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G.4 Two more theorems for the rotation matrix toolbox

Theorem 1: $\mathbf{R}\mathbf{J}\mathbf{R}^{-1} = \mathbf{R}^{-1}\mathbf{J}$ (G.4.1)

This is another vector/matrix notation theorem which makes a claim about rotating a vector of matrices. The above notation is a shorthand for the following,

$$RJ_iR^{-1} = [\mathbf{R}^{-1}\mathbf{J}]_i = R^{-1}{}_{ij}J_j \qquad i = 1,2,3\ . \tag{G.4.2}$$

The object on the left is a product of three 3x3 matrices, while the right side is a linear combination of 3x3 matrices, so at least the theorem's claim is dimensionally reasonable. More generally J_i might be an abstract "operator" and R a rotation which acts on that operator, see ** below.

Proof: Start with the general rotation form given in (G.2.6),

$$R = \exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = 1 + (\cos\theta - 1)\,T + \sin\theta[-i(\mathbf{n}\bullet\mathbf{J})] \quad \text{where } T_{ab} = (\delta_{ab} - n_a n_b)\ . \tag{G.4.3}$$

Sandwich this rotation around J_i and then use the sandwich formula (G.3.3),

$$\begin{aligned} RJ_iR^{-1} &= \exp(-i\theta\mathbf{n}\bullet\mathbf{J})\ J_i\ \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) \\ &= \cos\theta\, J_i + \sin\theta\, \varepsilon_{ijk}n_k J_j + (1-\cos\theta)\, n_i\ (\mathbf{n}\bullet\mathbf{J}) \\ &= \cos\theta\, \delta_{ij}J_j + \sin\theta\, \varepsilon_{ijk}n_k J_j + (1-\cos\theta)\, n_i n_j J_j \\ &= [\,\cos\theta\, \delta_{ij} + \sin\theta\, \varepsilon_{ijk}n_k + (1-\cos\theta)\, n_i n_j\,]\, J_j\ . \end{aligned} \tag{G.4.4}$$

Our theorem is proved if, comparing (G.4.2) and (G.4.4), we can show that

$$R^{-1}{}_{ij} = \cos\theta\, \delta_{ij} + \sin\theta\, \varepsilon_{ijk}n_k + (1-\cos\theta)\, n_i n_j\ . \tag{G.4.5}$$

Looking back at (G.4.3), one has

$$R^{-1} = \exp(+i\theta\mathbf{n}\bullet\mathbf{J}) = 1 + (\cos\theta - 1)\,T + \sin\theta[+i(\mathbf{n}\bullet\mathbf{J})]$$

so

$$\begin{aligned} R^{-1}{}_{ij} &= \delta_{ij} + (\cos\theta - 1)\,T_{ij} + \sin\theta[+n_k\,(iJ_k)_{ij}] \\ &= \delta_{ij} + (\cos\theta - 1)\,(\delta_{ij} - n_i n_j) + \sin\theta\, n_k\, \varepsilon_{kij} \qquad \text{// (G.1.3) and def of } T_{ij} \\ &= \delta_{ij}\cos\theta + (1-\cos\theta)\, n_i n_j + \sin\theta\, \varepsilon_{ijk}n_k \end{aligned} \tag{G.4.6}$$

But this is the same as (G.4.5) so the theorem is proved.

Theorem 2: $R \exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1} = \exp(-i\theta \mathbf{n}'\bullet\mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$ (G.4.7)

On the left we have a product of three 3x3 matrices, while the right side is a 3x3 matrix. The matrix R is an arbitrary rotation. One proof of this theorem might be to say "what else could it be? ". We shall provide a more substantial proof below.

One preliminary test is that we can take the determinant of both sides to make sure we get the same thing. Below in ** we shall prove that $\det(e^{A}) = e^{tr(A)}$ in any number of dimensions, so our check is then

$\det[R \exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1}] = \det[\exp(-i\theta \mathbf{n}'\bullet\mathbf{J})]$?

$\det[\exp(-i\theta \mathbf{n}\bullet\mathbf{J})] = \det[\exp(-i\theta \mathbf{n}'\bullet\mathbf{J})]$?

$\exp[\mathrm{tr}(-i\theta \mathbf{n}\bullet\mathbf{J})] = \exp[\mathrm{tr}(-i\theta \mathbf{n}'\bullet\mathbf{J})]$?

$\exp[-i\theta n_k (\mathrm{tr} J_k)] = \exp[-i\theta n'_k (\mathrm{tr} J_k)]$? // the J_i generators (G.1.2) are traceless

$\exp[0] = \exp[0]$? $1 = 1$? yes

So we are at least encouraged (and we got to apply the fancy trace theorem).

Proof: Start by inserting the general rotation form (G.2.6) into the left side of (G.4.7),

$$R \exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1} = R\,[\, 1 + (\cos\theta - 1)\, T - i \sin\theta\, (\mathbf{n}\bullet\mathbf{J})\,]\, R^{-1}$$

$$= 1 + (\cos\theta - 1) R T R^{-1} - i \sin\theta\; n_k R J_k R^{-1} . \qquad \text{(G.4.8)}$$

Now consider

$$[RTR^{-1}]_{ad} = R_{ab} T_{bc} R^{-1}_{cd} = R_{ab}(\delta_{bc} - n_b n_c) R^{T}_{cd}$$

$$= \delta_{ad} - R_{ab}\, n_b n_c R_{dc} = \delta_{ad} - (R_{ab}\, n_b)(R_{dc} n_c) = \delta_{ad} - n'_a n'_d \qquad \mathbf{n}' = R\mathbf{n}$$

$$\equiv T'_{ad} . \qquad \text{// in other words, } RTR^{-1} = T' \qquad \text{(G.4.9)}$$

From Theorem (G.4.2) we know that

$$R J_k R^{-1} = R^{-1}_{kj} J_j . \qquad \textit{(G.4.2)}$$

Inserting this last item and (G.4.9) into (G.4.8) gives

$$R \exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1} = 1 + (\cos\theta - 1) T' - i \sin\theta\; n_k R^{-1}_{kj} J_j$$

$$= 1 + (\cos\theta - 1)T' - i \sin\theta \, R_{jk} n_k J_j = 1 + (\cos\theta - 1)T' - i \sin\theta \, [Rn]_j J_j$$

$$= 1 + (\cos\theta - 1)T' - i \sin\theta \, n'_j J_j = 1 + (\cos\theta - 1)T' - i \sin\theta \, (\mathbf{n}' \bullet \mathbf{J})$$

$$= \exp(-i\theta \mathbf{n}' \bullet \mathbf{J}) \qquad \text{// using (G.2.6) with } \mathbf{n} \rightarrow \mathbf{n}' \qquad \text{(G.4.10)}$$

Thus the theorem is proved.

The fact (G.4.9) that $RTR^{-1} = T'$ is in fact just the standard rule for the transformation of a rank-2 tensor under rotations, as shown for example in Lucht *Tensor* (5.7.1). That is to say, if an Observer in Frame S sees T, an Observer in rotated Frame S' will see $T' = RTR^{-1}$. This same statement applies at the higher level of our theorem

$$R \exp(-i\theta \mathbf{n} \bullet \mathbf{J}) R^{-1} = \exp(-i\theta \mathbf{n}' \bullet \mathbf{J})$$

or

$$R \, R_n(\theta) \, R^{-1} = R_{n'}(\theta) \qquad \mathbf{n}' = R\mathbf{n} \qquad \text{// Theorem 2 restated} \qquad \text{(G.4.11)}$$

The rotation $R_n(\theta)$ seen by an Observer in Frame S appears as $R_{n'}(\theta)$ in Frame S' where $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$ as in (1.1.1) . Like T, the rotation $R_n(\theta)$ transforms as a rank-2 tensor under rotations.

An equivalent *active* statement of the theorem is the following:

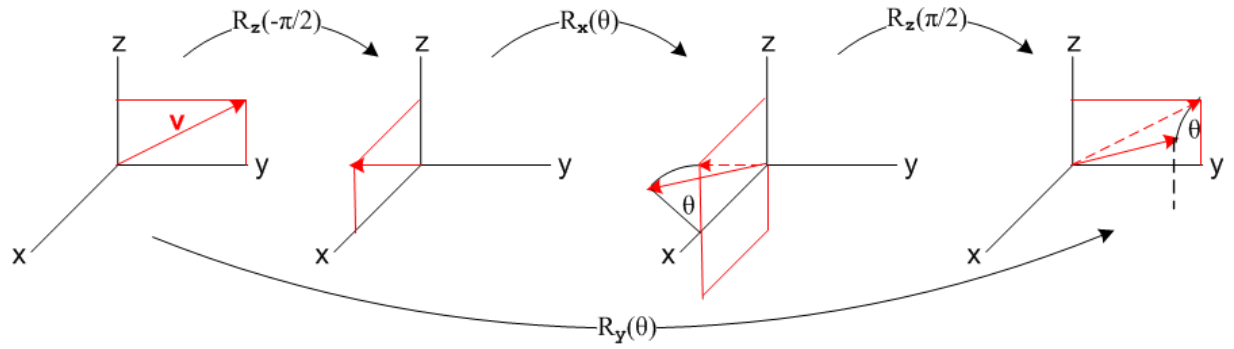
$$R \, R_n(\theta) \, R^{-1}\mathbf{v} = R_{n'}(\theta)\mathbf{v} \qquad \text{where } \mathbf{n}' = R\mathbf{n} \qquad \text{(G.4.12)}$$

Instead of rotating a vector **v** by amount θ about axis $\hat{\mathbf{n}}'$, one can first back-rotate the vector **v**, then rotate by amount θ about axis $\hat{\mathbf{n}}$, the forward rotate the result.

Example: $R_z(\pi/2)R_x(\theta)R_z(-\pi/2) = ??$ Here $\mathbf{n} = \hat{\mathbf{x}}$ and $\mathbf{n}' = R_z(\pi/2)\,\hat{\mathbf{x}} = \hat{\mathbf{y}}$ so we conclude that

$$R_z(\pi/2)R_x(\theta)R_z(-\pi/2) = R_y(\theta) \; . \qquad \text{(G.4.13)}$$

Here is a graphical interpretation of (G.4.13) applied to a particular vector **v** :



(G.4.14)