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G.9 Generalizations of the Rotation Group

Here we consider some generalizations of the ideas presented above in Section G.1

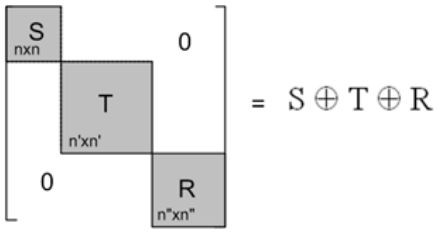
N dimensional representations of the rotation group

The generators J_i shown in (G.1.2) are a special case of a more general idea which starts with the commutation relation

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \ . \qquad \text{// for example, } [J_1,J_2] = iJ_3 \qquad \textit{(G.1.4)} \qquad \text{(G.9.1)}$$

One first thinks of the J_i as abstract "operators" in some abstract "operator space". One can show that it is possible to find a set of three NxN matrices of any integer dimension N which satisfy (G.1.4). The matrices are not unique, so (G.1.2) for the J_i in three dimensions is not unique, but it is a standard form.

The three NxN generator matrices J_i are said to form an N-dimensional "irreducible representation" of the abstract generators J_i. One can always create new viable generator matrices by taking a "direct sum" of existing viable generator matrices, such as in this block-diagonal-form picture



(G.9.2)

This generator matrix is "reducible" into a direct sum of S, T and R. An "irreducible" representation is one that cannot be reduced in this manner. The same comment applies to rotation matrices.

The commutation relation (G.9.1) is an example of a **Lie Algebra**. Our particular Lie Algebra is called so(3), so we can represent the algebra elements J_i of this algebra by three NxN matrices J_i. It is possible to write down a formula analogous to (G.1.3) $(iJ_a)_{bc} = \varepsilon_{abc}$ which works for any N, but (G.1.3) itself only applies to N = 3. This is so because ε_{abc} has no meaning for $N \neq 3$. But it always has meaning in (G.9.1) because there are only three generators regardless of the value of N.

For general N, the object $R_i(\theta) = \exp(-i\theta \mathbf{J}_i)$ is an NxN matrix which represents the action of a rotation of an N-vector in a Euclidean space E^N. The set of such rotation matrices forms an "irreducible representation" of the rotation group SO(3) in N dimensions. The integer N is usually written N = 2j+1 where j = 0, 1/2, 1, 3/2 ... and this j then serves as a label for a given matrix representation.

The value of j in N = 2j+1 is associated with "angular momentum" or "spin". In the case N=2 (having j=1/2) the generator matrices are the 2x2 "Pauli matrices". In this case the 2x2 matrices $\exp(-i\theta J_i)$ describe the rotations of spin-1/2 particles such as electrons or protons. Here are the details for N=2 :

$$J_x = (1/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_y = (1/2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad J_z = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_x = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \qquad R_z = \begin{pmatrix} \exp(-i\frac{\theta}{2}) & 0 \\ 0 & \exp(+i\frac{\theta}{2}) \end{pmatrix} \tag{G.9.3}$$

The "vectors" for spin-1/2 particles have two components $\begin{pmatrix} u \\ d \end{pmatrix}$. The special case $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is called "spin up" and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is "spin down".

For the Lie Algebra so(3) one can show that $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ and any particular J_i commute with each other, so $[J^2, J_i] = 0$. J^2 is called a Casimir operator of this Algebra, and fancier Lie Algebras can have several such Casimirs.

A Differential operator representation of the rotation group

It is also possible to "represent" the three rotation generators J_i by three differential operators in spherical coordinates θ and φ. These operators satisfy $[J_i, J_j] = i\varepsilon_{ijk}J_k$ and in this context they are usually called L_i but we shall stick with J_i. In this case, one can compute the differential operator J^2 and one can ponder differential equations which take the form $J^2 f_{jm}(\theta,\varphi) = j(j+1) f_{jm}(\theta,\varphi)$ and $J_z f_{jm}(\theta,\varphi) = m f_{jm}(\theta,\varphi)$. [The facts that the eigenvalue of J^2 is j(j+1) and not j^2, and that m runs from -j to j, derive from the structure of the Lie Algebra.] The solutions $f_{jm}(\theta,\varphi)$ are able to have well-defined eigenvalues j(j+1) and m *because* $[J^2,J_3] = 0$. If we instead had $[J^2,J_3] \neq 0$, then $[J^2,J_3] f_{jm}(\theta,\varphi) = j(j+1)m - mj(j+1) = 0$ is a contradiction and the two eigenfunction equations could not exist. The solutions $f_{jm}(\theta,\varphi)$ of these equations are called the **spherical harmonics** and are usually written $Y_{jm}(\theta,\varphi)$. Just for the record, here is what the differential operators look like, where C = cos and S = sin (for example, $S_\varphi = \sin\varphi$ and $\partial_\varphi = \partial/\partial\varphi$) :

$$\begin{aligned} J_1 &= i\,[\, S_\varphi \partial_\theta + \cot\theta\, C_\varphi \partial_\varphi] & J_\pm &= e^{\pm i\varphi}\,[\, \pm\partial_\theta + i\,\cot\theta\,\partial_\varphi\,] \\ J_2 &= -i\,[\, C_\varphi \partial_\theta - \cot\theta\, S_\varphi \partial_\varphi] & J^2 &= -[\, \partial_\theta^2 + \cot\theta\,\partial_\theta + (1/S_\theta)^2 \partial_\varphi^2] \\ J_3 &= -i\,\partial_\varphi & J^2 &= -[\, (1/S_\theta)\partial_\theta [\, S_\theta \partial_\theta\,] + (1/S_\theta)^2 \partial_\varphi^2\,] \end{aligned} \tag{G.9.4}$$

Reader Exercise: Verify that the three operators on the left satisfy the Lie Algebra $[J_i, J_j] = i\varepsilon_{ijk}J_k$.

For the hydrogen atom with a spinless electron, there are three mutually commuting quantities H, J^2 and J_3 where H is the Hamiltonian. This means that the solution eigenfunctions can have well defined E, j and m values and these eigenfunctions are those painful "orbitals" appearing in chemistry books. When the Hamiltonian commutes with some other operator like J^2, that operator is called a "symmetry". Solution functions then bear a label for each such symmetry, such as j for J^2.

The Lie Algebra so(3) is isomorphic (one-to-one related) to another Lie Algebra called su(2).
The Lie Group SO(3) is isomorphic to another Lie Group called SU(2).

In the above we discuss only the Lie Group SO(3) with its three generators J_i. There are many other Lie Groups which have physics applications.

Some Other Lie Groups of Interest

The group SO(n) is the **orthogonal group** in n dimensions and it has n(n-1)/2 generators.

The group SO(3,1) is the **Lorentz Group** which has 6 generators J_i and K_i which generate 3 rotations and 3 "boosts" (velocity transformations). The Lie Algebra is this,

$$\begin{aligned} [\, J_i, J_j] &= +i\, \varepsilon_{ijk} J_k \\ [\, J_i, K_j] &= +i\, \varepsilon_{ijk}\, K_k \\ [\, K_i, K_j] &= -i\, \varepsilon_{ijk} J_k \end{aligned} \qquad \text{(G.9.5)}$$

There are two Casimirs : $J^2 - K^2$ and $\mathbf{J}\bullet\mathbf{K}$.

In the Lorentz Group "vector representation" known as 1/2⊗1/2 the generators are represented as 4x4 matrices. When these are exponentiated, one obtains the finite rotation and boost matrices used in special relativity. For example, with space-time vectors ordered x^μ = (ct,x,y,z) one has

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(r) & -\sin(r) \\ 0 & 0 & \sin(r) & \cos(r) \end{pmatrix} = \exp(-irJ_1) \qquad \text{where } (J_1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \text{// rotation } R_x(r)$$

$$\begin{pmatrix} \cosh(b) & \sinh(b) & 0 & 0 \\ \sinh(b) & \cosh(b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp(-ibK_1) \quad \text{where } (K_1)^\mu{}_\nu = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{// boost } B_x(b) \qquad \text{(G.9.6)}$$

The **Poincare Group** is basically the Lorentz group SO(3,1) bolted onto the group T(4) of translations in four directions ct,x,y,z. It thus has 10 generators J_i, K_i and P_μ where these last four are momentum and energy. The Poincare algebra has two Casimir operators whose eigenvalues are associated with mass and spin. For example, the mass Casimir is $P^\mu P_\mu$. The irreducible representations of the Poincare Group are associated with "elementary particles" which have well defined mass and spin.

The group **SU(3)** has 8 generators and 2 Casimirs. In one key representation the generators are represented by 3x3 matrices which act on 3-vectors. Instead of having up and down states as with spin-1/2 noted above, these vectors have up, down and strange states called u,d.s which are associated with quarks.

The full symmetry group for the Standard Model of elementary particles is SU(3) x SU(2) x U(1) in which SU(3) plays its part.

Special Groups have Traceless Generators

It is desirable that "rotation matrices" have unit determinant because such matrices then do not change the "length" of a vector on which they act. When the representation matrices are restricted to have unit determinant, they are called "special" and the group name is prefixed by the letter S, as in SO(3) for the rotation group. For the 3x3 representation of the rotation group we know that the matrices are real orthogonal which means $RR^{T} = 1$ which in turn means $[\det(R)]^2 = 1$ and in SO(3) we select only those R with det(R) = +1. A "rotation" which just negates z (reflection) still satisfies $RR^{T} = 1$ but has det(R) = -1.

An elegant theorem concerning exponentiated square matrices is this (proved below):

$$\det(e^{A}) = e^{tr(A)} \qquad \text{(G.9.7)}$$

where det is the determinant and $tr(A) \equiv \Sigma_i A_{ii}$ is the "trace" or "spur" of the matrix A -- the sum of the diagonal elements. If we want the matrix $\exp(-i\,\theta\,\hat{\mathbf{n}} \bullet \mathbf{J})$ to have unit determinant so it is "Special", the exponent must be traceless, and in this case that means that the generators J_i must all be traceless. One can see from examples (G.1.2) and (G.9.3) and (G.9.6) that this is indeed the case.

Offhand, the matrix identity $\det(e^{A}) = e^{tr(A)}$ seems very unlikely and almost too simple. For that reason, we include here a straightforward proof which we feel is "one for the Book".

Proof of (G.9.7) There are no implied sums in this proof!

Write $A = \Sigma_{ij} A_{ij}\, s^{(ij)}$ where $[s^{(ij)}]_{ab} \equiv \delta_{ia}\delta_{jb}$. To verify,

$$A_{ab} = \Sigma_{ij} A_{ij}\, [s^{(ij)}]_{ab} = \Sigma_{ij} A_{ij} \delta_{ia}\delta_{jb} = A_{ab} \quad .$$

The matrix $s^{(ij)}$ is all zeros except for a single 1 located in row a and column b. Using $e^{x+y\cdots} = e^{x}e^{y}$ ... we can write

$$e^{A} = \exp(\Sigma_{ij} A_{ij}\, s^{(ij)}) = \Pi_{i,j} \exp(A_{ij}\, s^{(ij)}) \quad .$$

Then using det(XY..) = det(X)det(Y).... ,

$$\det(e^{A}) = \det\{\Pi_{i,j} \exp(A_{ij}\, s^{(ij)})\} = \Pi_{i,j} \det[\exp(A_{ij}\, s^{(ij)})] \quad . \qquad \text{(G.9.8)}$$

[Case i ≠j :] We first note that $[s^{(ij)}]^2 = 0$:

$$[s^{(ij)}]^2_{ac} = \Sigma_b [s^{(ij)}]_{ab}\, [s^{(ij)}]_{bc} = \Sigma_b \delta_{ia}\delta_{jb}\, \delta_{ib}\delta_{jc} = \delta_{ia}\delta_{ji}\delta_{jc} = 0 \text{ since } i \neq j$$

Then $[s^{(ij)}]^n = 0$ for $n \geq 2$. In this case we have

$$\exp(A_{ij}\, s^{(ij)}) = \Sigma_{n=0}^{\infty} (A_{ij})^n [\, s^{(ij)}]^n/n! = 1 + A_{ij} [\, s^{(ij)}]$$

and so

$$\det [\exp(A_{ij}\, s^{(ij)}) \,] = \det [\, 1 + A_{ij}\, s^{(ij)}] = 1 \,.$$

This is so because the matrix in question has all 1's on the diagonal and one non-vanishing off-diagonal element at (i,j). In fact, *any* triangular matrix (one side all zeros) with 1's on the diagonal has det = 1.

[Case i=j :] In this case we have $[s^{(ii)}]^2 = [s^{(ii)}]$:

$$[s^{(ii)}]^2_{ac} = \Sigma_b [s^{(ii)}]_{ab} [s^{(ii)}]_{bc} = \Sigma_b \delta_{ia}\delta_{ib}\, \delta_{ib}\delta_{ic} = \delta_{ia}\delta_{ii}\delta_{ic} = \delta_{ia}\delta_{ic} = [s^{(ii)}]_{ac} \quad .$$

Note that $[s^{(ii)}]_{ab} = \delta_{ia}\delta_{ib} = 1$ only when a = b = i, so $s^{(ii)}$ is an all-zero matrix with a single 1 at location i on the diagonal.

Since $[s^{(ii)}]^2 = s^{(ii)}$ it follows that $[s^{(ii)}]^n = s^{(ii)}$ for $n \geq 1$. Then,

$$\exp(A_{ii}\, s^{(ii)}) = \Sigma_{n=0}^{\infty} (A_{ii})^n [\, s^{(ii)}]^n \,/\, n! = 1 + s^{(ii)} \, [\Sigma_{n=1}^{\infty} (A_{ii})^n \,/\, n!]$$

$$= 1 + s^{(ii)} \, [\, -1 + \Sigma_{n=0}^{\infty} (A_{ii})^n/n! \,] = 1 + s^{(ii)} [-1 + \exp(A_{ii}) \,] \,.$$

This last item is the unit matrix with the i^{th} diagonal 1 replaced by $\exp(A_{ii})$. Therefore,

$$\det [\exp(A_{ii}\, s^{(ii)}) \,] = \det \{\, 1 + s^{(ii)} [-1 + \exp(A_{ii})] \,\} = \exp(A_{ii}) \,.$$

Now go back to (G.9.8),

$$\det(e^{A}) = \Pi_{i,j} \det [\exp(A_{ij}\, s^{(ij)}) \,]$$

$$= (\Pi_{i \neq j} \det [\exp(A_{ij}\, s^{(ij)}) \,] \,) \; * \; (\Pi_{i=j} \det [\exp(A_{ij}\, s^{(ij)}) \,] \,)$$

$$= (1*1*1*1..... \quad ....*1*1) * (\; \exp(A_{11})\exp(A_{22}) ..... \quad)$$

$$= 1 * \exp(A_{11} + A_{22} + ...) = \exp (\mathrm{tr}(A))$$

$$= e^{tr(A)} \,. \qquad \text{QED}$$