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Appendix G: Rotation Matrices and Goldstein's body-frame ω

Here we present still more information on rotation matrices and demonstrate some typical manipulations done with such matrices leading to a derivation of Goldstein body-frame **ω** vector expressions.

G.1 Generators and finite rotation matrices

The general active rotation matrix for rotation by angle θ about some **n** *unit vector* axis is given by,

$$R_{\hat{\mathbf{n}}}(\varphi) = \exp(-i\,\theta\,\mathbf{n}\bullet\mathbf{J})\,, \qquad \text{(G.1.1)}$$

where the J_k are 3x3 matrices known as the rotation **generator** matrices,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \qquad \text{(G.1.2)}$$

The numbers in these three matrices can be summarized in this single statement,

$$(J_a)_{bc} = -i\,\varepsilon_{abc} \qquad \text{// for example, } (J_1)_{23} = -i\,\varepsilon_{123} = -i$$

or

$$(iJ_a)_{bc} = \varepsilon_{abc} \qquad \text{(G.1.3)}$$

where the ε_{abc} permutation tensor is described in (1.5.3).

Defining the commutator of two square matrices X,Y as [X, Y] = XY-YX, one can show that the above three matrices satisfy this "commutation relation",

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \qquad \text{// for example, } [J_1,J_2] = iJ_3 \qquad \text{(G.1.4)}$$

Proof: These easily demonstrated facts will be used in the proof (implied sum on repeated indices) ,

$$\varepsilon_{abc}\varepsilon_{ABc} = \delta_{aA}\delta_{bB} - \delta_{aB}\delta_{bA} \qquad \delta_{ab} \equiv \delta_{a,b} \qquad \varepsilon_{abc} = \varepsilon_{bca} = \varepsilon_{cab} \qquad \text{(G.1.5)}$$

$$LHS_{ac} = (J_i)_{ab}(J_j)_{bc} - (J_j)_{ab}(J_i)_{bc} = (-i)^2[\varepsilon_{iab}\varepsilon_{jbc} - \varepsilon_{jab}\varepsilon_{ibc}] = -[\varepsilon_{iab}\varepsilon_{cjb} - \varepsilon_{jab}\varepsilon_{cib}]$$

$$= -[(\delta_{ic}\delta_{aj} - \delta_{ij}\delta_{ac}) - (\delta_{jc}\delta_{ai} - \delta_{ji}\delta_{ac})] = \delta_{jc}\delta_{ai} - \delta_{ic}\delta_{aj} \quad .$$

$$RHS_{ac} = i\varepsilon_{ijk}(J_k)_{ac} = i\varepsilon_{ijk}[-i\varepsilon_{kac}] = \varepsilon_{ijk}\varepsilon_{kac} = \varepsilon_{ijk}\varepsilon_{ack} = \delta_{ia}\delta_{jc} - \delta_{ja}\delta_{ic} \, . \qquad \text{QED}$$

The expression shown in (G.1.1) involves the exponentiation of a square matrix to produce a new square matrix of the same dimension. This notion of exponentiating a matrix is straightforward as we demonstrate with a simple example:

$$R_x(\theta) = \exp(-i\,\theta\,\hat{\mathbf{x}} \bullet \mathbf{J}) = \exp(-i\,\theta J_1) \equiv \Sigma_{n=0}^{\infty} (-i\theta)^n (J_1)^n / n!$$

$$= 1 + \Sigma_{2,4,6..} (-i\theta)^n (J_1)^n / n! + \Sigma_{1,3,5..} (-i\theta)^n (J_1)^n / n!$$

It is easy to show that $(J_1)^n = J_1$ for odd n, while $(J_1)^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for even $n > 0$. Thus,

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Sigma_{2,4,6..} (-i\theta)^n/n! + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \Sigma_{1,3,5..} (-i\theta)^n/n! \, .$$

But

$$\Sigma_{2,4,6..} (-i\theta)^n/n! = -\theta^2/2! + \theta^4/4! + ... = (\cos\theta - 1)$$
$$\Sigma_{1,3,5..} (-i\varphi)^n/n! = (-i\theta) + (-i\theta)^3/3! + ... = (-i)\,[\,\theta - \theta^3/3! + ..] = -i\sin\theta \qquad \text{(G.1.6)}$$

and therefore

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\cos\theta - 1) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} (-i\sin\theta)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \qquad \text{(G.1.7)}$$

The three axis-aligned rotations are found in this manner to be

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \, . \qquad \text{(G.1.8)}$$

The general rotation (G.1.1) applied to a vector **v** produces a **forward** right-hand-rule rotation of vector **v** by an angle θ about an arbitrary $\hat{\mathbf{n}}$ axis to create **v**' = R**v**. We call this an **active rotation**. For example, if

one applies $R_z(\theta)$ shown in (G.1.8) to the unit vector $\hat{\mathbf{x}} = (1,0,0)$ one gets $(\cos\theta,\sin\theta,0)$ which for small θ is a vector in the first quadrant of the x,y plane.

Some authors define rotation matrices which rotate vectors **backwards** according to the right hand rule, with the connection to our matrices then being $\theta \leftrightarrow -\theta$. The motivation for doing this is the fact that, whereas $\mathbf{v'} = R\mathbf{v}$ in the active view where the vector moves and the axes stay put, one can instead do $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$ and have the vector stay put and the axes are back-rotated, which is the **passive** viewpoint. The definition of the rotation matrices is just a convention and (G.1.8) shows our definitions.

One says that the matrices J_i "generate" the finite rotations when they are exponentiated.

An alternative approach: First, one can show that $R_i(d\theta) \approx 1 - id\theta J_i$ describes a 3x3 matrix which, when applied to a vector, causes that vector to rotate by the small amount $d\theta$ about the i axis (an "infinitesimal rotation"). Setting $d\theta = \theta/n$, one shows that $\lim_{n\to\infty} (1-i[\theta/n]J_i)^n = \exp(-i\theta J_i)$. This last limit is analogous to $\lim_{n\to\infty}(1+x/n)^n = e^x$ for a scalar value x.

G.2 About the general rotation matrix

<u>An explicit expression for the general rotation matrix</u>

We present this subsection as a Reader Exercise with a set of steps. The result is stated in (G.2.8).

We are interested in the following general rotation matrix:

$$R_n(\theta) = \exp(-i\,\theta\,\mathbf{n}\bullet\mathbf{J}) \equiv \Sigma_{n=0}^{\infty}(-i\theta)^n(\mathbf{n}\bullet\mathbf{J})^n/n! \quad . \quad \mathbf{n} = \hat{\mathbf{n}} = \text{a unit vector} \tag{G.2.1}$$

(a) using (G.1.3) and (G.1.5), show that $(\mathbf{n}\bullet\mathbf{J})^2 = T$ where $T_{ab} = (\delta_{ab} - n_a n_b)$. (G.2.2)

(b) show that

$$(\mathbf{n}\bullet\mathbf{J})^n = (\mathbf{n}\bullet\mathbf{J})\,T \qquad \text{for } n = 3,5,7,....$$
$$(\mathbf{n}\bullet\mathbf{J})^n = T \qquad \text{for } n = 2,4,6... \tag{G.2.3}$$

(c) show that $\exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = \cos(\theta\mathbf{n}\bullet\mathbf{J}) - i\sin(\theta\mathbf{n}\bullet\mathbf{J})$ where each term is defined by its series

(d) show that

$$\cos(\theta\mathbf{n}\bullet\mathbf{J}) = 1 + T(\cos\theta - 1)$$

$$\sin(\theta\mathbf{n}\bullet\mathbf{J}) = (\theta\mathbf{n}\bullet\mathbf{J}) + (\mathbf{n}\bullet\mathbf{J})T(\sin\theta - \theta) \tag{G.2.4}$$

(e) show therefore that

$$\exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = 1 + T(\cos\theta - 1) - i(\mathbf{n}\bullet\mathbf{J})\{\theta + T(\sin\theta - \theta)\} \tag{G.2.5}$$

(f) show that the terms linear in θ all cancel leaving this simpler result

$$\exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = 1 + (\cos\theta - 1)\,T + \sin\theta[\,-i(\mathbf{n}\bullet\mathbf{J})] \qquad \text{(G.2.6)}$$

(g) show that

$$T = \begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} \qquad -i(\mathbf{n}\bullet\mathbf{J}) = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \qquad \text{(G.2.7)}$$

giving this final result,

$$R_{\mathbf{n}}(\theta) = \exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (\cos\theta - 1)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}. \qquad \text{(G.2.8)}$$

Note that the first two terms are symmetric matrices, while the third is antisymmetric.

(h) verify the special cases shown in (G.1.8) above. These cases have **n** = (1,0,0), (0,1,0, (0,0,1).

Finding **n** and θ

Problem: One is handed a 9-element rotation matrix R and one wants to find **n**,θ so R = exp(-iθ**n**•**J**) .

Solution: We are unaware of a single magic formula that solves this problem, so we use brute force.

One can break R into the sum of two pieces, one symmetric and one antisymmetric by constructing

$$S_{ab} \equiv (R_{ab}+R_{ba})/2 \qquad A_{ab} \equiv (R_{ab}-R_{ba})/2 \qquad R_{ab} = S_{ab} + A_{ab} \qquad R = S + A\,. \qquad \text{(G.2.9)}$$

Looking at (G.2.9) one then has,

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (\cos\theta - 1)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} = \sin\theta\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \qquad \text{(G.2.10)}$$

which can be written as these nine equations

$$A_{12} = -n_3\sin\theta \qquad S_{11} = 1 + (\cos\theta-1)(1-n_1^2) \qquad S_{12} = (\cos\theta-1)(-n_1n_2)$$
$$A_{13} = n_2\sin\theta \qquad S_{22} = 1 + (\cos\theta-1)(1-n_2^2) \qquad S_{13} = (\cos\theta-1)(-n_1n_3)$$

$$A_{23} = -n_1\sin\theta \qquad S_{33} = 1 + (\cos\theta - 1)(1 - n_3^2) \qquad S_{23} = (\cos\theta - 1)(-n_2 n_3) \ . \tag{G.2.11}$$

• If $\theta = 0$, then the original matrix must be R = 1 and then we have no work to do.

• If $\theta = \pi$, the entire A matrix vanishes according to (G.2.9) leaving only S. The equations above are then

$$\begin{aligned} S_{11} &= 1 + (-2)(1 - n_1^2) & S_{12} &= 2n_1n_2 \\ S_{22} &= 1 + (-2)(1 - n_2^2) & S_{13} &= 2n_1n_3 \\ S_{33} &= 1 + (-2)(1 - n_3^2) & S_{23} &= 2n_2n_3 \ . \end{aligned}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$, not all three n_i can vanish. The left column of equations says

$$n_i^2 = 1 - (1 - S_{ii})/2 \qquad i = 1,2,3 \ .$$

Inspect the three n_i^2 and select one n_r^2 which is non-zero. Then select the plus sign to get,

$$n_r = +\sqrt{1 - (1 - S_{rr})/2} \ .$$

To be specific, assume $n_r = n_1$. Then from the right column, $n_2 = S_{12}/(2n_1)$ and $n_2 = S_{13}/(2n_1)$ and we are done. So at least one of the following solutions must be viable:

$$\begin{array}{llll} \theta = \pi & n_1 = \sqrt{1 - (1 - S_{11})/2} & n_2 = S_{12}/(2n_1) & n_3 = S_{13}/(2n_1) \\ \theta = \pi & n_2 = \sqrt{1 - (1 - S_{22})/2} & n_1 = S_{12}/(2n_2) & n_3 = S_{23}/(2n_2) \\ \theta = \pi & n_3 = \sqrt{1 - (1 - S_{33})/2} & n_1 = S_{13}/(2n_3) & n_2 = S_{23}/(2n_3) \end{array} \tag{G.2.12}$$

• If $\theta \neq 0$ or π, then note from (G.2.10) that

$$\begin{aligned} &A_{12}^2 + A_{13}^2 + A_{23}^2 = \sin^2\theta \\ &S_{11} + S_{22} + S_{33} = 1 + 2\cos\theta \ . \end{aligned} \tag{G.2.13}$$

Using the sign ambiguity in the direction of **n**, we can assume that $0 < \theta < \pi$ so $\sin\theta > 0$. Then from (G.2.13) and the left column of (G.2.11) we have this solution,

$$\begin{aligned} &\sin\theta = \sqrt{A_{12}^2 + A_{13}^2 + A_{23}^2} \\ &\cos\theta = [\, S_{11} + S_{22} + S_{33} - 1]/2 \\ &n_1 = -A_{23}/\sin\theta \\ &n_2 = A_{13}/\sin\theta \\ &n_3 = -A_{12}/\sin\theta \end{aligned} \tag{G.2.14}$$

and thus a viable θ and **n** have been found such that $R = \exp(-i\theta \mathbf{n}\bullet\mathbf{J})$.

Example: Let $R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so then $S = \begin{pmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & -\sin\alpha & 0 \\ \sin\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\sin\theta = \sqrt{A_{12}^2 + A_{13}^2 + A_{23}^2} = \sqrt{\sin\alpha^2 + 0 + 0} = \sin\alpha$

$\cos\theta = [\, S_{11} + S_{22} + S_{33} - 1]/2 = [\cos\alpha + \cos\alpha + 1 - 1]/2 = \cos\alpha \quad \Rightarrow \theta = \alpha$

$n_1 = -A_{23}/\sin\theta = 0$

$n_2 = A_{13}/\sin\theta = 0$

$n_3 = -A_{12}/\sin\theta = \sin\alpha/\sin\theta = 1 \quad \Rightarrow \quad \mathbf{n} = (0,0,1)$

Therefore $R = \exp(-i\alpha J_3) = R_z(\alpha)$, as verified in (G.1.8).

G.3 The Baker-Campbell-Hausdorff and Sandwich Formulas

Statement and proof of the Baker-Campbell-Hausdorff (BCH) formula

We now quote a fascinating fact known as the Baker-Campbell-Hausdorff formula involving two square matrices A and B :

$$e^{-A}Be^{A} = B + [B,A]/1! + [[B,A],A]/2! + [[[B,A],A],A]/3! + \ldots\ldots \qquad (G.3.1)$$

Outline of BCH proof: LHS = RHS

(a) Show that LHS $= \Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} (-A)^n B A^m / (n!m!)$.

(b) Set k = n+m to rewrite as LHS $= \Sigma_{k=0}^{\infty}\Sigma_{n=0}^{k} (-A)^n B A^{k-n} / (n![k-n]!)$.

(c) Rewrite again as LHS $= \Sigma_{k=0}^{\infty} T_k/k!$ where $T_k \equiv \Sigma_{n=0}^{k} \binom{k}{n} (-A)^n B A^{k-n}$.

(d) Define $C_0 = B$, $C_1 = [B,A]$, $C_2 = [[B,A],A]$, etc. so that RHS $= \Sigma_{k=0} C_k/k!$.
Note that $[C_k,A] = C_{k+1}$.

The proof LHS = RHS is complete if one can show that $T_k = C_k$.

(e) Show $T_k = C_k$ by induction: show $T_0 = C_0$ and $T_k = C_k \Rightarrow T_{k+1} = C_{k+1}$. QED

Statement and proof of the Sandwich Formula

Now *using* this BCH formula, along with the commutation relation (G.1.4) that $[J_i, J_j] = i\varepsilon_{ijk}J_k$, one can show that

$$\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\ \mathbf{J}\ \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) = \cos\theta\ \mathbf{J} + \sin\theta\ \mathbf{J} \times \mathbf{n} + (1 - \cos\theta)\ \mathbf{n}(\mathbf{n}\bullet\mathbf{J}) . \qquad (G.3.2)$$

This "vector of matrices" notation is just a shorthand for the following equations for k = 1,2,3 :

$$\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\ J_k\ \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) = \cos\theta J_k + \sin\theta[\mathbf{J} \times \mathbf{n}]_k + (1 - \cos\theta)\, n_k\, (\mathbf{n}\bullet\mathbf{J})$$

$$= \cos\theta\, J_k + \sin\theta\, \varepsilon_{kms} n_s J_m + (1 - \cos\theta)\, n_k\, (\mathbf{n}\bullet\mathbf{J}) \ . \qquad \text{(G.3.3)}$$

This is the "sandwich formula" since J_k on the left is sandwiched between two rotations.

Outline of Sandwich proof: LHS = RHS

(a) We will use the BCH formula with $A = i\theta\, \mathbf{n}\bullet\mathbf{J}$ and $B = J_k$. First, define C_k as in (d) above.

(b) show that $C_0 = J_k$, $C_1 = -\theta n_i \varepsilon_{kij} J_j$, $C_2 = \theta^2 (n_k n_i J_i - J_k)$, $C_3 = -\theta^2 C_1$, $C_4 = -\theta^2 C_2$

(c) deduce (or use induction to show) that in general,

$C_n = -(-1)^{n/2}\, \theta^n\, (n_k n_i J_i - J_k)$ n = 2,4,6....

$C_n = -(-1)^{(n-1)/2}\, \theta^n\, n_i \varepsilon_{kij} J_j$ n = 1,3,,5...

At this point we have from the BCH formula,

$\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\ J_k \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) = \Sigma_{n=0}^{\infty} C_n/n!$ with C_n as in (c) above

(d) Show that $\Sigma_{n=0}^{\infty} C_n/n! =$

$= J_k - n_i \varepsilon_{kij} J_j \Sigma_{n=1,3,5..} (-1)^{(n-1)/2}\, \theta^n/n! - (n_k n_i J_i - J_k) \Sigma_{n=2,4,6..} (-1)^{n/2}\, \theta^n/n!$

$= J_k - n_i \varepsilon_{kij} J_j [\theta - \theta^3/3! + ...] - (n_k n_i J_i - J_k) [-\theta^2/2! + \theta^4/4! + ...]$

$= J_k - n_i \varepsilon_{kij} J_j \sin\theta - (n_k n_i J_i - J_k)(\cos\theta - 1)$

$= J_k \cos\theta + \varepsilon_{kji} J_j n_i \sin\theta + n_k (\mathbf{n}\bullet\mathbf{J})(1-\cos\theta)$. QED

Comments

A vector **v** (rank-1 tensor) transforms ("rotates") according to $\mathbf{v}' = R\mathbf{v}$.
For a matrix M (rank-2 tensor) the corresponding transformation is $M' = RMR^{-1}$, and this is what one sees on the left side of (G.3.3) where $M = J_k$ and $R = \exp(-i\theta\, \hat{\mathbf{n}} \bullet \mathbf{J})$. In the expression RJ_kR^{-1} the rotation generator J_k is "sandwiched" between the two rotations.

The above sandwich formulas play a major role in Magnetic Resonance Imaging. The connection is that protons in your body have magnetic moments (spins) which can be lined up by a strong magnetic field. When the proton spins are slammed with a certain radio frequency pulse, they do conical rotation (precession) about the magnetic field axis at the so-called Larmor frequency. After the pulse this proton precession decays away (T1) and bulk-decoheres (T2) producing a certain return RF signal which can be analyzed. These return signals are sensitive to the local environment of the protons. The location of a

particular response is determined by giving the magnetic field a spatial gradient which affects the Larmor frequency. In this manner, an image can be formed. The sandwich formulas are not applied directly to individual spin angular momenta **J**, but to the average spin (polarization) density in the object being scanned. The analysis is quite complicated since it must take into account thermal and statistical effects which are managed with the use of the density matrix formalism. See the excellent text of Levitt for all the details.

Special cases of the sandwich formula

We shall have our own purposes for the sandwich formulas in the next Section. For rotations about the i axis we set

$$n_s = \delta_{s,i} \quad \Rightarrow \quad \mathbf{n} \bullet \mathbf{J} = J_i \qquad \text{and} \qquad \varepsilon_{kms} J_m n_s = \varepsilon_{kmi} J_m = \varepsilon_{ikm} J_m \tag{G.3.4}$$

Then from the sandwich formula (G.3.3),

$$\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\ J_k \exp(+i\theta \mathbf{n}\bullet\mathbf{J}) = \cos\theta\, J_k + \sin\theta\, \varepsilon_{kms} n_s J_m + (1 - \cos\theta)\, n_k (\mathbf{n}\bullet \mathbf{J}) \qquad \textit{(G.3.3)}$$

we find that (there is no implied sum on i in the rightmost term and $\varepsilon_{kmi} = \varepsilon_{ikm}$)

$$\exp(-i\theta J_i)\ J_k \exp(+i\theta J_i) = \cos\theta\, J_k + \sin\theta \varepsilon_{ikm} J_m + (1 - \cos\theta)\, \delta_{k,i} J_i$$

so then

$$R_i(\theta)\, J_k\, R_i(-\theta) = \cos\theta\, J_k + \sin\theta\, \varepsilon_{ikm}\, J_m + (1 - \cos\theta)\, \delta_{k,i}\, J_i\ . \tag{G.3.5}$$

In the case i = k we know that the left side is just J_k since everything then commutes. This is verified on the right since $\varepsilon_{iim} = 0$ and the other two terms then add up to J_k. So although obvious, we state:

$$R_k(\theta)\, J_k\, R_k(-\theta) = J_k \qquad\qquad i = k \tag{G.3.6}$$

In the case $i \neq k$ the third term in (G.3.5) does not contribute and we have

$$R_i(\theta)\, J_k\, R_i(-\theta) = \cos\theta\, J_k + \varepsilon_{ikm} \sin\theta\, J_m \qquad\qquad i \neq k$$

We shall now construct a table of all the cases for which $i \neq k$: (G.3.7)

$$\begin{aligned}
R_1(\theta)\, J_2 R_1(-\theta) &= \cos\theta\, J_2 + \varepsilon_{12m} \sin\theta\, J_m \quad &= \cos\theta\, J_2 + \sin\theta\, J_3 \\
R_1(\theta)\, J_3 R_1(-\theta) &= \cos\theta\, J_3 + \varepsilon_{13m} \sin\theta\, J_m \quad &= \cos\theta\, J_3 - \sin\theta\, J_2 \\
\\
R_2(\theta)\, J_1 R_2(-\theta) &= \cos\theta\, J_1 + \varepsilon_{21m} \sin\theta\, J_m \quad &= \cos\theta\, J_1 - \sin\theta\, J_3 \\
R_2(\theta)\, J_3 R_2(-\theta) &= \cos\theta\, J_3 + \varepsilon_{23m} \sin\theta\, J_m \quad &= \cos\theta\, J_3 + \sin\theta\, J_1 \\
\\
R_3(\theta)\, J_1 R_3(-\theta) &= \cos\theta\, J_1 + \varepsilon_{31m} \sin\theta\, J_m \quad &= \cos\theta J_1 + \sin\theta\, J_2
\end{aligned}$$

$$R_3(\theta)\, J_2 R_3(-\theta) \;=\; \cos\theta\, J_2 \;+\; \varepsilon_{32m} \sin\theta\, J_m \qquad = \cos\theta J_2 - \sin\theta\, J_1 \;. \tag{G.3.8}$$

G.4 Two more theorems for the rotation matrix toolbox

Theorem 1: $\quad R\mathbf{J}R^{-1} = R^{-1}\mathbf{J}$ (G.4.1)

This is another vector/matrix notation theorem which makes a claim about rotating a vector of matrices. The above ambiguous notation is a shorthand for the following,

$$RJ_iR^{-1} = [R^{-1}\mathbf{J}]_i \;= R^{-1}{}_{ij}J_j \qquad i = 1,2,3\;. \tag{G.4.2}$$

The object on the left is a product of three 3x3 matrices, while the right side is a linear combination of 3x3 matrices, so at least the theorem's claim is dimensionally reasonable. More generally J_i might be an abstract "operator" and R a rotation which acts on that operator, see ** below.

Proof: Start with the general rotation form given in (G.2.6),

$$R = \exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = \;1 + \;\; (\cos\theta - 1)\; T \;+ \sin\theta[\; -i(\mathbf{n}\bullet\mathbf{J})] \;\; \text{where } T_{ab} = (\delta_{ab} - n_a n_b) \;\; . \tag{G.4.3}$$

Sandwich this rotation around J_i and then use the sandwich formula (G.3.3),

$$\begin{aligned} RJ_iR^{-1} &= \exp(-\,i\theta\mathbf{n}\bullet\mathbf{J})\;\; J_i \exp(+\,i\theta\mathbf{n}\bullet\mathbf{J}) \\ &= \cos\theta\, J_i + \;\sin\theta\, \varepsilon_{ijk}n_k\, J_j \;+\; (1 - \cos\theta)\, n_i \;\; (\mathbf{n}\bullet\mathbf{J}) \\ &= \;\cos\theta\, \delta_{ij}J_j + \;\sin\theta\, \varepsilon_{ijk}n_k\, J_j \;+\; (1 - \cos\theta)\, n_i n_j J_j \\ &= \;[\;\cos\theta\, \delta_{ij} + \;\sin\theta\, \varepsilon_{ijk}n_k \;+\; (1 - \cos\theta)\, n_i n_j\;]\, J_j \quad . \end{aligned} \tag{G.4.4}$$

Our theorem is proved if, comparing (G.4.2) and (G.4.4), we can show that

$$R^{-1}{}_{ij} \;=\; \cos\theta\, \delta_{ij} + \;\sin\theta\, \varepsilon_{ijk}n_k \;+\; (1 - \cos\theta)\, n_i n_j \;. \tag{G.4.5}$$

Looking back at (G.4.3), one has

$$R^{-1} = \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) = \;1 + (\cos\theta - 1)\; T \;+ \sin\theta[\; +i(\mathbf{n}\bullet\mathbf{J})]$$

so

$$\begin{aligned} R^{-1}{}_{ij} &= \;\delta_{ij} + (\cos\theta - 1)\; T_{ij} \;+ \sin\theta[+n_k\,(iJ_k)_{ij}] \\ &= \;\delta_{ij} + \;\; (\cos\theta - 1)\;\; (\delta_{ij} - n_i n_j) \;\; + \sin\theta\; n_k\, \varepsilon_{kij} \qquad\qquad \text{// (G.1.3) and def of } T_{ij} \\ &= \delta_{ij}\cos\theta + (1\text{-}\cos\theta)\; n_i n_j + \;\sin\theta\, \varepsilon_{ijk}n_k \end{aligned} \tag{G.4.6}$$

But this is the same as (G.4.5) so the theorem is proved.

Theorem 2: $R\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\,R^{-1} = \exp(-i\theta \mathbf{n}'\bullet\mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$ (G.4.7)

or $R\,R_{\mathbf{n}}(\theta)\,R^{-1} = R_{\mathbf{n}'}(\theta)$ where $\mathbf{n}' = R\mathbf{n}$

The matrix R is an arbitrary rotation. On the left we have a product of three 3x3 matrices, while the right side is a 3x3 matrix. One proof of this theorem might be to say "what else could it be? ". We shall provide a more substantial proof below.

Proof: Start by inserting the general rotation form (G.2.6) into the left side of (G.4.7),

$$R\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\,R^{-1} = R\,[\,1 + (\cos\theta - 1)\,T - i\sin\theta\,(\mathbf{n}\bullet\mathbf{J})\,]\,R^{-1}$$

$$= 1 + (\cos\theta - 1)RTR^{-1} - i\sin\theta\; n_k\, RJ_kR^{-1}\,. \qquad \text{(G.4.8)}$$

Now consider

$$[RTR^{-1}]_{ad} = R_{ab}T_{bc}R^{-1}{}_{cd} = R_{ab}(\delta_{bc} - n_bn_c)R^{T}{}_{cd}$$

$$= \delta_{ad} - R_{ab}\,n_bn_cR_{dc} = \delta_{ad} - (\,R_{ab}\,n_b)(R_{dc}n_c) = \delta_{ad} - n'_an'_d \qquad \mathbf{n}' = R\mathbf{n}$$

$$\equiv T'_{ad}\,. \qquad \text{// in other words, } RTR^{-1} = T' \qquad \text{(G.4.9)}$$

From Theorem 1 (G.4.2) we know that

$$RJ_kR^{-1} = R^{-1}{}_{kj}J_j\,. \qquad \textit{(G.4.2)}$$

Inserting this last item and (G.4.9) into (G.4.8) gives

$$R\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\,R^{-1} = 1 + (\cos\theta - 1)T' - i\sin\theta\; n_kR^{-1}{}_{kj}J_j$$

$$= 1 + (\cos\theta - 1)T' - i\sin\theta\, R_{jk}n_k\, J_j = 1 + (\cos\theta - 1)T' - i\sin\theta\,[Rn]_j\, J_j$$

$$= 1 + (\cos\theta - 1)T' - i\sin\theta\, n'_jJ_j = 1 + (\cos\theta - 1)T' - i\sin\theta\,(\mathbf{n}'\bullet\mathbf{J})$$

$$= \exp(-i\theta \mathbf{n}'\bullet\mathbf{J}) \qquad \text{// using (G.2.6) with } \mathbf{n}\rightarrow\mathbf{n}' \qquad \text{(G.4.10)}$$

Thus the theorem is proved.

The fact (G.4.9) that $RTR^{-1} = T'$ is in fact just the standard rule for the transformation of a rank-2 tensor under rotations, as shown for example in Lucht *Tensor* (5.7.1). That is to say, if an Observer in Frame S sees T, an Observer in rotated Frame S' will see $T' = RTR^{-1}$. This same statement applies at the higher level of our theorem

$$R\,R_{\mathbf{n}}(\theta)\,R^{-1} = R_{\mathbf{n}'}(\theta) \qquad \mathbf{n}' = R\mathbf{n} \qquad \text{// Theorem 2 restated} \qquad \text{(G.4.11)}$$

The rotation $R_{\mathbf{n}}(\theta)$ seen by an Observer in Frame S appears as $R_{\mathbf{n}'}(\theta)$ in Frame S' where $\mathbf{e'}_{\mathbf{n}} = R^{-1}\mathbf{e}_{\mathbf{n}}$ as in (1.1.30) . Like T, the rotation $R_{\mathbf{n}}(\theta)$ transforms as a rank-2 tensor under rotations.

An equivalent *active* statement of the theorem is the following:

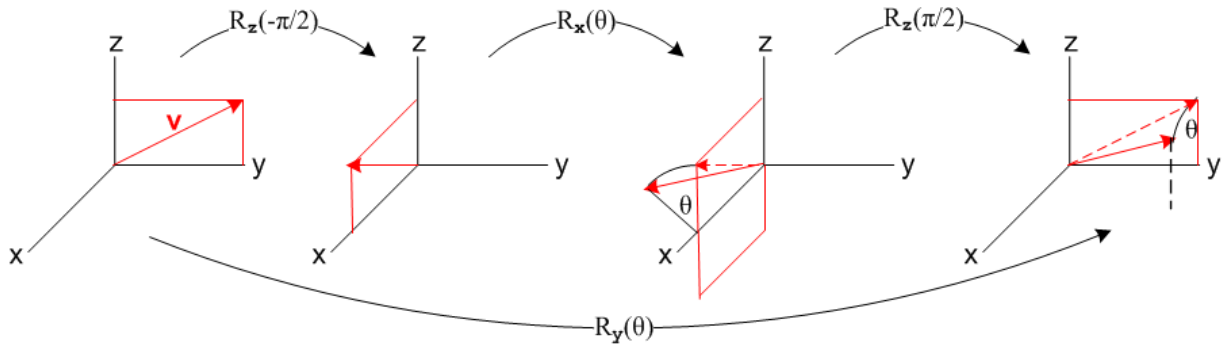
$$R\, R_{\mathbf{n}}(\theta)\, R^{-1}\mathbf{v} = R_{\mathbf{n}'}(\theta)\mathbf{v} \qquad \text{where } \mathbf{n}' = R\mathbf{n} \tag{G.4.12}$$

Instead of rotating a vector **v** by amount θ about axis $\hat{\mathbf{n}}'$, one can first back-rotate the vector **v**, then rotate by amount θ about axis $\hat{\mathbf{n}}$, the forward rotate the result.

Example: $R_{\mathbf{z}}(\pi/2)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(-\pi/2)$ = ?? Here $\mathbf{n} = \hat{\mathbf{x}}$ and $\mathbf{n}' = R_{\mathbf{z}}(\pi/2)\,\hat{\mathbf{x}} = \hat{\mathbf{y}}$ so we conclude that

$$R_{\mathbf{z}}(\pi/2)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(-\pi/2) = R_{\mathbf{y}}(\theta) \ . \tag{G.4.13}$$

Here is a graphical interpretation of (G.4.13) applied to a particular vector **v** :



(G.4.14)

G.5 Goldstein's Euler Angles

As the reader no doubt knows, it is possible to specify an arbitrary rotation in terms of three Euler angles,

$$R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi) \ . \tag{G.5.1}$$

As shown below, the matrix shown in (G.5.1) will be identified with R^{-1} of our Section 1 formalism.

Some authors use other letters for the angles, and some put $R_{\mathbf{y}}$ in the middle in place of $R_{\mathbf{x}}$. We have in mind that the matrices shown are specifically the active rotation matrices shown in (A.1).

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{G.5.2}$$

To get Maple warmed up for activities below, we enter the three matrices of (G.5.2),

```
Rx := x -> matrix(3,3,[1,0,0,0,cos(x),-sin(x),0,sin(x),cos(x)]):
Ry := x -> matrix(3,3,[cos(x),0,sin(x),0,1,0,-sin(x),0,cos(x)]):
Rz := x -> matrix(3,3,[cos(x),-sin(x),0,sin(x),cos(x),0,0,0,1]):
Rx(theta);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

```
Ry(theta);
```

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

```
Rz(theta);
```

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(G.5.3)}$$

These (G.5.2) matrices are "active" because, using the right-hand-rule, they rotate a vector forward by angle θ in the Active View described at the start of Section 1.3. For example, for small θ the rotation $R_z(\theta)$ acting on $\hat{x}$ produces a vector in the first quadrant of the x-y plane:

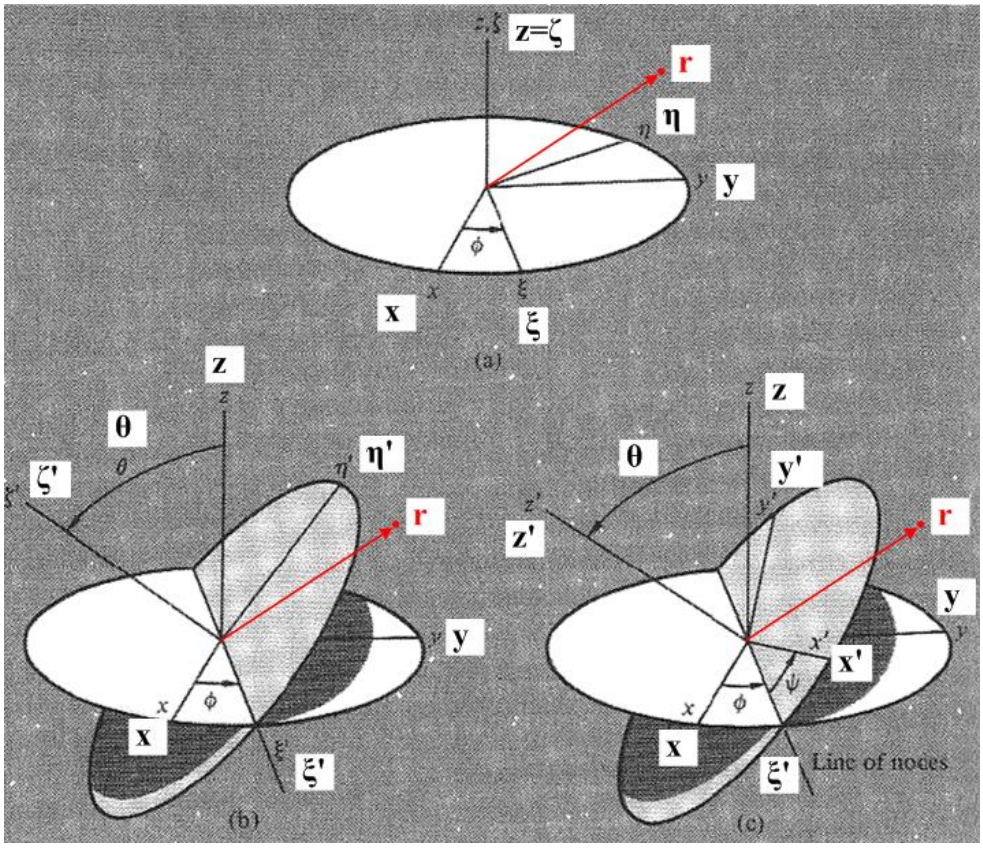
```
xhat := matrix(3,1,[1,0,0]);
```

$$xhat := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

```
evalm( Rz(theta) &* xhat);
```

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \qquad \text{(G.5.4)}$$

Here now is Goldstein's Euler Angle picture (G p 107, GPS p152), enhanced a bit for readability,



(G.5.5)

Astronomy Footnote: Imagine that the white disk is a "reference plane" (perhaps the equatorial plane of the Earth) and the grey disk perimeter denotes the orbit of some object about another (perhaps the Moon about the Earth). That orbiting object crosses the reference plane in two places called "nodes" (ascending and descending by some convention). The intersection of the two planes is called the "line of nodes". In Goldstein's drawing this coincides with the ξ' axis. In this application, the Euler angles θ,φ define the plane of the Moon's orbit, and Euler angle ψ shows the progress of the Moon in this orbit. As noted in Section 8.8, the Moon's orbital plane precesses, so the line of nodes rotates in the white disk plane.

Fig (G.5.5) shows how one can start with x,y,z axes at the top, and end up with x',y',z' axes on the lower right. There are really three sequential transformations occurring here, and we can just read off the effects on unit vectors by looking at the pictures: (ξ = xi = "zeye", η = eta = "ate'uh, ζ = zeta = "zee'ta")

top	$(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{\mathbf{z}}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$	where $\hat{\boldsymbol{\zeta}} = \hat{\mathbf{z}}$
left	$(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\boldsymbol{\xi}}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}})$	where $\hat{\boldsymbol{\xi}}' = \hat{\boldsymbol{\xi}}$
right	$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\boldsymbol{\zeta}'}(\psi)(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}')$	where $\hat{\mathbf{z}}' = \hat{\boldsymbol{\zeta}}'$

(G.5.6)

For example, the first of these nine equations says $\hat{\boldsymbol{\xi}} = R_{\mathbf{z}}(\varphi)\hat{\mathbf{x}}$ which seems clear from the top picture. The rotation $R_{\boldsymbol{\xi}}(\theta)$ is an active rotation of θ about the $\hat{\boldsymbol{\xi}}$ axis, and similarly for $R_{\boldsymbol{\zeta}'}(\psi)$.

We can combine transformations in various obvious ways :

$(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{\mathbf{z}}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ // each equation is like $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$

$(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\boldsymbol{\xi}}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{\boldsymbol{\xi}}(\theta)\,R_{\mathbf{z}}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\boldsymbol{\zeta}'}(\psi)\,(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\boldsymbol{\zeta}'}(\psi)\,R_{\boldsymbol{\xi}}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{\boldsymbol{\zeta}'}(\psi)\,R_{\boldsymbol{\xi}}(\theta)\,R_{\mathbf{z}}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \quad . \qquad \text{(G.5.7)}$$

For example, $\hat{\mathbf{x}}' = R_{\boldsymbol{\zeta}'}(\psi)\hat{\boldsymbol{\xi}}' = R_{\boldsymbol{\zeta}'}(\psi)\,R_{\boldsymbol{\xi}}(\theta)\,\hat{\boldsymbol{\xi}} = R_{\boldsymbol{\zeta}'}(\psi)\,R_{\boldsymbol{\xi}}(\theta)R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{x}}$.

These equations are all of the template form $\mathbf{e'}_n = (R^{-1})\mathbf{e}_n$ appearing in our Basis Theorem (1.1.30). The *meaning* is $|\mathbf{e'}_n\rangle = \mathcal{R}^{-1}|\mathbf{e}_n\rangle = |R^{-1}\mathbf{e}_n\rangle$. If one takes Frame S components of these equations, then in $\mathbf{e'}_n = (R^{-1})\mathbf{e}_n$ one can interpret (R^{-1}) as a matrix. For example, for $\hat{\boldsymbol{\xi}} = R_{\mathbf{z}}(\varphi)\hat{\mathbf{x}}$ one can write $(\hat{\boldsymbol{\xi}})_i = [R_{\mathbf{z}}(\varphi)]_{ij}(\hat{\mathbf{x}})_i$ in which case $[R_{\mathbf{z}}(\varphi)]_{ij}$ is the matrix shown in (G.5.2).

In going all the way from the Frame S basis $\mathbf{e}_n$ to the Frame S' basis $\mathbf{e'}_n$ we see from the last line in (G.5.7) that in order to interpret this last line as $\mathbf{e'}_n = (R^{-1})\mathbf{e}_n$, we must make the identification

$$R^{-1} = R_{\boldsymbol{\zeta}'}(\psi)\,R_{\boldsymbol{\xi}}(\theta)\,R_{\mathbf{z}}(\varphi)\,. \qquad \text{(G.5.8)}$$

The R symbol here is the R that appears in our Section 1 formalism. In particular, recall the Basis Theorem (1.1.29) and (1.1.30), and the alternate notation of (1.1.32),

$$\mathbf{e'}_n = R^{-1}\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \quad \text{or} \quad \begin{pmatrix}\mathbf{e'}_1\\ \mathbf{e'}_2\\ \mathbf{e'}_3\end{pmatrix} = R\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}. \qquad \textit{(1.1.29,30) + (1.1.32)}$$

Define $Q \equiv R^{-1}$ and rewrite the Basis Theorem as,

$$\mathbf{e'}_n = Q\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{e'}_n = (Q^{-1})_{nm}\,\mathbf{e}_m \quad \text{or} \quad \begin{pmatrix}\mathbf{e'}_1\\ \mathbf{e'}_2\\ \mathbf{e'}_3\end{pmatrix} = [Q]^{-1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}. \qquad \text{(G.5.9)}$$

This form of the Basis Theorem then serves as a template with which we can convert the equations of (G.5.7) to the corresponding linear combination equations of basis vectors :

$$\begin{pmatrix}\hat{\boldsymbol{\xi}}\\ \hat{\boldsymbol{\eta}}\\ \hat{\boldsymbol{\zeta}}\end{pmatrix} = [R_{\mathbf{z}}(\varphi)]^{-1}\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{pmatrix} = R_{\mathbf{z}}(-\varphi)\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{pmatrix} \qquad Q = R_{\mathbf{z}}(\varphi)$$

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta}' \\ \hat{\zeta} \end{pmatrix} = [R_{\xi}(\theta)\, R_{z}(\varphi)]^{-1} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{z}(-\varphi)\, R_{\xi}(-\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad Q = R_{\xi}(\theta)\, R_{z}(\varphi)$$

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = [R_{\zeta'}(\psi)\, R_{\xi}(\theta) R_{z}(\varphi)]^{-1} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{z}(-\varphi)\, R_{\xi}(-\theta) R_{\zeta'}(-\psi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} . \qquad \text{(G.5.10)}$$

We constantly use facts like $[ABC]^{-1} = C^{-1}B^{-1}A^{-1}$ and $R^{-1}{}_{s}(\alpha) = R_{s}(-\alpha)$.

These equations can be inverted in the obvious manner. The last one would give

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{\zeta'}(\psi)\, R_{\xi}(\theta) R_{z}(\varphi) \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} . \qquad \text{(G.5.11)}$$

If we install this equation on the right of the first two equations in (G.5.10), the results are

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = R_{z}(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{z}(-\varphi)\, R_{\zeta'}(\psi)\, R_{\xi}(\theta) R_{z}(\varphi) \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix}$$

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta}' \\ \hat{\zeta} \end{pmatrix} = R_{z}(-\varphi)\, R_{\xi}(-\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{z}(-\varphi)\, R_{\xi}(-\theta)\, R_{\zeta'}(\psi)\, R_{\xi}(\theta) R_{z}(\varphi) \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} . \qquad \text{(G.5.12)}$$

We know all about the rotations R_{x}, R_{y} and R_{z} since they are specifically stated in (G.5.2). But what about the strange rotations R_{ξ} and $R_{\zeta'}$ which dot the landscape above? In Theorem 3 below we shall show that each of these rotations may be written as a certain product of the R_{x}, R_{y} and R_{z}. Specifically, in Theorem 3 we shall prove the first three results below :

(1) $R_{\xi}(\theta) = R_{z}(\varphi)\, R_{x}(\theta)\, R_{z}(-\varphi)$

(2) $R_{\zeta'}(\psi) = R_{z}(\varphi)\, R_{x}(\theta) R_{z}(\psi) R_{x}(-\theta) R_{z}(-\varphi)$

(3) $R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{z}(\varphi) = R_{z}(\varphi) R_{x}(\theta) R_{z}(\psi) = R^{-1}$

Using these equations one can clear out all the strange rotations from (G.5.10,11,12) as follows :

(4) $R_{z}(-\varphi)\, R_{\xi}(-\theta) = R_{z}(-\varphi)\, [R_{z}(\varphi)\, R_{x}(-\theta)\, R_{z}(-\varphi)] = R_{x}(-\theta)\, R_{z}(-\varphi)$ (1)

(5) $R_{z}(-\varphi)\, R_{\xi}(-\theta) R_{\zeta'}(-\psi) = [R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{z}(\varphi)]^{-1} = [R_{z}(\varphi) R_{x}(\theta) R_{z}(\psi)]^{-1} = R_{z}(-\psi) R_{x}(-\theta) R_{z}(-\varphi)$ (3) (2) (1)

(6) $R_z(-\varphi)\, R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi) = R_z(-\varphi)\,[R_z(\varphi)\, R_x(\theta) R_z(\psi) R_x(-\theta) R_z(-\varphi)]\,[R_z(\varphi)\, R_x(\theta)\, R_z(-\varphi)] R_z(\varphi)$
$= R_x(\theta) R_z(\psi)$ // use $R_z(-\varphi)R_z(\varphi) = 1$ in three places, then $R_x(-\theta)R_x(\theta) = 1$

(7) $R_z(-\varphi)\, R_\xi(-\theta)\, R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)$
$= R_z(-\varphi)\,[R_z(\varphi)\, R_x(-\theta)\, R_z(-\varphi)]\,[R_z(\varphi)\, R_x(\theta) R_z(\psi) R_x(-\theta) R_z(-\varphi)][R_z(\varphi)\, R_x(\theta)\, R_z(-\varphi)] R_z(\varphi)$
$= R_z(\psi)$ (1) (3) (1)

(G.5.13)

One can then rewrite (G.5.10,11,12) as

(a) $\begin{pmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\eta}} \\ \hat{\boldsymbol{\zeta}} \end{pmatrix} = R_z(-\varphi) \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$

(b) $\begin{pmatrix} \hat{\boldsymbol{\xi}}' \\ \hat{\boldsymbol{\eta}}' \\ \hat{\boldsymbol{\zeta}}' \end{pmatrix} = R_x(-\theta)\, R_z(-\varphi) \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$ // using (4)

(c) $\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = R_z(-\psi) R_x(-\theta) R_z(-\varphi) \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$ // using (3)

(d) $\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = R_z(\varphi) R_x(\theta) R_z(\psi) \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix}$ // using (5)

(e) $\begin{pmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\eta}} \\ \hat{\boldsymbol{\zeta}} \end{pmatrix} = R_x(\theta) R_z(\psi) \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix}$ // using (6)

(f) $\begin{pmatrix} \hat{\boldsymbol{\xi}}' \\ \hat{\boldsymbol{\eta}}' \\ \hat{\boldsymbol{\zeta}}' \end{pmatrix} = R_z(\psi) \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix}$ // using (7) (G.5.14)

Using these equations, one has explicit formulas for writing any of the nine basis vectors either as a linear combination of $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ or as a linear combination of $\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}'$.

By inspection one can rewrite the above six equations in the form shown on the right side of the Basis Theorem (G.5.9),

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = [Q]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{e}'_n = Q\mathbf{e}_n \qquad \textit{(G.5.9)}$$

Therefore,

(a) $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_z(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$

(b) $(\hat{\boldsymbol{\xi}}', \hat{\boldsymbol{\eta}}', \hat{\boldsymbol{\zeta}}') = R_{\mathbf{z}}(\varphi) R_{\mathbf{x}}(\theta)\ (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$

(c) $(\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}') = R_{\mathbf{z}}(\varphi) R_{\mathbf{x}}(\theta) R_{\mathbf{z}}(\psi)\ (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$

(d) $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) = R_{\mathbf{z}}(-\psi) R_{\mathbf{x}}(-\theta) R_{\mathbf{z}}(-\varphi)\ (\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$

(e) $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}) = R_{\mathbf{z}}(-\psi) R_{\mathbf{x}}(-\theta)\ (\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$

(f) $(\hat{\boldsymbol{\xi}}', \hat{\boldsymbol{\eta}}', \hat{\boldsymbol{\zeta}}') = R_{\mathbf{z}}(-\psi)\ (\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$. (G.5.15)

We shall now prove the three facts quoted above as (G.5.13) (1), (2) and (3), and then we shall resume our discussion of the Euler Angles .

Theorem 3 : (G.5.16)

(1) $R_{\boldsymbol{\xi}}(\theta) = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(-\varphi)$

(2) $R_{\boldsymbol{\zeta}'}(\psi) = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta) R_{\mathbf{z}}(\psi) R_{\mathbf{x}}(-\theta) R_{\mathbf{z}}(-\varphi)$

(3) $R_{\boldsymbol{\zeta}'}(\psi)\, R_{\boldsymbol{\xi}}(\theta)\, R_{\mathbf{z}}(\varphi) = R_{\mathbf{z}}(\varphi) R_{\mathbf{x}}(\theta) R_{\mathbf{z}}(\psi)$ = the Euler angle rotation R^{-1} of (G.5.8)

Recall from (G.1.1) that a rotation of α about axis $\hat{\mathbf{n}}$ may be written $R_{\hat{\mathbf{n}}}(\alpha) = \exp(-i\,\alpha\, \mathbf{n} \bullet \mathbf{J})$.

Proof of (1) :

Recall Theorem 2 of (G.4.7) which says,

$R \exp(-i\theta \mathbf{n} \bullet \mathbf{J})\, R^{-1} = \exp(-i\theta \mathbf{n}' \bullet \mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$. *(G.4.7)*

Note from line 1 of (G.5.6) that $\hat{\boldsymbol{\xi}} = R_{\mathbf{z}}(\varphi)\, \hat{\mathbf{x}}$. We take $\mathbf{n}' = \hat{\boldsymbol{\xi}}$, $R = R_{\mathbf{z}}(\varphi)$, $\mathbf{n} = \hat{\mathbf{x}}$ to get

$$R_{\mathbf{z}}(\varphi) \exp(-i\theta \hat{\mathbf{x}} \bullet \mathbf{J})\, R_{\mathbf{z}}(-\varphi) = \exp(-i\theta \hat{\boldsymbol{\xi}} \bullet \mathbf{J})$$

or

$$R_{\boldsymbol{\xi}}(\theta) = \exp(-i\theta \hat{\boldsymbol{\xi}} \bullet \mathbf{J}) = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(-\varphi) .$$ QED (1)

and we have thus proved item (1).

One can see intuitively how this works, as in our example of (G.4.14). Instead of rotating θ about the $\hat{\boldsymbol{\xi}}$ axes, we first back-rotate around $\hat{\mathbf{z}}$ by -φ, use the aligned $R_{\mathbf{x}}(\theta)$ to create a tilted disk in the top drawing of Fig (G.5.5), then forward rotate that result by $R_{\mathbf{z}}(\varphi)$ to get the tilted disk in the left picture. The good news is that we don't have to rely on such visualizations to get the result right.

Proof of (2) :

Recall again Theorem 2 of (G.4.7) which says (now with dummy argument $\theta \rightarrow \psi$)

$$R \exp(-i\psi \mathbf{n}\bullet\mathbf{J})\, R^{-1} = \exp(-i\psi \mathbf{n}'\bullet\mathbf{J}) \quad \text{where } \mathbf{n}' = R\mathbf{n} \ . \qquad (G.4.7)$$

Note from lines 2,1 of (G.5.6) that that $\hat{\boldsymbol{\zeta}}' = R_{\boldsymbol{\xi}}(\theta)\hat{\boldsymbol{\zeta}} = R_{\boldsymbol{\xi}}(\theta)\hat{\mathbf{z}}$. We take $\mathbf{n}' = \hat{\boldsymbol{\zeta}}'$, $R = R_{\boldsymbol{\xi}}(\theta)$, $\mathbf{n} = \hat{\mathbf{z}}$ to get

$$R_{\boldsymbol{\xi}}(\theta)\exp(-i\psi\hat{\mathbf{z}}\bullet\mathbf{J})R_{\boldsymbol{\xi}}(-\theta) = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J}) = R_{\boldsymbol{\zeta}'}(\psi) \ .$$

Therefore

$$R_{\boldsymbol{\zeta}'}(\psi) = R_{\boldsymbol{\xi}}(\theta)R_{\mathbf{z}}(\psi)R_{\boldsymbol{\xi}}(-\theta) \ .$$

Then installing result (1) twice we get

$$R_{\boldsymbol{\zeta}'}(\psi) = [R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(-\varphi)]\, R_{\mathbf{z}}(\psi)\, [R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(-\theta)\, R_{\mathbf{z}}(-\varphi)]$$

$$= R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(\psi)\, R_{\mathbf{x}}(-\theta)\, R_{\mathbf{z}}(-\varphi) \qquad \text{QED (2)}$$

and we have thus proved item (2).

Reader Exercise: Interpret this result in terms of back-rotations and Fig (G.5.5).

Proof of (3) :

$$R_{\boldsymbol{\zeta}'}(\psi)\, R_{\boldsymbol{\xi}}(\theta)\, R_{\mathbf{z}}(\varphi) = \overset{(2)}{[R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)]}\ \overset{(1)}{[R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(-\varphi)]}\, R_{\mathbf{z}}(\varphi)$$

$$= R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(\psi)\, R_{\mathbf{x}}(-\theta)\, R_{\mathbf{z}}(-\varphi)\, R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(-\varphi)\, R_{\mathbf{z}}(\varphi)$$

$$= R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(\psi)\, R_{\mathbf{x}}(-\theta)\, R_{\mathbf{x}}(\theta)$$

$$= R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(\psi) \ . \qquad \text{QED (3)}$$

Resuming the Euler angle discussion, from (G.5.8) and Theorem 3 (3) we know that

$$R^{-1} = R_{\boldsymbol{\zeta}'}(\psi)\, R_{\boldsymbol{\xi}}(\theta)\, R_{\mathbf{z}}(\varphi) = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{x}}(\theta)\, R_{\mathbf{z}}(\psi)$$

so

$$R = R_{\mathbf{z}}(-\psi)\, R_{\mathbf{x}}(-\theta)\, R_{\mathbf{z}}(-\varphi) \ . \qquad (G.5.17a)$$

On Goldstein p 109 (GPS p 153) this last equation $R = R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)$ appears as A = BCD which is written out in detail in (4-46) (GPS 4.46) , and which Maple verifies,

```
R := evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
```

$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix} \qquad \text{(G.5.17b)}$$

The transpose $R^{-1} = R^T$ then appears in Goldstein (4-47) (GPS 4.47) .

Interpretation of the Goldstein's Triple Concatenation

In our Section 1 formalism, we discuss the idea of three concatenated transformations near (1.1.41). We can compare the equations there to those of (G.5.6),

$$\mathbf{e'''}_n = U^{-1}\mathbf{e''}_n \qquad \mathbf{e''}_n = S^{-1}\mathbf{e'}_n \qquad \mathbf{e'}_n = R^{-1}\mathbf{e}_n \qquad \text{(G.5.18a)}$$

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\,(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') \qquad (\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\xi}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) \qquad (\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{z}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \ .$$

It follows that the three "back-rotations" are

$$U^{-1} = R_{\zeta'}(\psi) \qquad S^{-1} = R_{\xi}(\theta) \qquad R^{-1} = R_{z}(\varphi) \ . \qquad \text{(G.5.18b)}$$

In Fig (1.3.4) we show an example where $R^{-1} = R_z(-\alpha)$ = "back-rotation" and we draw the figure for some small $\alpha > 0$. In Goldstein's back rotations, the role of α is played by $-\psi$,$-\theta$ and $-\varphi$. Figure (G.5.5) shows that the basis vector "back rotations" are really forward rotations by ψ, θ and φ.

Doing the triple concatenation one gets,

$$\mathbf{e'''}_n = U^{-1}S^{-1}R^{-1}\mathbf{e}_n = R_{\zeta'}(\psi)R_{\xi}(\theta)\,R_{z}(\varphi)\mathbf{e}_n$$

or

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\,R_{\xi}(\theta)\,R_{z}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \qquad \text{(G.5.19)}$$

in agreement with (G.5.7).

Transformation of Kinematic Vectors

The corresponding Passive View transformations of Kinematic Vectors for the three concatenations are,

$$(\mathbf{V})''' = U\mathbf{V}'' \qquad (\mathbf{V})'' = S\mathbf{V}' \qquad (\mathbf{V})' = R\mathbf{V} \ . \qquad \text{(G.5.20)}$$

Doing the concatenation and then changing the Section 1 triple-prime to Goldstein's single-prime, we get

$$(\mathbf{V})' = USR\,\mathbf{V}$$

$$= R_{\zeta'}(-\psi)R_{\xi}(-\theta)R_{z}(-\varphi)\,\mathbf{V} \ . \qquad \text{(G.5.21)}$$

If we now redefine R to be our notation for Goldstein's *overall* transformation, we get

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R^{-1}(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \qquad R^{-1} = R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{\mathbf{z}}(\varphi)$$
$$= R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{\mathbf{z}}(\varphi)\, (\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$$

$$(\mathbf{V})' = R\,\mathbf{V} \qquad R = [R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{\mathbf{z}}(\varphi)]^{-1}$$
$$= [R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{\mathbf{z}}(\varphi)]^{-1}\,\mathbf{V}\,. \qquad \text{(G.5.22)}$$

We can rewrite the above lines making use of Theorem 3 item (3) to get

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R^{-1}(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \qquad R^{-1} = R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)$$
$$= R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)\, (\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$$

$$(\mathbf{V})' = R\,\mathbf{V} \qquad R = [\,R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)]^{-1}$$
$$= [\,R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)]^{-1}\,\mathbf{V}$$

$$= R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)\,\mathbf{V}\,. \qquad \text{(G.5.23)}$$

This last item is the rule for finding the Frame S' components $(V)'_{\mathbf{i}}$ of a vector **V** in terms of its Frame S components $V_{\mathbf{i}}$.

Figure (G.5.5) shows the Kinematic Vector **V = r** which is the position of some point in Frame S .

Exercise 1:

Compute the Frame S' components of the vector **r** which in Frame S has components (x,y,z) .

$$(\mathbf{r})' = R\,\mathbf{r} = R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)\,\mathbf{r} \qquad \text{(G.5.24)}$$

```
R := evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
```

$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix} \qquad \text{(G.5.25)}$$

Therefore

$$(\mathbf{r})' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and so

$$x' = (\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)\, x + (\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)\, y + \sin\psi\sin\theta\, z$$

$$y' = (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)\, x + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)\, y + \cos\psi\sin\theta\, z$$

$$z' = \sin\theta\sin\varphi\ x - \sin\theta\cos\varphi\ y + \cos\theta\ z\ . \qquad \text{(G.5.26)}$$

Application: What are the Frame S' components of $\hat{\mathbf{x}}$?

Apply the previous equation to (x,y,z) = (1,0,0):

$$\begin{aligned} \hat{\mathbf{x}}_{\mathbf{x}'} &= \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi &&= (\hat{\mathbf{x}})'_1 \\ \hat{\mathbf{x}}_{\mathbf{y}'} &= -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi &&= (\hat{\mathbf{x}})'_2 \\ \hat{\mathbf{x}}_{\mathbf{z}'} &= \sin\theta\sin\varphi &&= (\hat{\mathbf{x}})'_3 \end{aligned} \qquad \text{(G.5.27)}$$

Inspect the basis vector $\hat{\mathbf{x}}$ in the lower right drawing of Fig (G.5.5). For the small Euler angles used in the figure, $\hat{\mathbf{x}}$ appears to have positive x' and z' components, but a negative y' component. This is confirmed by looking at (G.5.27).

We now reverse Exercise 1 to get Exercise 2.

Exercise 2:

Compute the Frame S components of the vector **r** which in Frame S' has components (x',y',z') .

$$\mathbf{r} = R^{-1}(\mathbf{r})' = R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)(\mathbf{r})' \ . \qquad \text{(G.5.28)}$$

The matrix $R^{-1} = R^{T}$ is just the transpose of the matrix shown above; Maple computes it anyway,

```
RINV := evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
```

$$RINV := \begin{bmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\phi)\sin(\psi) - \cos(\theta)\sin(\phi)\cos(\psi) & \sin(\theta)\sin(\phi) \\ \sin(\phi)\cos(\psi) + \cos(\theta)\cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\theta)\cos(\phi)\cos(\psi) & -\sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{bmatrix} \qquad \text{(G.5.29)}$$

Therefore

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R^{-1}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

and so

$$x = (\cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi)\ x' + (-\cos\varphi\sin\psi - \cos\theta\sin\varphi\cos\psi)\ y' + \sin\theta\sin\varphi\ z'$$

$$y = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\ x' + (-\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi)\ y' - \sin\theta\cos\varphi\ z'$$

$$z = \sin\theta\sin\psi\ x' + \sin\theta\cos\psi\ y' + \cos\theta\ z'\ . \qquad \text{(G.5.30)}$$

Application: What are the Frame S components of $\hat{\mathbf{x}}'$?

Apply the previous equation to (x',y',z') = (1,0,0):

$$\hat{\mathbf{x}}'_{\mathbf{x}} = \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi \qquad = (\hat{\mathbf{x}}')_1$$
$$\hat{\mathbf{x}}'_{\mathbf{y}} = \sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi \qquad = (\hat{\mathbf{x}}')_2$$
$$\hat{\mathbf{x}}'_{\mathbf{z}} = \sin\theta\sin\psi \qquad = (\hat{\mathbf{x}}')_3 \qquad \text{(G.5.31)}$$

Inspect the basis vector $\hat{\mathbf{x}}'$ in the lower right drawing of Fig (G.5.5). For the small Euler angles used in the figure, $\hat{\mathbf{x}}'$ appears to have positive x,y and z components. This is confirmed in (G.5.31).

Exercise 3: Express $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ in terms of $(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}')$

According to (G.5.14) (d) we know that

$$\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{pmatrix} = R_z(\varphi)R_x(\theta)R_z(\psi)\begin{pmatrix}\hat{\mathbf{x}}'\\ \hat{\mathbf{y}}'\\ \hat{\mathbf{z}}'\end{pmatrix} \qquad // = R^{-1}\begin{pmatrix}\hat{\mathbf{x}}'\\ \hat{\mathbf{y}}'\\ \hat{\mathbf{z}}'\end{pmatrix} \qquad \textit{(G.5.14) (d)}$$

where Maple computes the matrix,

```
evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
```

$$\begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

.

Therefore

$$\hat{\mathbf{x}} = (\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta)\,\hat{\mathbf{x}}' + (-\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta)\hat{\mathbf{y}}' + (\sin\varphi\sin\theta)\,\hat{\mathbf{z}}'$$

$$\hat{\mathbf{y}} = (\cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta)\,\hat{\mathbf{x}}' + (-\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta)\hat{\mathbf{y}}' + (-\cos\varphi\sin\theta)\,\hat{\mathbf{z}}'$$

$$\hat{\mathbf{z}} = (\sin\psi\sin\theta)\hat{\mathbf{x}}' + (\cos\psi\sin\theta)\hat{\mathbf{y}}' + (\cos\theta)\,\hat{\mathbf{z}}' \qquad \text{(G.5.32)}$$

And now we go the other direction:

Exercise 4: Express $(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}')$ in terms of $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$

Inverting (G.5.14) (d) quoted just above we find

$$\begin{pmatrix}\hat{\mathbf{x}}'\\ \hat{\mathbf{y}}'\\ \hat{\mathbf{z}}'\end{pmatrix} = R_z(-\psi)R_x(-\theta)R_z(-\varphi)\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{pmatrix} \qquad // = R\begin{pmatrix}\hat{\mathbf{x}}'\\ \hat{\mathbf{y}}'\\ \hat{\mathbf{z}}'\end{pmatrix} \qquad \textit{(G.5.14) (d) inverse}$$

We can then use the Exercise 3 result with $\varphi,\theta,\psi \rightarrow -\psi,-\theta,-\varphi$. But to avoid errors, we just use Maple again to get

```
evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
```

$$\begin{bmatrix} \cos(\psi)\cos(\phi)-\sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi)+\sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi)-\cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi)+\cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

Therefore,

$$\hat{\mathbf{x}}' = (\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta)\,\hat{\mathbf{x}} + (\cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta)\hat{\mathbf{y}} + (\sin\theta\sin\psi)\,\hat{\mathbf{z}}$$

$$\hat{\mathbf{y}}' = (-\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta)\,\hat{\mathbf{x}} + (-\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta)\hat{\mathbf{y}} + (\sin\theta\cos\psi)\,\hat{\mathbf{z}}$$

$$\hat{\mathbf{z}}' = (\sin\varphi\sin\theta)\hat{\mathbf{x}}' + (-\cos\varphi\sin\theta)\hat{\mathbf{y}} + (\cos\theta)\,\hat{\mathbf{z}} \qquad \text{(G.5.33)}$$

Exercise 5: Compute the unit vectors $\hat{\boldsymbol{\varphi}}$, $\hat{\boldsymbol{\psi}}$, $\hat{\boldsymbol{\theta}}$.

Looking at the polar coordinates equation (E.5.3), and then looking at Fig (G,.5.5), we can read off

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\psi}} = -\sin\psi\,\hat{\mathbf{x}}' + \cos\psi\,\hat{\mathbf{y}}'$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{z}} + \cos\theta\,\hat{\mathbf{y}} \qquad \text{(G.5.34)}$$

We could then use results (G.5.32) or (G.5.33) to express these all in terms of $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ or $(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}')$

G.6 Euler angles which change in time : computation of ω (Method 1)

Suppose now that all the Euler angles are changing in time. The combination of all these movements creates an overall **ω** angular rotation vector relating the relative motion of the two Frames (as in Fig 1). Looking at Fig (G.5.5) we see that **ω** will have three contributions, one from each Euler angle movement,

$$\boldsymbol{\omega}_{\varphi} = \dot{\varphi}\,\hat{\mathbf{z}}$$
$$\boldsymbol{\omega}_{\theta} = \dot{\theta}\,\hat{\boldsymbol{\xi}}'$$
$$\boldsymbol{\omega}_{\psi} = \dot{\psi}\,\hat{\mathbf{z}}' \quad . \qquad \text{(G.6.1)}$$

If we want to know these contributions in Frame S components, we have to replace $\hat{\boldsymbol{\xi}}'$ and $\hat{\mathbf{z}}'$ with their appropriate linear combinations of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. From (G.5.14b)

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = R_x(-\theta)\, R_z(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad (G.5.14b)$$

```
evalm(Rx(-theta) &* Rz(-phi));
```

$$\begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) & \sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

Therefore

$$\hat{\xi}' = \cos\varphi\, \hat{x} + \sin\varphi\, \hat{y} \; . \qquad \text{// this particular fact is obvious from Fig (G.5.5) top} \qquad (G.6.2)$$

From (G.5.33),

$$\hat{z}' = \sin\theta\sin\varphi\, \hat{x} - \sin\theta\cos\varphi\, \hat{y} + \cos\theta\, \hat{z} \; . \qquad (G.5.33)$$

Inserting these last two results into (G.6.1) gives,

$$\begin{aligned} \boldsymbol{\omega}_{\varphi} &= \dot{\varphi}\, \hat{z} \\ \boldsymbol{\omega}_{\theta} &= \dot{\theta}\cos\varphi\, \hat{x} + \dot{\theta}\sin\varphi\, \hat{y} \\ \boldsymbol{\omega}_{\psi} &= \dot{\psi}\sin\theta\sin\varphi\, \hat{x} - \dot{\psi}\sin\theta\cos\varphi\, \hat{y} + \dot{\psi}\cos\theta\, \hat{z} \; . \end{aligned} \qquad (G.6.3)$$

We add up to get

$$\begin{aligned} \omega &= \boldsymbol{\omega}_{\varphi} + \boldsymbol{\omega}_{\theta} + \boldsymbol{\omega}_{\psi} \\ &= [\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi]\, \hat{x} + [-\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi]\, \hat{y} + [\dot{\psi}\cos\theta + \dot{\varphi}]\, \hat{z} \end{aligned}$$

or

$$\begin{aligned} (\omega)_x &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ (\omega)_y &= -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi \\ (\omega)_z &= \dot{\psi}\cos\theta + \dot{\varphi} \; . \qquad \text{// Frame S} \end{aligned} \qquad (G.6.4)$$

These then are the Frame S components of the $\boldsymbol{\omega}$ vector.

Conversely, suppose we want (as Goldstein does want) the components of $\boldsymbol{\omega}$ in Frame S' components, Frame S' being the rotating frame in which a "rigid body" might lie. From (G.5.14e),

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = R_x(\theta) R_z(\psi) \begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} \qquad (G.5.14e)$$

```
evalm(Rx(theta) &* Rz(psi));
```

$$\begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi)\cos(\theta) & \cos(\psi)\cos(\theta) & -\sin(\theta) \\ \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Therefore

$$\hat{\xi}' = \hat{\xi} = \cos\psi\ \hat{\mathbf{x}}' - \sin\psi\ \hat{\mathbf{y}}' \ . \qquad \text{(G.6.5)}$$

This fact can be verified by staring for a while at the lower right drawing in Fig (G.5.5). There we see that $\hat{\xi}' = R_{z'}(-\psi)\hat{\mathbf{x}}'$ which implies the above. The author is prone to making errors staring at drawings and for this reason prefers the bulletproof Maple approach to computing things.

From (G.5.32),

$$\hat{\mathbf{z}} = \sin\theta\sin\psi\ \ \hat{\mathbf{x}}' + \sin\theta\cos\psi\ \hat{\mathbf{y}}' + \cos\theta\ \hat{\mathbf{z}}' \ . \qquad \textit{(G.5.32)}$$

Inserting these last two results into (G.6.1) gives,

$$\begin{aligned} \boldsymbol{\omega}_{\varphi} &= \dot{\varphi}\,\hat{\mathbf{z}} \ = \ \dot{\varphi}\sin\theta\sin\psi\ \ \hat{\mathbf{x}}' + \ \dot{\varphi}\sin\theta\cos\psi\ \hat{\mathbf{y}}' + \ \dot{\varphi}\cos\theta\ \hat{\mathbf{z}}' \\ \boldsymbol{\omega}_{\theta} &= \dot{\theta}\,\hat{\xi}' \ = \dot{\theta}\cos\psi\ \hat{\mathbf{x}}' - \dot{\theta}\sin\psi\ \hat{\mathbf{y}}' \\ \boldsymbol{\omega}_{\psi} &= \ \dot{\psi}\,\hat{\mathbf{z}}' \quad . \end{aligned} \qquad \text{(G.6.6)}$$

We add up to get

$$\begin{aligned} \omega \ &= \boldsymbol{\omega}_{\varphi} + \boldsymbol{\omega}_{\theta} + \boldsymbol{\omega}_{\psi} \\ &= [\ \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\ ]\ \hat{\mathbf{x}}' + [\ \dot{\varphi}\sin\theta\cos\psi\ - \dot{\theta}\sin\psi]\ \hat{\mathbf{y}}' \ + [\dot{\varphi}\cos\theta\ + \dot{\psi}]\ \hat{\mathbf{z}}' \end{aligned}$$

or

$$\begin{aligned} (\omega)'_{\mathbf{x}} &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \qquad \equiv \omega_{\mathbf{x}'} \\ (\omega)'_{\mathbf{y}} &= \dot{\varphi}\sin\theta\cos\psi\ - \dot{\theta}\sin\psi \qquad \equiv \omega_{\mathbf{y}'} \\ (\omega)'_{\mathbf{z}} &= \dot{\varphi}\cos\theta\ + \dot{\psi} \qquad \equiv \omega_{\mathbf{z}'} \qquad \text{// Frame S'} \end{aligned} \qquad \text{(G.6.7)}$$

These then are the Frame S' components of the $\boldsymbol{\omega}$ vector.

This result is in agreement with Goldstein page 134 (GPS page 174),

$$\begin{aligned} \omega_{x'} &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_{y'} &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_{z'} &= \dot{\phi}\cos\theta + \dot{\psi}. \end{aligned} \qquad \text{(4.87)}$$

A quick verification of (G.6.4)

From (G.5.19) we have

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \ . \qquad (G.5.19)$$

The rule for transformation of a kinematic vector is given by (1.3.3) as

$$\boldsymbol{\omega}' = (\boldsymbol{\omega})' = R\boldsymbol{\omega} \qquad (G.6.8)$$

which from (1.3.10) we can write as

$$(\omega')_i = (\omega)'_i = R_{ij}(\omega_j) = [\, R_z(-\psi)R_x(-\theta)R_z(-\varphi)\,]_{ij}(\omega_j) \ . \qquad (G.6.9)$$

Inverting,

$$(\omega)_i = [\, R_z(\varphi)R_x(\theta)R_z(\psi)]_{ij}\,(\omega)'_j \ . \qquad (G.6.10)$$

We enter the $(\omega)'_j$ components (G.6.7) into Maple and then compute the $(\omega)_i$ using (G.6.10),

```
wp1 := phid*sin(theta)*sin(psi) +thetad*cos(psi):
wp2 := phid*sin(theta)*cos(psi) -thetad*sin(psi):
wp3 := phid * cos(theta) + psid:
wp := matrix(3,1,[wp1,wp2,wp3]);
```

$$wp := \begin{bmatrix} phid\,\sin(\theta)\,\sin(\psi) + thetad\,\cos(\psi) \\ phid\,\sin(\theta)\,\cos(\psi) - thetad\,\sin(\psi) \\ phid\,\cos(\theta) + psid \end{bmatrix}$$

```
w := evalm(Rz(phi) &* Rx(theta) &* Rz(psi) &* wp):simplify(%);
```

$$\begin{bmatrix} \cos(\phi)\,thetad + \sin(\theta)\,\sin(\phi)\,psid \\ \sin(\phi)\,thetad - \sin(\theta)\,\cos(\phi)\,psid \\ phid + \cos(\theta)\,psid \end{bmatrix} \qquad (G.6.11)$$

Transcribing the result gives,

$$(\omega)_x = \cos\varphi\ \dot{\theta} + \sin\theta\sin\varphi\ \dot{\psi}$$
$$(\omega)_y = \sin\varphi\ \dot{\theta} - \sin\theta\cos\varphi\ \dot{\psi}$$
$$(\omega)_z = \dot{\varphi} + \cos\theta\ \dot{\psi} \qquad (G.6.12)$$

This agrees with (G.6.4) above, providing some verification for our Frame S components of **ω**.

G.7 Computation of ω (Method 2)

In this section our approach to computing $\boldsymbol{\omega}$ for the Euler Angle rotation is to find an equation which involves $\boldsymbol{\omega}$ and solve it for $\boldsymbol{\omega}$! More or less at random, we choose (1.7.4),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \; . \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

Unlike our Method 1 computation of $\boldsymbol{\omega}$ in the previous section, here we shall have no need for Goldstein's geometric Figure (G.5.5) or intermediate angles like ξ and ζ. We will, however, need various algebraic results developed in Section G.3.

This calculation is a bit slippery and requires care and precision in the use of notation -- it is easy to go astray. We shall use both matrix and Section 1.1 Dirac notations as seem convenient. A silver lining is that we shall be able to apply many of the results derived earlier in this document.

Recall from (G.5.17a) that the Goldstein Euler angle rotation is given by

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \; . \qquad \textit{(G.5.17a)} \qquad \text{(G.7.2)}$$

In order to avoid a hundred minus signs, we shall *temporarily negate* all three angles. Then when we are done, we will undo this negation. We therefore temporarily take R to be,

$$R(\psi,\theta,\varphi) = R_z(\psi)R_x(\theta)R_z(\varphi) \equiv R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi} \bullet \mathbf{J}) \; . \qquad \text{// temp} \qquad \text{(G.7.3)}$$

Comment: To find $\boldsymbol{\Phi}$ we could write out the matrix $R_z(\psi)R_x(\theta)R_z(\varphi)$,

```
R := evalm(Rz(psi) &* Rx(theta) &* Rz(phi));
```

$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

and decompose it into its symmetric and antisymmetric components S and A. *In theory* we could then compute Φ (=θ) and $\hat{\boldsymbol{\Phi}}$ (=$\hat{\mathbf{n}}$) from (G.2.14) and come up with an explicit expression for $\boldsymbol{\Phi} = \Phi\hat{\boldsymbol{\Phi}}$. The reader is just reminded that this is mechanically possible, but luckily we have no need for the result (which is quite complicated). We could also compute $d\boldsymbol{\Phi}$ from the following

$$R(\psi+d\psi,\theta+d\theta,\varphi+d\varphi) = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = \exp(-i\,[\boldsymbol{\Phi}+d\boldsymbol{\Phi}] \bullet \mathbf{J})$$

but again there is no need to do this. Note that $\boldsymbol{\Phi}$ and $d\boldsymbol{\Phi}$ will generally not be in the same direction.

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n(\boldsymbol{\Phi}) = R(\boldsymbol{\Phi})\mathbf{e}'_n \qquad \text{or} \qquad |\mathbf{e}_n(\boldsymbol{\Phi})\rangle = |R(\boldsymbol{\Phi})\mathbf{e}'_n\rangle = \mathcal{R}(\boldsymbol{\Phi})\,|\mathbf{e}'_n\rangle \; . \qquad \text{(G.7.4)}$$

Since we have in mind that $\boldsymbol{\Phi} = \boldsymbol{\Phi}(t)$, one sees that the Frame S basis vectors change in time as viewed from Frame S'. In contrast, the basis vectors $\mathbf{e'}_n$ are static. The basis vectors $|\mathbf{e}_n(\boldsymbol{\Phi})\rangle$ are complete at time t so we can write, in analogy with (1.1.20),

$$\boldsymbol{1} = |\mathbf{e}_n(\boldsymbol{\Phi})\rangle\langle\mathbf{e}_n(\boldsymbol{\Phi})| \qquad \text{completeness of the } \mathbf{e}_n \text{ at time t} \qquad \text{(a)}$$

$$\boldsymbol{1} = |\mathbf{e}_n(\boldsymbol{\Phi}+d\boldsymbol{\Phi})\rangle\langle\mathbf{e}_n(\boldsymbol{\Phi}+d\boldsymbol{\Phi})| \qquad \text{completeness of the } \mathbf{e}_n \text{ at time t+dt} \qquad \text{(b)}$$

$$\boldsymbol{1} = |\mathbf{e'}_n\rangle\langle\mathbf{e'}_n| \qquad \text{completeness of the } \mathbf{e'}_n \text{ at any time} \qquad \text{(c)} \qquad \text{(G.7.5)}$$

In (b) the rotation vector has changed from $\boldsymbol{\Phi}$ to some $\boldsymbol{\Phi}+d\boldsymbol{\Phi}$ as time moved from t to d+dt. A key point is that the $\mathbf{e}_n$ basis vectors are complete at any point in time. The rotation operator $\boldsymbol{\mathcal{R}}(\boldsymbol{\Phi})$ similarly is real orthogonal at any time, analogous to (1.1.37),

$$\boldsymbol{\mathcal{R}}^{\mathrm{T}}(\boldsymbol{\Phi}) = \boldsymbol{\mathcal{R}}^{-1}(\boldsymbol{\Phi}) \qquad \text{and similarly for matrices} \qquad R^{\mathrm{T}}(\boldsymbol{\Phi}) = R^{-1}(\boldsymbol{\Phi})\ . \qquad \text{(G.7.6)}$$

Now we close the Dirac equation in (G.7.4) on the left with $\langle\mathbf{e'}_m|$ to get

$$\langle\mathbf{e'}_m|\ \mathbf{e}_n(\boldsymbol{\Phi})\rangle = \langle\mathbf{e'}_m|\boldsymbol{\mathcal{R}}(\boldsymbol{\Phi})\ |\mathbf{e'}_n\rangle = [R(\boldsymbol{\Phi})]'_{mn}\ . \qquad \text{(G.7.7a)}$$

Since this is true for any $\boldsymbol{\Phi}$, one also has

$$\langle\mathbf{e'}_m|\ \mathbf{e}_n(\boldsymbol{\Phi}+d\boldsymbol{\Phi})\rangle = \langle\mathbf{e'}_m|\boldsymbol{\mathcal{R}}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})\ |\mathbf{e'}_n\rangle = [R(\boldsymbol{\Phi}+d\boldsymbol{\Phi})]'_{mn}\ . \qquad \text{(G.7.7.b)}$$

Recall from (1.1.35) that $(R)'_{ij} = R_{ij}$. Here we confirm that fact in the current fancier notation,

$$[R(\boldsymbol{\Phi})]_{mn} = \langle\mathbf{e}_m(\boldsymbol{\Phi})|\ \boldsymbol{\mathcal{R}}(\boldsymbol{\Phi})\ |\mathbf{e}_n(\boldsymbol{\Phi})\rangle$$

$$= \langle\mathbf{e}_m(\boldsymbol{\Phi})\ |\mathbf{e'}_i\rangle\langle\mathbf{e'}_i|\ \boldsymbol{\mathcal{R}}(\boldsymbol{\Phi})\ |\mathbf{e'}_j\rangle\langle\mathbf{e'}_j|\mathbf{e}_n(\boldsymbol{\Phi})\rangle = [R(\boldsymbol{\Phi})]'_{im}\ [R(\boldsymbol{\Phi})]'_{ij}[R(\boldsymbol{\Phi})]'_{jn}$$

$$= [\ R^{\mathrm{T}}(\boldsymbol{\Phi})R(\boldsymbol{\Phi})R(\boldsymbol{\Phi})]'_{mn} = [\ R^{-1}(\boldsymbol{\Phi})R(\boldsymbol{\Phi})R(\boldsymbol{\Phi})]'_{mn} = [R(\boldsymbol{\Phi})]'_{mn}\ . \qquad \text{(G.7.8a)}$$

Since this is true for any $\boldsymbol{\Phi}$, one also has

$$[R(\boldsymbol{\Phi}+d\boldsymbol{\Phi})]_{mn} = [R(\boldsymbol{\Phi}+d\boldsymbol{\Phi})]'_{mn}\ . \qquad \text{(G.7.8b)}$$

DETERMINATION OF THE FRAME S' COMPONENTS OF $\boldsymbol{\omega}$

First expression for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

We now examine (from Frame S') a small change in the basis vector $\mathbf{e}_n(\boldsymbol{\Phi})$,

$$|(d\mathbf{e}_n(\boldsymbol{\Phi}))_{S'}\rangle \equiv |\mathbf{e}_n(\boldsymbol{\Phi}+d\boldsymbol{\Phi})\rangle - |\mathbf{e}_n(\boldsymbol{\Phi})\rangle$$

$$= |\mathbf{e'}_i\rangle\langle\mathbf{e'}_i|\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle - |\mathbf{e'}_i\rangle\langle\mathbf{e'}_i|\mathbf{e}_n(\mathbf{\Phi})\rangle \qquad \text{// completeness twice}$$

$$= |\mathbf{e'}_i\rangle\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - |\mathbf{e'}_i\rangle[R(\mathbf{\Phi})]'_{in} \qquad \text{// (G.7.7b,a)}$$

$$= (\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - [R(\mathbf{\Phi})]'_{in}\,)\;|\mathbf{e'}_i\rangle \qquad \text{// reorder}$$

$$= (\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]_{in} - [R(\mathbf{\Phi})]_{in}\,)\;|\mathbf{e'}_i\rangle \qquad \text{// (G.7.8b,a) to remove primes}$$

$$= (\,R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})\,)_{in}\;|\mathbf{e'}_i\rangle$$

$$= (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi})\,)_{ni}\;|\mathbf{e'}_i\rangle\,. \qquad \text{(G.7.9)}$$

The next step is to replace $|\mathbf{e'}_i\rangle$ as follows

$$|\mathbf{e'}_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})|\,\mathbf{e'}_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle[R(\mathbf{\Phi})]'_{ij} = [R(\mathbf{\Phi})]_{ij}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle\,. \qquad \text{(G.7.10)}$$

Then

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi})\,)_{ni}[R(\mathbf{\Phi})]_{ij}\,|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$= (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - R^T(\mathbf{\Phi})R(\mathbf{\Phi})\,)_{nj}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$= (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)_{nj}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle\,. \qquad \text{(G.7.11)}$$

We then add dt/dt to the left side to get

$$dt\;|(d\mathbf{e}_n(\mathbf{\Phi})/dt\,)_{S'}\rangle = (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)_{nj}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle\,. \qquad \text{(G.7.12)}$$

We wish to evaluate the above vector equation in Frame S' components. To do this, we close both sides with $\langle\mathbf{e'}_i|$, obtaining

$$dt\;\langle\mathbf{e'}_i\;|(d\mathbf{e}_n(\mathbf{\Phi})/dt\,)_{S'}\rangle = (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)_{nj}\;\;\langle\mathbf{e'}_i\,|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

or

$$dt\;[(d\mathbf{e}_n/dt)_{S'}]'_i = (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)_{nj}\;R(\mathbf{\Phi})_{ij}$$

$$= (\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)_{nj}R^T(\mathbf{\Phi})_{ji} = [(\,R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1\,)R^T(\mathbf{\Phi})]_{ni}$$

$$= [\,R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi})\,]_{ni} = [\,R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})\,]_{in}$$

$$\equiv (dR)_{in} \quad \text{where} \quad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})\,. \qquad \text{(G.7.13)}$$

Divide both sides by dt to obtain

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = (dR/dt)_{in} \qquad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \quad . \tag{G.7.14}$$

Second expression for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

Recall (G.7.1),

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \textit{(G.7.1)}$$

We evaluate the above vector equation in Frame S' components,

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ikc}(\omega)'_k(\mathbf{e}_n)'_c = -\,\varepsilon_{ikc}(\omega)'_k\,[R(\mathbf{\Phi})]'_{cn} = -\varepsilon_{ikc}(\omega)'_k\,[R(\mathbf{\Phi})]_{cn} \,. \tag{G.7.15}$$

Equate the two expressions for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

At this point we have shown that doing component evaluations of $(d\mathbf{e}_n/dt)_{S'}$ in Frame S' gives,

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = (dR/dt)_{in} \qquad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \qquad \textit{(G.7.14)}$$

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ikc}(\omega)'_k\,[R(\mathbf{\Phi})]_{cn} \,. \qquad \textit{(G.7.15)}$$

Setting the right sides equal, one obtains,

$$-\,\varepsilon_{ikc}(\omega)'_k\,[R(\mathbf{\Phi})]_{cn} = (dR/dt)_{in} \,.$$

Multiply both sides on the right by $[R^{-1}(\mathbf{\Phi})]_{nj}$ to get

$$-\,\varepsilon_{ikc}(\omega)'_k\,[R(\mathbf{\Phi})]_{cn}\,[R^{-1}(\mathbf{\Phi})]_{nj} = (dR/dt)_{in}\,[R^{-1}(\mathbf{\Phi})]_{nj}$$

or

$$-\,\varepsilon_{ikc}(\omega)'_k\delta_{cj} = [\,(dR/dt)R^{-1}(\mathbf{\Phi})\,]_{ij}$$

or

$$-\,\varepsilon_{ikj}(\omega)'_k = [\,(dR/dt)R^{-1}(\mathbf{\Phi})\,]_{ij}$$

or

$$\varepsilon_{ijk}(\omega)'_k = A_{ij} \qquad \text{where} \qquad A \equiv (dR/dt)R^{-1}(\mathbf{\Phi}) \,. \tag{G.7.16}$$

It seems that the object A_{ij} must be antisymmetric in its two indices, so the matrix A_{ij} has only three significant elements. Setting ijk = 231 gives the first line below, then the next two lines follow from cyclic permutation :

$$(\omega)'_1 = A_{23}$$
$$(\omega)'_2 = A_{31}$$
$$(\omega)'_3 = A_{12} \,. \tag{G.7.17}$$

Thus we have succeeded in solving for the components of $\boldsymbol{\omega}$ in Frame S' . Recall that Frame S' is rotating at rate $\boldsymbol{\omega}$ relative to Frame S as in Fig 1. It remains to compute the three A_{ij} matrix elements so we can learn the specific expressions for the $(\omega)'_i$ in terms of Euler Angles.

Computation of A and Statement of Final Result

Our task is to compute $A \equiv (dR/dt)R^{-1}(\boldsymbol{\Phi})$ where our "temporary" $R(\boldsymbol{\Phi})$ is given by

$$R(\boldsymbol{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi) \qquad (G.7.3)$$

and where

$$dR = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi}) \ . \qquad (G.7.13)$$

The first step is to compute dR in terms of the Euler angles. We make use of the obvious fact that $R_i(\alpha +d\alpha) = R_i(\alpha)R_i(d\alpha)$ and then the fact (1.5.6) that $R_i(d\alpha) \approx 1 - id\alpha\, J_i$ for small $d\alpha$. Keeping only terms of first order in the differential angles, one finds

$$dR = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi})$$

$$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= R_z(\psi)\, R_z(d\psi)\, R_x(\theta)\, R_x(d\theta)\, R_z(\varphi)\, R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= R_z(\psi)\{1-id\psi J_3\}R_x(\theta)\{1-id\theta J_1\}R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= -id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3 \qquad (G.7.18)$$

where the leading terms exactly cancel. Dividing by dt one then has

$$(dR/dt) = -i\dot{\psi}\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - i\dot{\theta}\, R_z(\psi)R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)R_x(\theta)R_z(\varphi)J_3 \ . \qquad (G.7.19)$$

The next step is to compute $A \equiv (dR/dt)R^{-1}(\boldsymbol{\Phi})$ using $R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)R_z(-\psi)$. In doing so, we shall three times in blue use the fact that J_i commutes with R_i as formally stated in (G.3.6) :

$$A \equiv (dR/dt)\ R^{-1}(\boldsymbol{\Phi})$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3R_x(\theta)R_z(\varphi) - i\dot{\theta}\, R_z(\psi)R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)R_x(\theta)R_z(\varphi)J_3] \, R_z(-\varphi)R_x(-\theta)R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3R_z(-\varphi)\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ \ R_z(\psi)J_3\ \ -\ i\dot{\theta}\ \ R_z(\psi)\ R_x(\theta)J_1R_x(-\theta)\ \ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ \ ]\ \ R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ \ R_z(\psi)J_3\ \ -\ i\dot{\theta}\ \ R_z(\psi)\ J_1\ \ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ \ ]\ \ R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ \ R_z(\psi)J_3\ R_z(-\psi)\ \ -\ i\dot{\theta}\ \ R_z(\psi)\ J_1\ R_z(-\psi)\ \ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ R_z(-\psi)\ \ ]$$

$$= [\ -i\dot{\psi}\ \ J_3\ \ -\ i\dot{\theta}\ \ R_z(\psi)\ J_1\ R_z(-\psi)\ \ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ R_z(-\psi)\ \ ]$$

$$= [\ -i\dot{\psi}\ \ J_3\ \ -\ i\dot{\theta}\ \ M_1\ \ -\ i\dot{\varphi}\ M_2\ ]$$

where (G.7.20)

$$M_1 \equiv R_z(\psi)\ J_1R_z(-\psi) \qquad = [\ R_3(\psi)\ J_1R_3(-\psi)]$$

$$M_2 \equiv R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi) \qquad = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)\ .$$

We now call upon our non-trivial sandwich formulas in (G.3.8) to simplify thing further :

$$M_1 = \ R_3(\psi)\ J_1R_3(-\psi)\ = \cos\psi J_1 + \sin\psi\ J_2 \qquad \text{// (G.3.8) line 5}$$

$$M_2 = \ R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)$$

$$= R_3(\psi)\ [\ \cos\theta\ J_3 - \sin\theta\ J_2\ ]\ R_3(-\psi) \qquad \text{// (G.3.8) line 2}$$

$$= \cos\theta\ [\ R_3(\psi)\ J_3\ R_3(-\psi)]\ - \sin\theta\ [R_3(\psi)J_2R_3(-\psi)]$$

$$= \cos\theta\ J_3\ - \sin\theta\ (\cos\psi J_2 - \sin\psi\ J_1) \qquad \text{// (G.3.6) and (G.3.8) line 6}$$

$$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3\ . \qquad \text{(G.7.21)}$$

Then,

$$A\ = [\ -i\dot{\psi}\ \ J_3\ \ -\ i\dot{\theta}\ \ M_1\ \ -\ i\dot{\varphi}\ M_2\ ]$$

$$= [\ -i\ \dot{\psi}\ \ J_3\ \ -\ i\ \dot{\theta}\ (\cos\psi J_1 + \sin\psi\ J_2)\ \ -\ i\ \dot{\varphi}\ (\sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 +\ \cos\theta\ J_3)\ ]$$

$$= [\ -i\ \dot{\psi}\ \ J_3\ \ -\ i\ \dot{\theta}\ \cos\psi J_1\ \ -\ i\dot{\theta}\ \sin\psi\ J_2\ \ -\ i\ \dot{\varphi}\ \sin\theta\sin\psi J_1 +\ i\ \dot{\varphi}\sin\theta\cos\psi J_2 - i\ \dot{\varphi}\ \ \cos\theta\ J_3\ ]$$

$$= (-i)[\ \dot{\psi}\ \ J_3\ +\dot{\theta}\ \cos\psi J_1\ \ +\dot{\theta}\ \sin\psi\ J_2\ \ +\dot{\varphi}\ \sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2\ +\dot{\varphi}\ \ \cos\theta\ J_3\ ]$$

$$= (-i)[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)J_1 + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)J_2 + (\dot\psi + \dot\varphi\cos\theta)J_3\ ]$$

$$= -[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)(iJ_1) + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)(iJ_2) + (\dot\psi + \dot\varphi\cos\theta)(iJ_3)\ ]. \qquad (G.7.22)$$

We now take the ij element of this matrix using the fact (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$,

$$A_{ij} = -[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)\varepsilon_{1ij} + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot\psi + \dot\varphi\cos\theta)\varepsilon_{3ij}\ ]\ . \qquad (G.7.23)$$

The tensor ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A is antisymmetric (as conjectured earlier) and thus has only three distinct matrix elements. They are:

$$A_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)\varepsilon_{123} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)$$

$$A_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)\varepsilon_{231} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)$$

$$A_{12} = -(\dot\psi + \dot\varphi\cos\theta)\,\varepsilon_{312} = -(\dot\psi + \dot\varphi\cos\theta)\ . \qquad (G.7.24)$$

From (G.7.17) we then conclude that

$$(\omega)'_1 = A_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)$$
$$(\omega)'_2 = A_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)$$
$$(\omega)'_3 = A_{12} = -(\dot\psi + \dot\varphi\cos\theta)\ . \qquad \text{// angles still negated}$$

We now undo the temporary negation of the angles ψ,θ,φ enacted below (G.7.2). The velocities and sines then negate. Our final result for the Frame S' components of **ω** is then,

$$(\omega)'_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot\varphi\cos\theta + \dot\psi \equiv \omega_{z'}\ . \qquad (G.7.25)$$

This is in agreement with our Method 1 calculation (G.6.7) and with Goldstein's result which we again quote from Goldstein page 134 (GPS page 174),

$$\omega_{x'} = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\omega_{y'} = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\omega_{z'} = \dot\phi\cos\theta + \dot\psi. \qquad (4.87)$$

DETERMINATION OF THE FRAME S COMPONENTS OF **ω**

Here we shall review three different Plans for computing the Frame S components of $\boldsymbol{\omega}$.

Plan A: Since we just computed the $(\omega)'_i$ in Frame S', we just use $\boldsymbol{\omega} = R^{-1}(\boldsymbol{\omega})'$ to get the Frame S components. This was done at the end of Section G.6 with the result stated in (G.6.12).

Plan B: Start with (1.7.1) that $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega}$ x $\mathbf{e}'_n$ in place of (G.7.1) that $(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega}$ x $\mathbf{e}_n$, which adds an overall minus sign to the result. The new (G.7.4) becomes $\mathbf{e}'_n(\boldsymbol{\Phi}) = R^{-1}(\boldsymbol{\Phi})\mathbf{e}_n$ with $\mathbf{e}_n$ = constant in Frame S. Things go through as presented above, but since $R(\boldsymbol{\Phi}) \rightarrow R^{-1}(\boldsymbol{\Phi})$ and since $R(\boldsymbol{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi)$, one has $R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)\,R_z(-\psi)$. Thus, to convert the result (G.7.25) to the Frame S result, we have to make these changes: (1) $\varphi, \psi, \theta \rightarrow -\psi, -\varphi, -\theta$; (2) add the overall minus sign just noted. We do that right here:

$$\varphi, \psi, \theta \rightarrow -\psi, -\varphi, -\theta$$

$$\begin{aligned}
(\omega)'_1 &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \equiv \omega_{x'} \\
(\omega)'_2 &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \equiv \omega_{y'} \\
(\omega)'_3 &= \dot{\varphi}\cos\theta + \dot{\psi} \equiv \omega_{z'} \ . \qquad \textit{(G.7.25)}
\end{aligned}$$

$\rightarrow$

$$\begin{aligned}
-(\omega)_1 &= [-\dot{\psi}][-\sin\theta][-\sin\varphi] + [-\dot{\theta}]\cos\varphi \quad \Rightarrow \omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\
-(\omega)_2 &= [-\dot{\psi}][-\sin\theta]\cos\varphi - [-\dot{\theta}][-\sin\varphi] \quad \Rightarrow \omega_2 = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi \\
-(\omega)_3 &= [-\dot{\psi}]\cos\theta + [-\dot{\varphi}] \quad \Rightarrow \omega_3 = \dot{\psi}\cos\theta + \dot{\varphi} \qquad \text{(G.7.26)}
\end{aligned}$$

and this agrees with (G.6.12).

Plan C: Compute the ω_i directly using the machinery developed earlier in this section, but take Frame S components instead of Frame S' components. We regard this as a "stress test" of the machinery.

First expression for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

Start with (G.7.11)

$$|(d\mathbf{e}_n(\boldsymbol{\Phi}))_{S'}\rangle = \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} |\mathbf{e}_j(\boldsymbol{\Phi})\rangle \ . \qquad \textit{(G.7.11)}$$

Instead of closing on the left with $\langle\mathbf{e}'_i|$ to get Frame S' components, this time close on the left with $\langle\mathbf{e}_i|$ to get Frames S components,

$$[(d\mathbf{e}_n)_{S'}]_i = \langle\mathbf{e}_i|(d\mathbf{e}_n(\boldsymbol{\Phi}))_{S'}\rangle = \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} \langle\mathbf{e}_i(\boldsymbol{\Phi})|\mathbf{e}_j(\boldsymbol{\Phi})\rangle$$

$$= \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} \delta_{ji} \qquad \text{// (1.1.14)}$$

$$= \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} (R^{-1}(\boldsymbol{\Phi})R(\boldsymbol{\Phi}))_{ji} = [\left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)R^{-1}(\boldsymbol{\Phi})R(\boldsymbol{\Phi})]_{ni}$$

$$= [\left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R^T(\boldsymbol{\Phi}) \right) R(\boldsymbol{\Phi})]_{ni} = [R^T(\boldsymbol{\Phi})\left(R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi}) \right)]_{in}$$

$$= [R^{T}(\mathbf{\Phi})(dR)]_{in} = [R^{T}(\mathbf{\Phi})(dR)\, R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi})]_{in}$$

and dividing by dt,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [R^{T}(\mathbf{\Phi})(dR/dt)R^{-1}(\mathbf{\Phi})\, R(\mathbf{\Phi})]_{in} = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \qquad \text{(G.7.27)}$$

where recall that $A \equiv (dR/dt)R^{-1}(\mathbf{\Phi})$ from (G.7.16).

Second expression for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

Recall (G.7.1),

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \textit{(G.7.1)}$$

We evaluate the above vector equation in Frame S components,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\,\varepsilon_{ikc}\omega_k(\mathbf{e}_n)_c = -\,\varepsilon_{ikc}\omega_k\delta_{nc} = -\,\varepsilon_{ikn}\omega_k \,. \qquad \text{(G.7.28)}$$

Equate the two expressions for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

At this point we have shown that doing component evaluations of $(d\mathbf{e}_n/dt)_{S'}$ in Frame S gives,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \,. \qquad \textit{(G.7.27)}$$

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\,\varepsilon_{ikn}\omega_k \,. \qquad \textit{(G.7.28)}$$

Setting the right sides equal, we obtain

$$-\,\varepsilon_{ikn}\omega_k = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \equiv B_{in}$$

or

$$\varepsilon_{ink}\omega_k = B_{in}$$

or

$$\varepsilon_{ijk}\omega_k = B_{ij} \qquad B = R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi}) \,, \qquad \text{(G.7.29)}$$

Thus we arrive at (G.7.16) with $(\omega)'_i$ replaced by ω_i and with A replaced by B,

$$\omega_1 = B_{23}$$
$$\omega_2 = B_{31}$$
$$\omega_3 = B_{12} \,. \qquad \text{(G.7.30)}$$

It remains only to compute the B_{ij}.

Computation of B and Statement of Final Result

Recall from (G.7.22) that

$$A = -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)] \quad . \qquad (G.7.22)$$

We may write this as

$$A = a_i J_i$$

where (G.7.31)

$$a_1 = -i(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)$$

$$a_2 = -i(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$a_3 = -i(\dot{\psi} + \dot{\varphi}\cos\theta) \; .$$

Entering into Maple,

```
a1 := -I*(td*cos(psi)+phid*sin(theta)*sin(psi));
```

$$a1 := -I\,(td\cos(\psi) + phid\sin(\theta)\sin(\psi))$$

```
a2 := -I*(td*sin(psi)-phid*sin(theta)*cos(psi));
```

$$a2 := -I\,(td\sin(\psi) - phid\sin(\theta)\cos(\psi))$$

```
a3 := -I*(psid+phid*cos(theta));
```

$$a3 := -I\,(psid + phid\cos(\theta)) \quad . \qquad (G.7.32)$$

Then the matrix B in (G.7.29) can be written,

$$B = R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi}) \;=\; R^{-1}(\mathbf{\Phi})[a_i J_i]\,R(\mathbf{\Phi})$$

$$= a_i\,[R^{-1}(\mathbf{\Phi})J_i R(\mathbf{\Phi})[\;=\; a_i R(\mathbf{\Phi})_{ij} J_j \qquad \text{// Theorem 1 of (G.4.1) with } R \to R^{-1}$$

$$= a_i Q_i \qquad \text{where } Q_i \equiv R(\mathbf{\Phi})_{ij} J_j \; . \qquad (G.7.33)$$

Using our "temporary" R(**Φ**) in (G.7.3) we compute the vector Q as follows (a vector of matrices),

```
Q := evalm(Rz(psi) &* Rx(theta) &* Rz(phi) &* matrix(3,1,[J1,J2,J3]));
```

$$Q := \begin{bmatrix} (\cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi))\,J2 + \sin(\psi)\sin(\theta)\,J3 \\ (\sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi))\,J2 - \cos(\psi)\sin(\theta)\,J3 \\ \sin(\theta)\sin(\phi)\,J1 + \sin(\theta)\cos(\phi)\,J2 + \cos(\theta)\,J3 \end{bmatrix}$$

(G.7.34)

The quantity $B = a_i Q_i$ is then,

```
B := evalm(matrix(1,3,[a1,a2,a3]) &* Q):  B := simplify(B);
```

$$B := [-I\,psid\cos(\theta)\,J3 - I\,phid\,J3 - I\,td\,J1\cos(\phi) + I\,td\,J2\sin(\phi) - I\,psid\sin(\theta)\sin(\phi)\,J1 - I\,psid\sin(\theta)\cos(\phi)\,J2]$$

```
B := collect(B[1,1],[J1,J2,J3]);
```

$$B := (-I\,td\cos(\phi) - I\,psid\sin(\theta)\sin(\phi))\,J1 + (I\,td\sin(\phi) - I\,psid\sin(\theta)\cos(\phi))\,J2 + (-I\,psid\cos(\theta) - I\,phid)\,J3$$

(G.7.35)

We move the factors of i (Maple I) next to the J_k and transcribe the last line above:

$$B = (-\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi)(iJ_1) + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)(iJ_2) + (-\dot{\psi}\cos\theta - \dot{\varphi})(iJ_3) \quad \text{(G.7.36)}$$

Using (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$ the matrix elements of B are then,

$$B_{ij} = (-\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi)\varepsilon_{1ij} + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)\varepsilon_{2ij} + (-\dot{\psi}\cos\theta - \dot{\varphi})\varepsilon_{3ij} \ . \quad \text{(G.7.37)}$$

Therefore,

$$B_{23} = -\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi$$
$$B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$B_{12} = -\dot{\psi}\cos\theta - \dot{\varphi} \ . \quad \text{(G.7.38)}$$

Then from (G.7.30),

$$\omega_1 = B_{23} = -\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi$$
$$\omega_2 = B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$\omega_3 = B_{12} = -\dot{\psi}\cos\theta - \dot{\varphi} \ . \quad \text{// angles still negated}$$

We now undo the temporary negation of the Euler angles ψ,θ,φ enacted below (G.7.2). The velocities and sines then negate. Our final result for the Frame S components of **ω** is then,

$$\omega_1 = B_{23} = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi$$
$$\omega_2 = B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$\omega_3 = B_{12} = \dot{\psi}\cos\theta + \dot{\varphi} \quad \text{(G.7.39)}$$

This is in agreement which (G.6.11) which we now quote,

$$(\omega)_x = \cos\varphi\,\dot{\theta} + \sin\theta\sin\varphi\,\dot{\psi}$$
$$(\omega)_y = \sin\varphi\,\dot{\theta} - \sin\theta\cos\varphi\,\dot{\psi}$$
$$(\omega)_z = \dot{\varphi} + \cos\theta\,\dot{\psi} \quad \textit{(G.6.11)}$$

Review of this Section

Frame S and Frame S' are related by some angular velocity **ω** as shown in Fig 1. The actual rotation relating the two frames is $R = R_z(-\psi)R_x(-\theta)R_z(-\varphi)$ as in (G.7.2) and (G.5.17a). If the three Euler angles are all static, **ω** would be 0. It is the fact that one or more of these Euler angles varies in time which causes **ω** to be non-zero.

Our task was to solve the following equation for $\boldsymbol{\omega}$

$$(d\mathbf{e_n})_{S'} = -\boldsymbol{\omega} \times \mathbf{e_n}\, dt \ . \qquad (1.7.4) \qquad (G.7.1)$$

Acting as an Observer in Frame S', we studied the change in the basis vectors $(d\mathbf{e_n})_{S'}$ caused by the time-varying Euler angles. In (G.7.11) we obtained the following vector equation for this change,

$$(d\mathbf{e_n})_{S'} = [\, R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,]_{nj}\, \mathbf{e_j} \ . \qquad (G.7.11)$$

Equating the right sides of the above two equations, we obtained the following vector equation,

$$-\boldsymbol{\omega} \times \mathbf{e_n}\, dt = (\, R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,)_{nj}\, \mathbf{e_j} \ . \qquad (G.7.40)$$

We found that, to lowest order, the quantity $(R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,)_{nj}$ was linear in the differential angles $d\psi$,$d\theta$ and $d\varphi$ and when both sides were divided by dt, these became rates $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$.

Although we were observing things from Frame S', we were allowed to take components of any vector equation in either Frame S or in Frame S'.

By taking the Frame S' components of (G.7.40) we obtained the Frame S' components of $\boldsymbol{\omega}$ as in (G.7.25).

By taking the Frame S components of (G.7.40) we obtained the Frame S components of $\boldsymbol{\omega}$ as in (G.7.39).

Along the way we got to exercise some earlier results of this Appendix: the sandwich formulas (G.3.6) and (G.3.8) in the computation of A, Theorem 1 of (G.4.1) saying $RJ_iR^{-1} = R^{-1}{}_{ij}J_j$ in the computation of B, and the rotation generator matrix representations of (G.1.3). Finally, we were able to exercise the Dirac notation of Section 1.1. The approach of this section made use of the "linear combination" side of the Basis Theorem (1.1.29) rather than the "operator" side, so matrices appeared right from the get-go.

G.9 Generalizations of the Rotation Group

Here we consider some generalizations of the ideas presented above in Section G.1

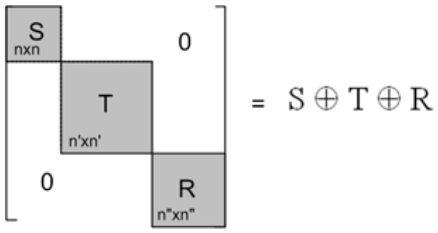
N dimensional representations of the rotation group

The generators J_i shown in (G.1.2) are a special case of a more general idea which starts with the commutation relation

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \ . \qquad \text{// for example, } [J_1,J_2] = iJ_3 \qquad (G.1.4) \qquad (G.9.1)$$

One first thinks of the J_i as abstract "operators" in some abstract "operator space". One can show that it is possible to find a set of three NxN matrices of any integer dimension N which satisfy (G.1.4). The matrices are not unique, so (G.1.2) for the J_i in three dimensions is not unique, but it is a standard form.

The three NxN generator matrices J_i are said to form an N-dimensional "irreducible representation" of the abstract generators J_i. One can always create new viable generator matrices by taking a "direct sum" of existing viable generator matrices, such as in this block-diagonal-form picture



(G.9.2)

This generator matrix is "reducible" into a direct sum of S, T and R. An "irreducible" representation is one that cannot be reduced in this manner. The same comment applies to rotation matrices.

The commutation relation (G.9.1) is an example of a **Lie Algebra**. Our particular Lie Algebra is called so(3), so we can represent the algebra elements J_i of this algebra by three NxN matrices J_i. It is possible to write down a formula analogous to (G.1.3) $(iJ_a)_{bc} = \varepsilon_{abc}$ which works for any N, but (G.1.3) itself only applies to N = 3. This is so because ε_{abc} has no meaning for $N \neq 3$. But it always has meaning in (G.9.1) because there are only three generators regardless of the value of N.

For general N, the object $R_i(\theta) = \exp(-i\theta \mathbf{J}_i)$ is an NxN matrix which represents the action of a rotation of an N-vector in a Euclidean space E^N. The set of such rotation matrices forms an "irreducible representation" of the rotation group SO(3) in N dimensions. The integer N is usually written N = 2j+1 where j = 0, 1/2, 1, 3/2 ... and this j then serves as a label for a given matrix representation.

The value of j in N = 2j+1 is associated with "angular momentum" or "spin". In the case N=2 (having j=1/2) the generator matrices are the 2x2 "Pauli matrices". In this case the 2x2 matrices $\exp(-i\theta J_i)$ describe the rotations of spin-1/2 particles such as electrons or protons. Here are the details for N=2 :

$$J_x = (1/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_y = (1/2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad J_z = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_x = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_z = \begin{pmatrix} \exp(-i\frac{\theta}{2}) & 0 \\ 0 & \exp(+i\frac{\theta}{2}) \end{pmatrix}$$

(G.9.3)

The "vectors" for spin-1/2 particles have two components $\begin{pmatrix} u \\ d \end{pmatrix}$. The special case $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is called "spin up" and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is "spin down".

For the Lie Algebra so(3) one can show that $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ and any particular J_i commute with each other, so $[J^2, J_i] = 0$. J^2 is called a Casimir operator of this Algebra, and fancier Lie Algebras can have several such Casimirs.

A Differential operator representation of the rotation group

It is also possible to "represent" the three rotation generators J_i by three differential operators in spherical coordinates θ and φ. These operators satisfy $[J_i, J_j] = i\varepsilon_{ijk}J_k$ and in this context they are usually called L_i but we shall stick with J_i. In this case, one can compute the differential operator J^2 and one can ponder differential equations which take the form $J^2\, f_{jm}(\theta,\varphi) = j(j+1)\, f_{jm}(\theta,\varphi)$ and $J_z\, f_{jm}(\theta,\varphi) = m\, f_{jm}(\theta,\varphi)$. [The facts that the eigenvalue of J^2 is j(j+1) and not j^2, and that m runs from -j to j, derive from the structure of the Lie Algebra.] The solutions $f_{jm}(\theta,\varphi)$ are able to have well-defined eigenvalues j(j+1) and m *because* $[J^2,J_3] = 0$. If we instead had $[J^2,J_3] \neq 0$, then $[J^2,J_3]\, f_{jm}(\theta,\varphi) = j(j+1)m - mj(j+1) = 0$ is a contradiction and the two eigenfunction equations could not exist. The solutions $f_{jm}(\theta,\varphi)$ of these equations are called the **spherical harmonics** and are usually written $Y_{jm}(\theta,\varphi)$. Just for the record, here is what the differential operators look like, where C = cos and S = sin (for example, $S_\varphi = \sin\varphi$ and $\partial_\varphi = \partial/\partial\varphi$) :

$$
\begin{aligned}
J_1 &= i\,[\, S_\varphi\, \partial_\theta + \cot\theta\, C_\varphi\, \partial_\varphi] & J_\pm &= e^{\pm i\varphi}\, [\, \pm\partial_\theta + i\, \cot\theta\, \partial_\varphi\,] \\
J_2 &= -i\,[\, C_\varphi\, \partial_\theta - \cot\theta\, S_\varphi\, \partial_\varphi] & J^2 &= -[\, \partial_\theta^2 + \cot\theta\, \partial_\theta + (1/S_\theta)^2\, \partial_\varphi^2] \\
J_3 &= -i\, \partial_\varphi & J^2 &= -[\, (1/S_\theta)\, \partial_\theta\, [\, S_\theta\, \partial_\theta\,] + (1/S_\theta)^2\, \partial_\varphi^2\,] \qquad \text{(G.9.4)}
\end{aligned}
$$

Reader Exercise: Verify that the three operators on the left satisfy the Lie Algebra $[J_i, J_j] = i\varepsilon_{ijk}J_k$.

For the hydrogen atom with a spinless electron, there are three mutually commuting quantities H, J^2 and J_3 where H is the Hamiltonian. This means that the solution eigenfunctions can have well defined E, j and m values and these eigenfunctions are those painful "orbitals" appearing in chemistry books. When the Hamiltonian commutes with some other operator like J^2, that operator is called a "symmetry". Solution functions then bear a label for each such symmetry, such as j for J^2.

The Lie Algebra so(3) is isomorphic (one-to-one related) to another Lie Algebra called su(2).
The Lie Group SO(3) is isomorphic to another Lie Group called SU(2).

In the above we discuss only the Lie Group SO(3) with its three generators J_i. There are many other Lie Groups which have physics applications.

Some Other Lie Groups of Interest

The group SO(n) is the **orthogonal group** in n dimensions and it has n(n-1)/2 generators.

The group SO(3,1) is the **Lorentz Group** which has 6 generators J_i and K_i which generate 3 rotations and 3 "boosts" (velocity transformations). The Lie Algebra is this,

$$
\begin{aligned}
[\, J_i, J_j] &= +i\, \varepsilon_{ijk}J_k \\
[\, J_i, K_j] &= +i\, \varepsilon_{ijk}\, K_k \\
[\, K_i, K_j] &= -i\, \varepsilon_{ijk}J_k \qquad \text{(G.9.5)}
\end{aligned}
$$

There are two Casimirs : $J^2 - K^2$ and $\mathbf{J}\bullet\mathbf{K}$.

In the Lorentz Group "vector representation" known as 1/2⊗1/2 the generators are represented as 4x4 matrices. When these are exponentiated, one obtains the finite rotation and boost matrices used in special relativity. For example, with space-time vectors ordered x^{μ} = (ct,x,y,z) one has

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(r) & -\sin(r) \\ 0 & 0 & \sin(r) & \cos(r) \end{pmatrix} = \exp(-irJ_1) \qquad \text{where } (J_1)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \text{// rotation } R_x(r)$$

$$\begin{pmatrix} \cosh(b) & \sinh(b) & 0 & 0 \\ \sinh(b) & \cosh(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(-ibK_1) \quad \text{where } (K_1)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{// boost } B_x(b) \tag{G.9.6}$$

The **Poincare Group** is basically the Lorentz group SO(3,1) bolted onto the group T(4) of translations in four directions ct,x,y,z. It thus has 10 generators J_i, K_i and P_{μ} where these last four are momentum and energy. The Poincare algebra has two Casimir operators whose eigenvalues are associated with mass and spin. For example, the mass Casimir is $P^{\mu}P_{\mu}$. The irreducible representations of the Poincare Group are associated with "elementary particles" which have well defined mass and spin.

The group **SU(3)** has 8 generators and 2 Casimirs. In one key representation the generators are represented by 3x3 matrices which act on 3-vectors. Instead of having up and down states as with spin-1/2 noted above, these vectors have up, down and strange states called u,d.s which are associated with quarks. The full symmetry group for the Standard Model of elementary particles is SU(3) x SU(2) x U(1) in which SU(3) plays its part.

Special Groups have Traceless Generators

It is desirable that "rotation matrices" have unit determinant because such matrices then do not change the "length" of a vector on which they act. When the representation matrices are restricted to have unit determinant, they are called "special" and the group name is prefixed by the letter S, as in SO(3) for the rotation group. For the 3x3 representation of the rotation group we know that the matrices are real orthogonal which means $RR^{T} = 1$ which in turn means $[\det(R)]^2 = 1$ and in SO(3) we select only those R with det(R) = +1. A "rotation" which just negates z (reflection) still satisfies $RR^{T} = 1$ but has det(R) = -1.

An elegant theorem concerning exponentiated square matrices is this (proved below):

$$\det(e^{A}) = e^{tr(A)} \tag{G.9.7}$$

where det is the determinant and $tr(A) \equiv \Sigma_i A_{ii}$ is the "trace" or "spur" of the matrix A -- the sum of the diagonal elements. If we want the matrix $\exp(-i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J})$ to have unit determinant so it is "Special", the exponent must be traceless, and in this case that means that the generators J_i must all be traceless. One can see from examples (G.1.2) and (G.9.3) and (G.9.6) that this is indeed the case.

Offhand, the matrix identity $\det(e^{\mathbf{A}}) = e^{\mathbf{tr(A)}}$ seems very unlikely and almost too simple. For that reason, we include here a straightforward proof which we feel is "one for the Book".

Proof of (G.9.7) There are no implied sums in this proof!

Write $A = \Sigma_{ij} A_{ij}\, s^{(ij)}$ where $[s^{(ij)}]_{ab} \equiv \delta_{ia}\delta_{jb}$. To verify,

$$A_{ab} = \Sigma_{ij} A_{ij}\, [s^{(ij)}]_{ab} = \Sigma_{ij} A_{ij} \delta_{ia}\delta_{jb} = A_{ab} \quad .$$

The matrix $s^{(ij)}$ is all zeros except for a single 1 located in row a and column b. Using $e^{x+y\cdots} = e^x e^y \ldots$ we can write

$$e^{\mathbf{A}} = \exp(\Sigma_{ij} A_{ij}\, s^{(ij)}) = \Pi_{i,j} \exp(A_{ij}\, s^{(ij)}) \quad .$$

Then using det(XY..) = det(X)det(Y).... ,

$$\det(e^{\mathbf{A}}) = \det \{\Pi_{i,j} \exp(A_{ij}\, s^{(ij)}) \} = \Pi_{i,j} \det [\exp(A_{ij}\, s^{(ij)})] \quad . \qquad \text{(G.9.8)}$$

[Case i ≠ j :] We first note that $[s^{(ij)}]^2 = 0$:

$$[s^{(ij)}]^2_{ac} = \Sigma_b [s^{(ij)}]_{ab}\, [s^{(ij)}]_{bc} = \Sigma_b \delta_{ia}\delta_{jb}\, \delta_{ib}\delta_{jc} = \delta_{ia}\delta_{ji}\delta_{jc} = 0 \text{ since } i \neq j$$

Then $[s^{(ij)}]^n = 0$ for $n \geq 2$. In this case we have

$$\exp(A_{ij}\, s^{(ij)}) = \Sigma_{n=0}^{\infty} (A_{ij})^n\, [\, s^{(ij)}]^n/n! = 1 + A_{ij}\, [\, s^{(ij)}]$$

and so

$$\det [\exp(A_{ij}\, s^{(ij)})] = \det [\, 1 + A_{ij}\, s^{(ij)}] = 1 \, .$$

This is so because the matrix in question has all 1's on the diagonal and one non-vanishing off-diagonal element at (i,j). In fact, *any* triangular matrix (one side all zeros) with 1's on the diagonal has det = 1.

[Case i=j :] In this case we have $[s^{(ii)}]^2 = [s^{(ii)}]$:

$$[s^{(ii)}]^2_{ac} = \Sigma_b [s^{(ii)}]_{ab}\, [s^{(ii)}]_{bc} = \Sigma_b \delta_{ia}\delta_{ib}\, \delta_{ib}\delta_{ic} = \delta_{ia}\delta_{ii}\delta_{ic} = \delta_{ia}\delta_{ic} = [s^{(ii)}]_{ac} \quad .$$

Note that $[s^{(ii)}]_{ab} = \delta_{ia}\delta_{ib} = 1$ only when a = b = i, so $s^{(ii)}$ is an all-zero matrix with a single 1 at location i on the diagonal.

Since $[s^{(ii)}]^2 = s^{(ii)}$ it follows that $[s^{(ii)}]^n = s^{(ii)}$ for $n \geq 1$. Then,

$$\exp(A_{ii}\, s^{(ii)}) = \Sigma_{n=0}^{\infty} (A_{ii})^n\, [\, s^{(ii)}]^n \,/\, n! = 1 + s^{(ii)}\, [\Sigma_{n=1}^{\infty} (A_{ii})^n \,/\, n!]$$

$= 1 + s^{(ii)} [-1 + \Sigma_{n=0}^{\infty} (A_{ii})^n/n!] = 1 + s^{(ii)} [-1 + \exp(A_{ii})]$.

This last item is the unit matrix with the i^{th} diagonal 1 replaced by $\exp(A_{ii})$. Therefore,

$\det [\exp(A_{ii}\, s^{(ii)})] = \det \{ 1 + s^{(ii)} [-1 + \exp(A_{ii})] \} = \exp(A_{ii})$.

Now go back to (G.9.8),

$\det(e^{A}) = \Pi_{i,j} \det [\exp(A_{ij}\, s^{(ij)})]$

$= (\Pi_{i \neq j} \det [\exp(A_{ij}\, s^{(ij)})]) * (\Pi_{i=j} \det [\exp(A_{ij}\, s^{(ij)})])$

$= (1*1*1*1..... \quad ....*1*1) * (\exp(A_{11})\exp(A_{22}) .....)$

$= 1 * \exp(A_{11} + A_{22} + ...) = \exp (tr(A))$

$= e^{tr(A)}$. QED