [- \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + [\dot{\theta}'\sin\theta'\cos\varphi'] \hat{\boldsymbol{\varphi}}'\}$$

The sum of these terms will be

$$\text{sum} = -m_1\omega r'_1\mathbf{Q}$$

where

$$\mathbf{Q} = [\omega\sin^2\theta'\cos^2\varphi' - \omega] \hat{\mathbf{r}}'_1 + [\omega\sin\theta'\cos\theta'\cos^2\varphi'] \hat{\boldsymbol{\theta}}' + [- \omega\sin\theta'\cos\varphi' \sin\varphi'] \hat{\boldsymbol{\varphi}}'$$

$$+ [2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] \hat{\mathbf{r}}'_1 + [- 2 \dot{\varphi}' \sin^2\theta'\cos\varphi'] \hat{\boldsymbol{\theta}}' + [2\dot{\theta}'\sin\theta'\cos\varphi'] \hat{\boldsymbol{\varphi}}'$$

$= [\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\ \hat{\mathbf{r}}'_1$

$+ [\ \omega\sin\theta'\cos\theta'\cos^2\varphi' - 2\ \dot{\varphi}'\ \sin^2\theta'\cos\varphi'\ ]\ \hat{\boldsymbol{\theta}}'$

$+ [\ -\ \omega\sin\theta'\cos\varphi'\ \sin\varphi' + 2\dot{\theta}'\sin\theta'\cos\varphi'\ ]\ \hat{\boldsymbol{\varphi}}'$

$= [\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\ \hat{\mathbf{r}}'_1$

$+ \sin\theta'\cos\varphi'[\ \omega\cos\theta'\cos\varphi' - 2\ \dot{\varphi}'\ \sin\theta'\ ]\ \hat{\boldsymbol{\theta}}'$

$+ \sin\theta'\cos\varphi'\ [\ -\ \omega\sin\varphi'\ + 2\dot{\theta}'\ ]\ \hat{\boldsymbol{\varphi}}'$

Finally we can assemble all the terms:

$\mathbf{F}'_{\mathtt{eff},1} = m_1\ \ddot{\mathbf{r}}'_1$

$= \{\ \omega^2 b m_1\ (\ -\ (b^3/r_1^3) + 1)\ \cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}\ \hat{\mathbf{r}}'_1 + [-\ \omega^2 b m_1\ (\ -(b^3/r_1^3) + 1)\ \sin\theta']\ \hat{\boldsymbol{\theta}}'$

$-m_1\omega r'_1\mathbf{Q}$

Take it one component at a time. First the $\hat{\mathbf{r}}'_1$ component:

$\{\ \omega^2 b m_1\ (\ -\ (b^3/r_1^3) + 1)\ \cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$

$-m_1\omega r'_1\ \ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\}$

$\approx\ \ \{\ \omega^2 b m_1\ \ (3\ r'_1\cos\theta'/b)\ \cos\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$ // approx from above

$-m_1\omega r'_1\ \ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\}$

$\approx\ \ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ -\ [\ (\omega^2 b^3 r'_1 m_1/r_1^3) + T\ ]\ \}$

$-m_1\omega r'_1\ \ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\}$

$\approx\ \ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ -\ [\ (\omega^2 r'_1 m_1) + T\ ]\ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1\ \ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\}$

$\approx\ \ \{\ \omega^2 m_1\ 3\ r'_1\cos^2\theta'\ -\ \omega^2 r'_1 m_1 - T\ ]\ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1\ \ \{[\ \omega\sin^2\theta'\cos^2\varphi' - \omega\ + 2\dot{\theta}'\sin\varphi' + \ 2\dot{\varphi}'\ \sin\theta'\ \cos\theta'\ \cos\varphi'\ ]\}$

$\approx \quad \{ \omega^2 m_1 r'_1(3\cos^2\theta'-1) - T \ ] \ \}$ // set $r_1 \approx b$ in ratio

$-m_1\omega r'_1 \ \{[\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi']\}$

$\approx \quad \omega^2 m_1 r'_1(3\cos^2\theta'-1) - T \ -m_1\omega r'_1 \{[\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi']\}$

This is going to get set equal to $m_1 a'_1$ which has this $\hat{\mathbf{r}}'_1$ component

$m_1 r'_1[- \dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta']$

So here is the equation of motion in the $\hat{\mathbf{r}}'_1$ component:

$- [\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1)] - \omega \{[\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi']\}$

$= - [\dot{\theta}'^2 + \dot{\varphi}'^2 \sin^2\theta']$

or

$\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1) + \omega [\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] - \dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta' = 0$

$\omega^2(3\cos^2\theta'-2) + T/(m_1 r'_1) + \omega [\omega\sin^2\theta'\cos^2\varphi' + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}' \sin\theta' \cos\theta' \cos\varphi'] - \dot{\theta}'^2 - \dot{\varphi}'^2 \sin^2\theta' = 0$

The unknowns here are: $\theta'(t)$, $\varphi'(t)$, $T(t)$

This ODE has the θ' and φ' stuff cross coupled, and it is non-linear because it has $\dot{\theta}'^2$.

Let's get the other two equations at least written down. First here is the $\hat{\boldsymbol{\theta}}'$ component:

$[- \omega^2 b m_1 (-(b^3/r_1^3) + 1) \sin\theta'] \ -m_1\omega r'_1 \sin\theta'\cos\varphi'[\omega\cos\theta'\cos\varphi' - 2 \dot{\varphi}' \sin\theta']$

$= m_1 r'_1 [\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta']$

I will approx the first term as before, leaving then

$- \omega^2 m_1 3 r'_1\cos\theta' \sin\theta' \ -m_1\omega r'_1 \sin\theta'\cos\varphi'[\omega\cos\theta'\cos\varphi' - 2 \dot{\varphi}' \sin\theta']$

$= m_1 r'_1 [\ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta']$

$- \omega^2 3 \cos\theta' \sin\theta' \ -\omega \sin\theta'\cos\varphi'[\omega\cos\theta'\cos\varphi' - 2 \dot{\varphi}' \sin\theta'] = \ddot{\theta}' - \dot{\varphi}'^2 \sin\theta' \cos\theta'$

What happens here if θ' is very small?

$$-3\omega^2\,\theta' - \omega\theta'\cos\varphi'\,[\,\omega\cos\varphi' - 2\,\dot{\varphi}'\,\theta'] = \ddot{\theta}' - \dot{\varphi}'^2\,\theta'$$

$$-3\omega^2\,\theta' - [\,\omega\theta'\cos\varphi'\,\omega\cos\varphi' - \omega\theta'\cos\varphi'\,2\,\dot{\varphi}'\,\theta'] = \ddot{\theta}' - \dot{\varphi}'^2\,\theta'$$

$$-3\omega^2\,\theta' - [\,\omega^2\theta'\cos^2\varphi' \quad \text{-order}(\theta')^2] = \ddot{\theta}' - \dot{\varphi}'^2\,\theta'$$

$$-3\omega^2\,\theta' - \omega^2\theta'\cos^2\varphi' = \ddot{\theta}' - \dot{\varphi}'^2\,\theta'$$

$$0 = \ddot{\theta}' - \dot{\varphi}'^2\,\theta' + 3\omega^2\,\theta' + \omega^2\theta'\cos^2\varphi'$$

$$\ddot{\theta}' + [\,3\omega^2 - \dot{\varphi}'^2 + \omega^2\cos^2\varphi'\,]\,\theta' = 0$$

Now make the ansatz that $\varphi' = \pi/2$ so we have a fixed planar situation in paper. Then $\cos\varphi' = 0$ so

$$\ddot{\theta}' + [\,3\omega^2\,]\;\theta' = 0$$

and this agrees with what I got in the torque analysis! But there I had to assume ω large I think. But this basic resulting is still lurking in the works !

What about the $\hat{\boldsymbol{\varphi}}'$ equation just to fill this out. Only **Q** has such a component and we get

$$\mathbf{F}'_{\mathbf{eff},1} \bullet \hat{\boldsymbol{\varphi}}' = -m_1\omega r'_1 \mathbf{Q} \bullet \hat{\boldsymbol{\varphi}}' = -m_1\omega r'_1 \sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi' \;\; 2\dot{\theta}'\,]$$

and the right side is $m_1 a'_{\varphi 1}$ so we get

$$-m_1\omega r'_1 \sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi' + 2\dot{\theta}'\,] = m_1 r'_1\,[\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta']$$

$$-\omega\sin\theta'\cos\varphi'\,[\,-\omega\sin\varphi' + 2\dot{\theta}'\,] = [\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta']$$

$$-\,[\,-\omega\sin\theta'\cos\varphi'\omega\sin\varphi' + \omega\sin\theta'\cos\varphi' 2\dot{\theta}'\,] = [\,2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta']$$

$$\omega^2\sin\theta'\cos\varphi'\sin\varphi' - \omega\sin\theta'\cos\varphi' 2\dot{\theta}' = 2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'$$

$$\ddot{\varphi}'\sin\theta' + 2\,\dot{\theta}'\,\dot{\varphi}'\cos\theta' + \omega\sin\theta'\cos\varphi' 2\dot{\theta}' - \omega^2\sin\theta'\cos\varphi'\sin\varphi' = 0$$

What does this one do for small θ' ?

$$\ddot{\varphi}'\theta' + 2\,\dot{\theta}'\,\dot{\varphi}' + \omega\theta'\cos\varphi' 2\dot{\theta}' - \omega^2\theta'\cos\varphi'\sin\varphi' = 0$$

$$\ddot{\varphi}' + 2(\dot{\theta}'/\theta')\dot{\varphi}' + \omega\cos\varphi' 2\dot{\theta}' - \omega^2\cos\varphi'\sin\varphi' = 0$$

Suppose again that $\varphi' = \pi/2$, then it says

$$\ddot{\varphi}' + 2(\dot{\theta}'/\theta')\dot{\varphi}' = 0$$

and φ' = constant is a fine solution to this equation. So we have a self-consistent solution. Now recall

$$\theta' = A\sin(\omega' t + \alpha) \qquad \dot{\theta}' = A\omega\cos(\omega' t + \alpha) \qquad (\dot{\theta}'/\theta') = \omega\cos(\omega' t + \alpha)/\sin(\omega' t + \alpha)$$

Then

$$\sin(\omega' t + \alpha)\ddot{\varphi}' + 2\omega\cos(\omega' t + \alpha)\dot{\varphi}' = 0$$

which is some strange but well defined first order ODE for $\dot{\varphi}'$.

Summary of resulting Equations of Motion (the order is r,θ,φ)

$$\omega^2(3\cos^2\theta' - 1) + T/(m_1 r'_1) + \omega[\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2\sin^2\theta'$$

$$-\omega^2 3\cos\theta'\sin\theta' - \omega\sin\theta'\cos\varphi'[\omega\cos\theta'\cos\varphi' - 2\dot{\varphi}'\sin\theta'] = \ddot{\theta}' - \dot{\varphi}'^2\sin\theta'\cos\theta'$$

$$\omega^2\sin\theta'\cos\varphi'\sin\varphi' - \omega\sin\theta'\cos\varphi' 2\dot{\theta}' = 2\dot{\theta}'\dot{\varphi}'\cos\theta' + \ddot{\varphi}'\sin\theta'$$

What does the first equation say for small θ' ?

$$\omega^2(3-2) + T/(m_1 r'_1) + \omega[\omega\theta'^2\cos^2\varphi' + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2\theta'$$

$$\omega^2 + T/(m_1 r'_1) + \omega[2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\theta'\cos\varphi'] = \dot{\theta}'^2 + \dot{\varphi}'^2\theta'$$

$$2\omega^2 + T/(m_1 r'_1) + 2\omega\dot{\theta}'\sin\varphi' + 2\omega\dot{\varphi}'\theta'\cos\varphi' = \dot{\theta}'^2 + \dot{\varphi}'^2\theta'$$

$$\omega^2 + T/(m_1 r'_1) + 2\omega\dot{\theta}'\sin\varphi' = \dot{\theta}'^2 + [\dot{\varphi}'^2 - 2\omega\dot{\varphi}'\cos\varphi']\theta'$$

Suppose it is static. Then equation says

$$\omega^2 + T/(m_1 r'_1) = 0$$

$$T = -m_1 r'_1 \omega^2$$ what I get here

This is the first time I have had any estimate for T. But what happened to

$$T = 3\,(mMG/r_0^3)(\Delta r) \quad . \qquad (8.6.9)$$

If I set $\Delta r = r'_1$ for mass m_1 and $r_0 = b$, this says

$$T = 3\,(m_1 r'_1 M_E G/b^3) \quad .$$

But as usual

$$GM_E = \omega^2 b^3$$

so my main text says

$$T = 3\,(m_1 r'_1\,\omega^2) \qquad \text{what I get in the main text}$$

and things are not right, though dimensions are correct.

Start again with the r equation

$$\omega^2(3\cos^2\theta'-1) + T/(m_1 r'_1) + \omega\,[\omega\sin^2\theta'\cos^2\varphi' - \omega + 2\dot{\theta}'\sin\varphi' + 2\dot{\varphi}'\sin\theta'\cos\theta'\cos\varphi'\,] = \dot{\theta}'^2 + \dot{\varphi}'^2\sin^2\theta'$$

Set $\theta' = 0$ and $\dot{\theta}' = 0$ to get

$$2\omega^2 + T/(m_1 r'_1) + \omega\,[\,-\omega\,] = 0$$

$$\omega^2 + T/(m_1 r'_1) = 0$$

Same bad result. I think T has to be positive, so I have a bug in the r equation which of course is the messiest. I will review this tomorrow and hope it repairs itself.