How do you get the tension in the stick? PhL 1.5.17

This is a very important question for two reasons: (1) will the tether stay taut? (2) I want the tension T in the rest vertical position in order to verify my claims in the main text where the factor 2 or 3 is in question.

The only way I know to proceed is in the way I start off Section D.10 which is sitting right now in App F v 6. Keep in mind that I am in swap coordinates where Frame S is rotating.

A. COMPUTE TRUE FORCES

The only forces on a dumbbell mass are gravity and stick tension T. From ** we then write

$$\mathbf{F}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b}+\mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r}'_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2$$

Given that b is the distance to CMS, the above equations are exact no matter where COG lies.

B. COMPUTE FICTITIOUS FORCES

We must now compute the fictitious force on each mass seen in Frame S:

$$\mathbf{F}_{fict,1} \approx -m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1 \qquad \text{// swap}$$
$$\mathbf{F}_{fict,2} \approx -m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2 \qquad \text{// swap}$$

What is the meaning of $\boldsymbol{\omega}$ here? Frame S is rotating about the Frame S' origin, but in non-swap notation we would say that Frame S' is rotating around Frame S origin, so this would be Special Case #1, but I have to translate the results! Here are some translated results from frames doc:

$$\dot{\mathbf{b}}_S \equiv (d\mathbf{b}/dt)_S = 0 \qquad \text{seems OK} \qquad \text{(swap)} \qquad (4.4.2)_s$$

$$\dot{\mathbf{b}}_{S'} \equiv (d\mathbf{b}/dt)_{S'} = \boldsymbol{\omega}\times\mathbf{b}\,. \qquad \text{// Special Case \#1} \qquad \text{(swap)} \qquad (4.4.3)_s$$

I then have this non-swap claims in (7.13),

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b})\,. \qquad \text{// Special Case \#1} \qquad \text{(non-swap)} \qquad (7.13)$$

which I can translate to

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}). \qquad \text{// Special Case \#1} \qquad \text{(swap)} \qquad (7.13)_s$$

If this is correct, then I can restate things above as

$$\mathbf{F}_{fict,1} \approx -m_1[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$

$$\mathbf{F}_{fict,2} \approx -m_2[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) - 2m_2\,\boldsymbol{\omega} \times \mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}} \times \mathbf{r}_2$$

or

$$\mathbf{F}_{fict,1}/m_1 \approx -[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2\,\boldsymbol{\omega} \times \mathbf{v}_1 - \dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$

$$\mathbf{F}_{fict,2}/m_2 \approx -[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) - 2\,\boldsymbol{\omega} \times \mathbf{v}_2 - \dot{\boldsymbol{\omega}} \times \mathbf{r}_2$$

or

$$\mathbf{F}_{fict,1}/m_1 \approx -[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}_1+\mathbf{b}])] - 2\,\boldsymbol{\omega} \times \mathbf{v}_1 - \dot{\boldsymbol{\omega}} \times (\mathbf{r}_1+\mathbf{b})$$

$$\mathbf{F}_{fict,2}/m_2 \approx -[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}_2+\mathbf{b}])] - 2\,\boldsymbol{\omega} \times \mathbf{v}_2 - \dot{\boldsymbol{\omega}} \times (\mathbf{r}_2+\mathbf{b})$$

or

$$\mathbf{F}_{fict,1}/m_1 \approx -[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1)] - 2\,\boldsymbol{\omega} \times \mathbf{v}_1 - \dot{\boldsymbol{\omega}} \times \mathbf{r'}_1$$

$$\mathbf{F}_{fict,2}/m_2 \approx -[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2)] - 2\,\boldsymbol{\omega} \times \mathbf{v}_2 - \dot{\boldsymbol{\omega}} \times \mathbf{r'}_2$$

Now remove the Euler term to get

$$\mathbf{F}_{fict,1}/m_1 \approx -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) - 2\,\boldsymbol{\omega} \times \mathbf{v}_1$$

$$\mathbf{F}_{fict,2}/m_2 \approx -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2) - 2\,\boldsymbol{\omega} \times \mathbf{v}_2$$

Here then are the F = ma equations of motion in Frame S"

$$m_1\mathbf{a}_1 = \{ -(GM_E m_1/r'_1{}^3)\,\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1\} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) - 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1$$

$$m_2\mathbf{a}_2 = \{ -(GM_E m_2/r'_2{}^3)\,\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2\} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2) - 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_2$$

or

$$-m_1\mathbf{a}_1 = (GM_E m_1/r'_1{}^3)\,\mathbf{r'}_1 + T\,\hat{\mathbf{r}}_1 + m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) + 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1$$

$$-m_2\mathbf{a}_2 = (GM_E m_2/r'_2{}^3)\,\mathbf{r'}_2 + T\,\hat{\mathbf{r}}_2 + m_2\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2) + 2m_2\,\boldsymbol{\omega} \times \mathbf{v}_2$$

or

$$-\mathbf{a}_1 = (GM_E/r'_1{}^3)\,\mathbf{r'}_1 + (T/m_1)\,\hat{\mathbf{r}}_1 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) + 2\,\boldsymbol{\omega} \times \mathbf{v}_1$$

$$-\mathbf{a}_2 = (GM_E/r'_2{}^3)\,\mathbf{r'}_2 + (T/m_2)\,\hat{\mathbf{r}}_2 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2) + 2\boldsymbol{\omega} \times \mathbf{v}_2$$

or

$$-\mathbf{a}_1 = (GM_E/r'_1{}^3)\,\mathbf{r'}_1 + (T/m_1)\,\hat{\mathbf{r}}_1 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_1) + 2\,\boldsymbol{\omega} \times \mathbf{v}_1$$

$$-\mathbf{a}_2 = (GM_E/r'_2{}^3)\,\mathbf{r'}_2 - (T/m_2)\,\hat{\mathbf{r}}_1 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}_2) + 2\boldsymbol{\omega} \times \mathbf{v}_2$$

Here $\mathbf{a}_1$ is the acceleration of mass m_1 in non-inertial Frame S! I am hoping that I can use just the a_1 equation and somehow obtain T from it! But I have to compute these cross products, and I am not sure I have done them elsewhere. So put this off to tomorrow! I don't offhand know a simpler way to get T.

Back on Fri 1.6.17.

Go back to an earlier statement of forces

$\mathbf{F_1} = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r_1}) - T\,\hat{\mathbf{r}}_1$ // true

$\mathbf{F}_{fict,1} \approx -m_1[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_1}) - 2\, m_1\boldsymbol{\omega} \times \mathbf{v_1}$ // fictitious

$\mathbf{F}_{eff,1} = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r_1}) - T\,\hat{\mathbf{r}}_1 - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_1}) - 2\, m_1\boldsymbol{\omega} \times \mathbf{v_1}$

$\mathbf{F}_{eff,1} = m_1\mathbf{a_1}$ // Newton's law in rotating frame

I really need to know all of these terms in spherical unit vectors, then I can look at the $\hat{\mathbf{r}}_1$ equation and determine T. So dot Newton's effective law into $\hat{\mathbf{r}}_1$. Here is Newton's Law:

$\mathbf{a_1} = -(GM_E/r'^3_1)(\mathbf{b} + \mathbf{r_1}) - (T/m_1)\,\hat{\mathbf{r}}_1 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_1}) - 2\,\boldsymbol{\omega} \times \mathbf{v_1}$

Dot Newton into $\hat{\mathbf{r}}_1$:

$\mathbf{a_1} \bullet \hat{\mathbf{r}}_1 = -(GM_E/r'^3_1)(\mathbf{b}\bullet\hat{\mathbf{r}}_1 + r_1) - T/m_1$

$-\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - \hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_1})] - 2\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times \mathbf{v_1}]$

We know that

$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ and $\mathbf{b} = b\hat{\mathbf{z}}$

Work from left to right. First

$\mathbf{a_1} \bullet \hat{\mathbf{r}}_1 = [a_{1r}\,\hat{\mathbf{r}}_1 + a_{1\theta}\hat{\boldsymbol{\theta}} + a_{1\varphi}\,\hat{\boldsymbol{\varphi}}] \bullet \hat{\mathbf{r}}_1 = a_{1r} = -r_1\dot{\theta}^2 - r_1\,\dot{\varphi}^2 \sin^2\theta = r_1(\dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta)$

Next:

$\mathbf{b}\bullet\hat{\mathbf{r}}_1 = b\,\hat{\mathbf{z}}\bullet\hat{\mathbf{r}}_1 = b\cos\theta$

Next:

$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C}$

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = (\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} - \omega^2\mathbf{b} = \omega^2 b\,\hat{\mathbf{x}}\bullet\,\hat{\mathbf{z}} - \omega^2 b\hat{\mathbf{z}} = -\omega^2 b\hat{\mathbf{z}}$

$-\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] = -\hat{\mathbf{r}}_1 \bullet [-\omega^2 b\hat{\mathbf{z}}] = \omega^2 b\,\hat{\mathbf{r}}_1 \bullet \hat{\mathbf{z}} = \omega^2 b\cos\theta$

Next:

$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C}$

$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = (\boldsymbol{\omega} \bullet \mathbf{r}_1)\boldsymbol{\omega} - \omega^2 \mathbf{r}_1 = \omega^2 r_1(\hat{\mathbf{x}} \bullet \hat{\mathbf{r}}_1)\hat{\mathbf{x}} - \omega^2 r_1 \hat{\mathbf{r}}_1 = \omega^2 r_1[\sin\theta\cos\varphi\, \hat{\mathbf{x}} - \hat{\mathbf{r}}_1]$

$-\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)] = -\omega^2 r_1[\sin\theta\cos\varphi\, \hat{\mathbf{r}}_1 \bullet \hat{\mathbf{x}} - 1] = -\omega^2 r_1[\sin\theta\cos\varphi\sin\theta\cos\varphi - 1]$

$= -\omega^2 r_1[\sin^2\theta\cos^2\varphi - 1]$

Next:

$\boldsymbol{\omega} \times \mathbf{v}_1 = [\omega\hat{\mathbf{x}}] \times [v_{1\theta}\hat{\boldsymbol{\theta}} + v_{1\varphi}\hat{\boldsymbol{\varphi}}]$

$= \omega v_{1\theta}\hat{\mathbf{x}} \times \hat{\boldsymbol{\theta}} + \omega v_{1\varphi}\hat{\mathbf{x}} \times \hat{\boldsymbol{\varphi}} = \omega v_{1\theta}[\sin\theta\cos\varphi\, \hat{\boldsymbol{\varphi}} + \sin\varphi\, \hat{\mathbf{r}}_1] + \omega v_{1\varphi}[-\sin\theta\cos\varphi\, \hat{\boldsymbol{\theta}} + \cos\theta\cos\varphi\, \hat{\mathbf{r}}_1]$

$= [\omega v_{1\theta}\sin\varphi + \omega v_{1\varphi}\cos\theta\cos\varphi]\hat{\mathbf{r}}_1 + [-\omega v_{1\varphi}\sin\theta\cos\varphi]\hat{\boldsymbol{\theta}} + [\omega v_{1\theta}\sin\theta\cos\varphi]\hat{\boldsymbol{\varphi}}$

Then,

$-2\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times \mathbf{v}_1] = -2[\omega v_{1\theta}\sin\varphi + \omega v_{1\varphi}\cos\theta\cos\varphi] = -2\omega[v_{1\theta}\sin\varphi + v_{1\varphi}\cos\theta\cos\varphi]$

$= -2\omega[r_1\sin\varphi\,\dot{\theta} + r_1\sin\theta\,\dot{\varphi}\cos\theta\cos\varphi] = -2\omega r_1[\sin\varphi\,\dot{\theta} + \sin\theta\cos\theta\cos\varphi\,\dot{\varphi}]$

Now finally put all the pieces together:

$\mathbf{a}_1 \bullet \hat{\mathbf{r}}_1 = -(GM_E/r'^3_1)(\mathbf{b} \bullet \hat{\mathbf{r}}_1 + r_1) - T/m_1$

$-\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - \hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)] - 2\hat{\mathbf{r}}_1 \bullet [\boldsymbol{\omega} \times \mathbf{v}_1]$

says :

$r_1(\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta) = -(GM_E/r'^3_1)(b\cos\theta + r_1) - T/m_1 \qquad // GM_E = \omega^2 b^3$

$+ \omega^2 b\cos\theta - \omega^2 r_1[\sin^2\theta\cos^2\varphi - 1] - 2\omega r_1[\sin\varphi\,\dot{\theta} + \sin\theta\cos\theta\cos\varphi\,\dot{\varphi}]$

Again:

$r_1(\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta) = -(\omega^2 b^3/r'^3_1)(b\cos\theta + r_1) - T/m_1$

$+ \omega^2 b\cos\theta - \omega^2 r_1[\sin^2\theta\cos^2\varphi - 1] - 2\omega r_1[\sin\varphi\,\dot{\theta} + \sin\theta\cos\theta\cos\varphi\,\dot{\varphi}]$

Divide by b :

$(r_1/b)(\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta) = -(\omega^2 b^3/r'^3_1)(\cos\theta + (r_1/b)) - T/(bm_1)$

$$+ \omega^2 \cos\theta - \omega^2(r_1/b)[\sin^2\theta\cos^2\varphi - 1] - 2\omega(r_1/b)[\sin\varphi\,\dot{\theta} + \sin\theta\cos\theta\cos\varphi\,\dot{\varphi}]$$

Maybe I can just drop all terms which are proportional to (r_1/b) . The result would then be

$$0 = -(\omega^2 b^3/r'^3_1)(\cos\theta) - T/(bm_1) + \omega^2\cos\theta$$

or

$$0 = -T/(bm_1) + \omega^2\cos\theta\,[1 - b^3/r'^3_1]$$

or

$$T/(bm_1) = \omega^2\cos\theta\,[1 - b^3/r'^3_1]$$

or

$$T = bm_1\omega^2\cos\theta\,[1 - b^3/r'^3_1]$$

So this is my first rough estimate for stick tension T. I know that

$$r'^2_1 = b^2 + r_1^2 + 2br_1\cos\theta$$

$$(r'_1/b)^2 = 1 + (r_1/b)^2 + 2(r_1/b)\cos\theta$$

$$(r'_1/b)^3 = [1 + (r_1/b)^2 + 2(r_1/b)\cos\theta]^{3/2}$$

$$b^3/r'^3_1 = [1 + (r_1/b)^2 + 2(r_1/b)\cos\theta]^{-3/2}$$

Now in our smallness limit this is roughly

$$b^3/r'^3_1 = [1 + 2(r_1/b)\cos\theta]^{-3/2} \approx 1 + (-3/2)2(r_1/b)\cos\theta \qquad //(1+a)^n = 1 + na + ...$$

$$[1 - b^3/r'^3_1] = 1 - 1 - (-3/2)2(r_1/b)\cos\theta = (3/2)2(r_1/b)\cos\theta = 3(r_1/b)\cos\theta$$

So then I get

$$T \approx bm_1\omega^2\cos\theta\, 3(r_1/b)\cos\theta = m_1\omega^2\cos\theta\, 3r_1\cos\theta = 3m_1r_1\omega^2\cos^2\theta$$

For an aligned dumbbell have $\theta = 0$ so get

$$T \approx 3m_1r_1\omega^2 \qquad // T = ML/T^2 \text{ , rhs} = M\,L/T^2 \quad \text{check}$$

What did I get inside frames doc where the Swedish controversy lives? If I replace

$$GM_E/b^3 = \omega^2$$

the above becomes

$$T \approx 3m_1r_1GM_E/b^3 = 3(m_1M_EG/b^3)r_1$$

which compare to frames doc

$$= 3\,(mMG/r_0^3)(\Delta r) \quad . \tag{8.6.9}$$

I think I have finally recovered what I wanted!! More pondering is needed, but this is very promising.