**Analysis of App G section on computing ω** PhL 2.17.17

I start with the equation,

$\mathbf{e}_n = R\,\mathbf{e}'_n$ *(1.1.1)* (G.4.1)

$R = R_z(\psi)\,R_x(\theta)\,R_z(\varphi) = R(\mathbf{\Phi}) = \exp(-i\,\mathbf{\Phi}\bullet\mathbf{J}) = \exp(-i\,\Phi\,\mathbf{n}\bullet J)$ (G.4.2)

which then determines MY meaning of the Euler angles. If I wanted to do this from the Start in Goldstein notation I would write

$R = R_z(-\psi)\,R_x(-\theta)\,R_z(-\varphi) = R(-\mathbf{\Phi}) = \exp(+i\,\mathbf{\Phi}\bullet\mathbf{J}) = \exp(+i\,\Phi\,\mathbf{n}\bullet J)$ (G.4.2)

$R = B\,C\,D$ // Goldstein page 109

For now I will stick with <u>my original notation</u>. I then quote an earlier equation which contains **ω**,

$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n$ . *(1.7.4)* (G.4.5)

Is this a general vector equation? Each side is a vector, you could take components on either side! At least you should be able to! Is it a covariant equation? That is, is it a "true vector equation". I am very unsure of this.

<u>Note added:</u> My analysis below says that the vector nature of this equation is very suspicious. Since $\mathbf{e}_n$ is not a true tensorial vector (not a true vector), I think neither side is a true vector. It is *not* a true vector equation of the form **a** = **b** in the covariance sense. I don't really know its tensor nature.

Since the equation is not a true vector equation like **a** = **b**, it is unclear whether you are allowed to evaluate the sides of the equation in either frame. But I did decide that for $\mathbf{e}_n = R\mathbf{e}'_n$ that IT was a non-vector equation, but I WAS allowed to evaluate both sides in either frame.

<u>Note added 1.3.17.</u>

(1) We have $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$ and $(\mathbf{V})' = R\mathbf{V}$ . The second is not really a vector equation, it is just a shorthand notation for $(V)'_i = R_{ij}V_j$. Thus, you cannot evaluate $(\mathbf{V})' = R\mathbf{V}$ in either frame, that is true.

(2) $(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n$ really is a vector equation. Yes, on the right you have a mix of a Kinematic Vector with a Basis vector. But they are still vectors, and you can still take components in both Frame S and in Frame S'.

(3) I was looking perhaps for reasons to explain my ω computation Paradox which was giving the same results for ω components computed in Frame S or Frame S', see below,

Only someone in Frame S' can measure $(d\mathbf{e}_n/dt)_{S'}$, but they could express that measured result in either set of coordinates I think. I then write (I will use Q here to avoid bias).

$\mathbf{e}_n = R(\mathbf{\Phi})\,\mathbf{e}'_n$ *(G.4.4)*

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'}_n - R(\mathbf{\Phi})\mathbf{e'}_n \equiv dQ\ \mathbf{e'}_n \qquad (G.4.8)$$

$$dQ \equiv [\ R_z(\psi+d\psi)\ R_x(\theta+d\theta)\ R_z(\varphi+d\varphi) - R_z(\psi)\ R_x(\theta)\ R_z(\varphi)\ ]\ . \qquad (G.4.9)$$

Certainly the 6 matrices shown are exactly as I define them, not rotated versions or any such thing. I then write

$$(d\mathbf{e}_n)_{S'} = dQ\ \mathbf{e'}_n = dQ[R^{-1}(\mathbf{\Phi})\mathbf{e}_n\ ] = [dQ\ R^{-1}(\mathbf{\Phi})\ ]\ \mathbf{e}_n \qquad (G.4.10)$$

I then divide by dt to get

$$(d\mathbf{e}_n/dt)_{S'} = [(dQ/dt)\ R^{-1}(\mathbf{\Phi})\ ]\ \mathbf{e}_n = A\ e_n$$

I am working in Frame S', but it still seems to be a true vector equation. I then show that

$$A = -[\ (\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\ ]$$

So I then have these two equations I want to compare :

$$(d\mathbf{e}_n/dt)_{S'} = -\ \mathbf{\omega} \times \mathbf{e}_n$$

$$(d\mathbf{e}_n/dt)_{S'} = A\ \mathbf{e}_n \qquad [\ A \text{ should have a prime on it to be } A'\ ]$$

Now we come to the Nubbins of the matter. I want to claim that we then have this vector equation

$$-\ \mathbf{\omega} \times \mathbf{e}_n = A\ \mathbf{e}_n$$

I would like to say you can take components here in either Frame S or Frame S'. So we then have

Frame S : $\quad -[\ \mathbf{\omega} \times \mathbf{e}_n]_i = [A\ \mathbf{e}_n]_i$

$$-\varepsilon_{ikj}(\omega)_k(\mathbf{e}_n)_j = A_{ij}(\mathbf{e}_n)_j \qquad [\ = [R^{-1}A'R]_{ij}(\mathbf{e}_n)_j\ ]$$

I have in mind that the $e_n$ are just the Cartesian basis vectors in Frame S, so I continue

$$-\varepsilon_{ikj}(\omega)_k\delta_{n,j} = A_{ij}\delta_{n,j}$$

$$-\varepsilon_{ikn}(\omega)_k = A_{in} \qquad [\ = [R^{-1}A'R]_{in}\ ]$$

Now let's do this in Frame S':

Frame S' : $\quad -[\ \mathbf{\omega} \times \mathbf{e}_n]'_i = [A\ \mathbf{e}_n]'_i \qquad [$ note that $[A\ \mathbf{e}_n]'_i = A'_{ij}(\mathbf{e}_n)'_j\ ]$

$$- \varepsilon_{ikj}(\omega)'_k(\mathbf{e}_n)'_j = A_{ij}(\mathbf{e}_n)'_j$$

$$[\, A_{ij} + \varepsilon_{ikj}(\omega)'_k](\mathbf{e}_n)'_j = 0$$

One way to make this equation be true is to require that

$$- \varepsilon_{ikn}(\omega)'_k = A_{in}$$ [ this A should be A' ]

But this is the SAME as the previous equation, so we conclude that

$(\omega)'_k = (\omega)_k$ **contradiction and paradox**

which we know is not true. So my current effort is to provide a clean explanation of this Paradox without doing arm-waving BS. I have had similar problems before in other Worlds, and I guess I never nailed down an explanation of the confusion. [ I think correct A labeling resolves the above paradox ]

I am now going to push the stack and ponder related but hopefully similar issues.

Let's take a simpler Case. Suppose you have

$\mathbf{a} = A\mathbf{b}$

which is a "vector equation" in Frame S. If it is a physics equation, it should be covariant, so in Frame S' it should look like this

$\mathbf{a}' = A'\mathbf{b}'$ [ assuming **a**, **b** and A are true tensors ]

where

$\mathbf{a}' = R\mathbf{a}$ $\mathbf{b}' = R\mathbf{b}$ $A' = RAR^{-1}$ $A = R^{-1}A' R$

Now go back to our equations of interest,

$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$

$(d\mathbf{e}_n/dt)_{S'} = A\mathbf{e}_n$

In what frame are these equations true as stated? That is a Nubbins question. Does the second equation have one of these forms??

$\mathbf{a} = B\mathbf{b}$

$\mathbf{a}' = B'\mathbf{b}'$

Let's back up and search for ambiguity. Consider again the basic relation

$\mathbf{e}_n = R\ \mathbf{e}'_n$

What can be said about the vector nature of this equation? Is it a true vector equation? Is it a covariant equation? Does it look some way in each frame? Lots more Nubbins questions! [ these questions are answered below in a box ]

I think this is NOT a true vector equation since it has "one foot in each space" as in tensor doc. There I had ?? I am having trouble locating the analog of this equation in tensor doc! It might be this in dev not,

$\mathbf{e}_n \equiv S\mathbf{e}'_n$ . (3.2.4)

$\mathbf{e}'_n = R(\mathbf{x})\ \mathbf{e}_n$ . (3.3.2)

$(\mathbf{e}'_n)_i = \Sigma_j R_{ij}\ (\mathbf{e}_n)_j \quad \Rightarrow \quad \delta_{n,i} = \Sigma_j R_{ij} S_{jn}$ (3.3.3)

This looks pretty close. The idea is that R is not a tensor here because it links two worlds. But if both worlds are Cartesian, they both have the same up/down and g = g. But I don't think it is a tensor even in this case, so I have said the wrong thing on this somewhere, calling R a tensor.

So: if $\mathbf{e}_n = R\ \mathbf{e}'_n$ is NOT a true tensor equation, it makes no sense to ask if it is covariant! Thus, the issue of **a** = B**b** versus **a'** = B' **b'** is not relevant to this equation. State clearly:

$\mathbf{e}_n = R\ \mathbf{e}'_n$

1. This is not a true-tensor equation, it is a frames-linker equation
2. There is no meaning to asking whether it is **a** = B**b** versus **a'** = B' **b'**
3. The equation is not covariant. [ all correct ]

I still have more questions. The equation is of the form **a** = **b** where each side is a vector. Are you allowed to evaluate this equation in either Frame S or Frame S' ?? Another Nubbins question! (this word is my own word, means corn bits in general). [I think the answer is yes ]

Well, I can certainly dot the equation with either $\mathbf{e}_m$ or $\mathbf{e}'_m$. Let's start with $\mathbf{e}_m$ :

$\mathbf{e}_n = R\ \mathbf{e}'_n$

$\mathbf{e}_n \bullet \mathbf{e}_m = [R\ \mathbf{e}'_n] \bullet \mathbf{e}_m$

$\delta_{nm} = [R\ \mathbf{e}'_n] \bullet \mathbf{e}_m$

How do you evaluate the dot product? Can you do it in either frame? Is this dot product a scalar? More Nubbins questions!!! [ yes, you can evaluate a dot product in either frame ]

**Fact**: Even though $\mathbf{e}_n$ does not transform as a tensorial vector, the dot product $\mathbf{e}_n \bullet \mathbf{e}_m$ really is a scalar.

The reason is that they both transform backwards and this compensates. So

$$\mathbf{e}_n \bullet \mathbf{e}_m = [\, R\, \mathbf{e}'_n \,] \bullet [\, R\, \mathbf{e}'_m \,] = \mathbf{e}'_n \bullet \mathbf{e}'_m$$ [ but really a metric tensor rank 2, not scalar]

Now $\mathbf{e}_n = R\, \mathbf{e}'_n$ I initially think of as valid in Frame S where $\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{nm}$. It then defines the new vectors $\mathbf{e}'_n \equiv R^{-1}\mathbf{e}_n$ within Frame S. Both $\mathbf{e}_n$ and $\mathbf{e}'_n$ are vectors in Frame S. So it seems that I should be able to evaluate

$$[R\, \mathbf{e}'_n] \bullet \mathbf{e}_m = [R\, \mathbf{e}'_n]_i\, (\mathbf{e}_m)_i$$

$$= R_{ij}(\mathbf{e}'_n)_j\, (\mathbf{e}_m)_i = R_{ij}(\mathbf{e}'_n)_j\, \delta_{im} = R_{mj}(\mathbf{e}'_n)_j$$

I then get

$$\delta_{nm} = R_{mj}(\mathbf{e}'_n)_j$$

In general in Section 1 of frames doc I always "work in Frame S" when doing things with components. I do allow as you can expand a vector on either basis, however

$$\begin{array}{lll} \mathbf{a} = (a)_i\, \mathbf{e}_i = (a)'_i\, \mathbf{e}'_i & (a)_i = \mathbf{a} \bullet \mathbf{e}_i & (a)'_i = \mathbf{a} \bullet \mathbf{e}'_i \\ \mathbf{a}' = (a')_i\, \mathbf{e}_i = (a')'_i\, \mathbf{e}'_i & (a')_i = \mathbf{a}' \bullet \mathbf{e}_i & (a')'_i = \mathbf{a}' \bullet \mathbf{e}'_i \quad . \end{array} \qquad (1.2.2)$$

I think I need to back up some more, in particular to these equations above. Consider

$$\mathbf{a} = (a)_i\, \mathbf{e}_i \qquad (a)_i = \mathbf{a} \bullet \mathbf{e}_i$$

How is that dot product defined? It is defined in S space! Namely

$$\mathbf{a} \bullet \mathbf{e}_i = (a)_j\, (\mathbf{e}_i)_j = (a)_j \delta_{ij} = (a)_i$$

Next Nubbins question: Is $\mathbf{a} \bullet \mathbf{e}_i$ a scalar? [ answer is negative, see box below] This is a major recurring question in my efforts of the last decade. I never seem to nail down the answer. I think I have at least written words about this question in several places. The question is really: is this a rotational scalar? That of course gets down to our various "experiments". I even have a section right in frames doc on this subject. I have comments in active vs passive, and then more comments as the core of Section 3 on the Observer. I talk about things like (1.2.2) above, but I don't talk about whether or not $\mathbf{a} \bullet \mathbf{e}_i$ is a scalar. Nothing else in frames doc.

How about wedge doc? The word "experiment" does not appear, but a lot of dot products do appear. For example,

$$\mathbf{a} \bullet \mathbf{b} = g_{ij}a^i b^j = g^{ij}a_i b_j = a_i b^i = a^i b_i$$ x-space

$$\mathbf{a}' \bullet \mathbf{b}' = g'_{ij}a'^{i}b'^{j} = g'^{ij}a'_{i}b'_{j} = a'_{i}b'^{i} = a'^{i}b'_{i} \qquad \text{x'-space} \quad . \tag{2.2.5}$$

The dot product is a scalar (rank-0 tensor) so it must be the same in either space

$$\mathbf{a}' \bullet \mathbf{b}' = \mathbf{a} \bullet \mathbf{b} \; . \tag{2.2.6}$$

This is true of **a** and **b** are "general vectors" which transform according to $\mathbf{a}' = R\mathbf{a}$. But basis vectors do not transform in this way, so they are not "general vectors". That is a key initial fact to get clear. The basis vectors are not "tensorial vectors". If they were, you could write $\mathbf{e}'_n = R\mathbf{e}_n$ but we have $\mathbf{e}_n = R\,\mathbf{e}'_n$ so these particular vectors transform the wrong way to be normal tensorial vectors.

Conclusion #1: the following is NOT true: $\mathbf{a}' \bullet \mathbf{e}'_n = \mathbf{a} \bullet \mathbf{e}_n$

Conclusion #2: the dot product $\mathbf{a} \bullet \mathbf{e}_n$ is NOT a rotational scalar inasmuch as the above is not true. This thing is a scalar only in the sense that it is 1-tuple.

What about this question: Can the dot product $\mathbf{a} \bullet \mathbf{e}_n$ be evaluated in either Frame S or Frame S' . Lots of Nubbins questions!!! [ tentative answer in box below] Assume S and S' are both Cartesian. Let's try and see what happens

$$\mathbf{a} \bullet \mathbf{e}_n = (a)_i(\mathbf{e}_n)_i = (a)_n$$

$$\mathbf{a} \bullet \mathbf{e}_n = (a)'_i(\mathbf{e}_n)'_i$$

Well what is the value of $(\mathbf{e}_n)'_i$ ? I show in table (1.2.3) that it is

$$(\mathbf{e}_n)'_i = \mathbf{e}'_i \bullet \mathbf{e}_n = (R^{-1})_{ni} = R^{T}_{ni} = R_{in}$$

so then we have

$$\mathbf{a} \bullet \mathbf{e}_n = (a)'_i(\mathbf{e}_n)'_i = (a)'_i R_{in} = R^{T}_{ni}\,(a)'_i = [R^{T}\mathbf{a}]'_i \qquad ??$$

Tentative Conclusion #3: The expression $\mathbf{a} \bullet \mathbf{e}_n$ is NOT a rotational scalar, but still you can evaluate the dot product either in Frame S or in Frame S'. The results are

$$\mathbf{a} \bullet \mathbf{e}_n = (a)_i(\mathbf{e}_n)_i = (a)_n$$

$$\mathbf{a} \bullet \mathbf{e}_n = (a)'_i(\mathbf{e}_n)'_i = (a)'_i R_{in} = R^{T}_{ni}\,(a)'_i = [R^{T}\mathbf{a}]'_i$$

The second method of evaluation is (I think) valid, but does not seem very useful. Notice that the vector **a**' does not appear anywhere here, nor do any of its components. In the rightmost expression, I treat the

object $R^T$ as a matrix of numbers, not as any kind of tensor. Recall that it is not in either space, it is the R matrix, so it is NOT a tensor here or in tensor doc.

Fact: The R matrix is not a rank-2 tensor under transformation. It is a frame-linker matrix.

[ I address these dot product issues now in frames doc in Section 1.3. Yes $\mathbf{a} \bullet \mathbf{e}_n$ is not a scalar because the first vector is a Kinematic Vector while the second is a Basis Vector.]

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Now that I have tried to nail down some classic nightmare questions, I return to the original question

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

$$(d\mathbf{e}_n/dt)_{S'} = A\,\mathbf{e}_n$$

$$A = -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\,]$$

In what frame are these equations true as stated? That is a Nubbins question. Does the second equation have one of these forms??

$$\mathbf{a} = B\mathbf{b}$$

$$\mathbf{a}' = B'\,\mathbf{b}'$$

Partial answer: Since $\mathbf{e}_n$ does not transform as a tensorial vector, the object $A\,\mathbf{e}_n$ does not fit into the topic of $\mathbf{a} = B\mathbf{b}$ vs $\mathbf{a}' = B'\,\mathbf{b}'$ [ agreed ]

Question: Suppose you have this equation

$$\mathbf{v} = A\mathbf{e}_n$$

where A is basically a derived rotation matrix of some sort (as in our case above). What can you say about the vector nature of this equation?

1. $\mathbf{e}_n$ is not a tensorial vector. [ it is not a Kinematic Vector ]
2. A being a rotation like R is not a tensorial rank-2 tensor. [ I think in fact A *is* a rank-2 tensor ]
3. It seems unlikely that $\mathbf{v}$ would be a tensorial vector. [ not sure of this! ]
4. Each side of this equation is a 3-tuple, that is true.

Therefore if we have the equation.

$$(d\mathbf{e}_n/dt)_{S'} = A\,\mathbf{e}_n$$

I would say that the LHS is probably not a tensorial vector.

So this is a very *suspicious* equation in terms of its tensor nature . [ agreed, suspicious ]

Question: Suppose you have this equation

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

Since $\mathbf{e}_n$ is not a tensorial vector, I don't think the right side is a tensorial vector. I think $\boldsymbol{\omega}$ probably is such a vector and you can say $\boldsymbol{\omega}' = R\boldsymbol{\omega}$ (but not sure) [I think yes] . So probably the left side is NOT a tensorial vector. It also involves $\mathbf{e}_n$ which is not a true vector.

Let's now back up from here a bit and look at

$$d\mathbf{v} = d\boldsymbol{\varphi} \times \mathbf{v} \qquad d\boldsymbol{\varphi} \equiv d\varphi\, \hat{\mathbf{n}} \tag{1.5.8}$$

$$(d\mathbf{v}/dt) = \boldsymbol{\omega} \times \mathbf{v} \qquad \text{where } \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt \quad . \tag{1.6.1}$$

I think everything is a true vector here, *assuming* that we take $\mathbf{v}$ to be a true vector. [ ok ]

Question: for a true vector, do you have to write $(d\mathbf{v}/dt)_S$ and $(d\mathbf{v}/dt)_{S'}$ to distinguish two cases? This is a Nubbins question. Well, if $\mathbf{v}$ is the velocity vector, we know that

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \tag{6.6a}$$

Since we don't just have $\mathbf{v}' = R\mathbf{v}$, I would say that <u>$\mathbf{v}$ is NOT a true vector</u>. It does not transform between the frames as a rotational vector.

[ Wrong! We don't have $\mathbf{v}' = R\mathbf{v}$ because the *name* $\mathbf{v}'$ is already defined be something else, so we cannot just define $\mathbf{v}' \equiv R\mathbf{v}$ for our Passive View work. But I think (6.6a) above is at least a vector equation which you could evaluate in Frame S or Frame S'. Probably it is also a tensorial vector equation. ]

Rewrite the above as

$$(d\mathbf{r}/dt)_S = (d\mathbf{r}'/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$$

If the two frames had a static relationship (both inertial) then $\boldsymbol{\omega} = 0$ and $\dot{\mathbf{b}}_S = 0$ and then $\mathbf{r} = \mathbf{r}' + \mathbf{b}$ says that $d\mathbf{r} = d\mathbf{r}'$ and then the above just becomes

$$(d\mathbf{r}/dt)_S = (d\mathbf{r}/dt)_{S'}$$

and you get the same thing measuring in either inertial frame. So it is the fact that at least one frame is non-inertial that is causing our need for the $()_S$ labeling. They are "rotating frames of reference". [ ok ]

So vector senses of things are very suspect here! The usual tensor "analysis" of *rotations* assumes that you are just comparing two static frames of reference, and then $\mathbf{v}' = R\mathbf{v}$ for normal vectors and all is well.

OK, go back to the conical motion idea for vector **a** which is not a true tensor.

$$(d\mathbf{a}/dt) = \boldsymbol{\omega} \times \mathbf{a}$$

Picture of this in (1.6.2).

I open section 1.7 by saying "let's apply the above to $\mathbf{e}'_n$" (which is NOT a true vector). I then write

$$(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n$$

I say "we are implicitly computing this derivative in Frame S". I write

$$\mathbf{q} \equiv (d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n \quad . \qquad (1.7.1)$$

$$\mathbf{c} \equiv (d\mathbf{e}'_n/dt)_{S'} = 0 \qquad (1.7.2)$$

so everyone must agree, there is a big difference between these two left hand sides!

Nubbins question: Once again, can you evaluate either of these equations in both frames or not? This is like asking if you can do this for **q** in the first equation,

$$\mathbf{q} = (\mathbf{q})_i \mathbf{e}_i = (\mathbf{q})'_i \mathbf{e}'_i$$

Note there is no vector **q**' mentioned. It seems that you MUST be able to do either expansion *even though* **q** is not a true vector, so we don't have $\mathbf{q}' = R\mathbf{q}$. So then we would have

$$(q)_i = [\, (d\mathbf{e}'_n/dt)_S \,]_i = \varepsilon_{ijk}\, \omega_j\, (\mathbf{e}'_n)_k \qquad (\mathbf{e}'_n)_k = R_{nk} \qquad \text{Frame S}$$

$$(q)'_i = [\, (d\mathbf{e}'_n/dt)_S \,]'_i = \varepsilon_{ijk}\, (\omega)'_j\, (\mathbf{e}'_n)'_k \qquad (\mathbf{e}'_n)'_k = \delta_{n,k} \qquad \text{Frame S'}$$

This is a big Nubbins question right now. Can you really do both these evaluations?? I tentatively think that yes you can, only because I see no reason why you cannot [ agreed ] . I even state this explicitly in (1.8.10) !!! I see no contradiction that is obvious.

OK, let's go with that conclusion for a while. It would then also apply to this equation,

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \quad . \qquad (1.7.4)$$

You can evaluate the equation in either frame. So

$$(\mathbf{d})_i = [\, (d\mathbf{e}_n/dt)_{S'} \,]_i = \varepsilon_{ijk}\, \omega_j\, (\mathbf{e}_n)_k \qquad (\mathbf{e}_n)_k = \delta_{n,k} \qquad \text{Frame S}$$

$(\mathbf{d})'_i = [\ (d\mathbf{e}_n/dt)_{S'}\ ]'_i = \varepsilon_{ijk}\,(\omega)'_j\,(\mathbf{e}_n)'_k \qquad (\mathbf{e}_n)'_k = R_{kn}$ Frame S'

Tentative Conclusion : The equation $(d\mathbf{e}_n/dt)_{S'} = -\ \boldsymbol{\omega} \times \mathbf{e}_n$ is valid in both Frame S and Frame S' components. You can "evaluate it in either Frame". [ correct ]

So now we come to my OTHER equation which is a bit more delicate.

$\mathbf{e}_n = R(\boldsymbol{\Phi})\ \mathbf{e}'_n$ // not a true vector equation, but can eval in either Frame

$(d\mathbf{e}_n)_{S'} = dQ\ \mathbf{e}'_n$ // not a true vector equation, note that $(d\mathbf{e}'_n)_{S'} = 0$

$dQ = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi}) = [\ R_z(\psi+d\psi)\,R_x(\theta+d\theta)\,R_z(\varphi+d\varphi) - R_z(\psi)\,R_x(\theta)\,R_z(\varphi)\ ]$

This dQ is a 3x3 matrix of numbers, it is made of basic rotations, it is NOT a tensor.

[ Well, we know that R by itself is a basis-linking tensor-like object, so dQ would be the difference of two such objects, so it would seem that dQ is not just a matrix of numbers like π. ]

First Nubbins question: can you evaluate $(d\mathbf{e}_n)_{S'} = dQ\ \mathbf{e}'_n$ in either frame??

$(\mathbf{q})_i = [(d\mathbf{e}_n)_{S'}]_i = [dQ\ \mathbf{e}'_n]_i$ Frame S

???

$(\mathbf{q})'_i = [(d\mathbf{e}_n)_{S'}]'_i = [dQ\ \mathbf{e}'_n]'_i$ Frame S'

Second Nubbins question: if you CAN do the above, can you say (dQ is "just numbers")

$[dQ\ \mathbf{e}'_n]_i = (dQ)_{ij}\,(\mathbf{e}'_n)_j = (dQ)_{ij}R_{nj}$

$[dQ\ \mathbf{e}'_n]'_i = (dQ)_{ij}\,(\mathbf{e}'_n)'_j = (dQ)_{ij}\delta_{nj}$

[ I think this is wrong, I think the second line's matrix should be $(dQ)'_{ij}$, while first line is OK. ]

These evaluations are at least *different* from each other. I think both equations are OK. [ not as stated ]

Let's move down a little more in the calculation.

$(d\mathbf{e}_n/dt)_{S'} = [(dQ/dt)\ R^{-1}(\boldsymbol{\Phi})\ ]\ \mathbf{e}_n = A\ \mathbf{e}_n$

Again, it seems that A is just a matrix of "numbers" and not a tensor. So then

$(\mathbf{q})_i = [(d\mathbf{e}_n/dt)_{S'}]_i = [A\ \mathbf{e}_n]_i = A_{ij}(\mathbf{e}_n)_j = A_{ij}\ \delta_{nj}$

$(\mathbf{q})'_i = [(d\mathbf{e}_n/dt)_{S'}]'_i = [A\ \mathbf{e}_n]'_i = A_{ij}(\mathbf{e}_n)'_j = A_{ij}R_{jn}$

Again these two evaluations are *different* from each other.

So let's gather up results so far, assuming you can evaluate everything in either frame.

$(\mathbf{d})_i = [\ (d\mathbf{e}_n/dt)_{S'}\ ]_i = \varepsilon_{ijk}\,\omega_j\,(\mathbf{e}_n)_k = \varepsilon_{ijk}\,\omega_i\,\delta_{n,k}$ Frame S

$(\mathbf{d})'_i = [\ (d\mathbf{e}_n/dt)_{S'}\ ]'_i = \varepsilon_{ijk}\,(\omega)'_j\,(\mathbf{e}_n)'_k = \varepsilon_{ijk}\,(\omega)'_i R_{kn}$ Frame S'

$(\mathbf{q})_i = [(d\mathbf{e}_n/dt)_{S'}]_i = [A\ \mathbf{e}_n]_i = A_{ij}(\mathbf{e}_n)_j = A_{ij}\ \delta_{n,j}$ Frame S

$(\mathbf{q})'_i = [(d\mathbf{e}_n/dt)_{S'}]'_i = [A\ \mathbf{e}_n]'_i = A_{ij}(\mathbf{e}_n)'_j = A_{ij}R_{jn}$ Frame S'

The vector $\mathbf{q} = \mathbf{d}$ is the same on both sides. So equating right sides, we get

$\varepsilon_{ijk}\,\omega_j\,\delta_{n,k} = A_{ij}\ \delta_{n,j}$ Frame S

$\varepsilon_{ijk}\,(\omega)'_j R_{kn} = A_{ij}R_{jn}$ Frame S'

The two equations appear to be different. I can rewrite them this way

$\varepsilon_{ijn}\,\omega_j = A_{in}$ Frame S

$\varepsilon_{ikj}\,(\omega)'_k\ R_{jn} = A_{ij}R_{jn}$ Frame S'

Nubbins question. Can the second equation be trivially simplified? It does say'

$[\varepsilon_{ikj}\,(\omega)'_k - A_{ij}]\,R_{jn} = 0$

BUT I cannot say anything like "the $R_{jn}$ form a complete basis and therefore....". I just know that the terms in this sum add up to zero, that is ALL I know. It is true that ONE WAY this could happen is if

$\varepsilon_{ikj}\,(\omega)'_k - A_{ij} = 0 \quad \Rightarrow \quad \varepsilon_{ikj}\,(\omega)'_k = A_{ij}$

But in Frame S these seems little uncertainty about the conclusion:

$\varepsilon_{ijn}\,\omega_j = A_{in}$

and this indeed gives the WRONG ANSWER!!!!

Is it possible that $\boldsymbol{\omega}$ has the same components in both frames??? [no] I want a simple example. I did one on paper, no they are NOT the same in both frames.

Conclusion: It must be that at least one of my two equations is not allowed to be evaluated in Frame S.

$(\mathbf{d})_i = [\ (d\mathbf{e}_n/dt)_{S'}\ ]_i = \varepsilon_{ijk}\,\omega_j\,(\mathbf{e}_n)_k = \varepsilon_{ijk}\,\omega_i\,\delta_{n,k}$ Frame S ?

$(\mathbf{d})'_i = [\ (d\mathbf{e}_n/dt)_{S'}\ ]'_i = \varepsilon_{ijk}\,(\omega)'_j\,(\mathbf{e}_n)'_k = \varepsilon_{ijk}\,(\omega)'_i R_{kn}$ Frame S' [ both OK ]

$(\mathbf{q})_i = [(d\mathbf{e}_n/dt)_{S'}]_i = [A\ \mathbf{e}_n]_i = A_{ij}(\mathbf{e}_n)_j = A_{ij}\ \delta_{n,j}$ Frame S ? [ok]

$(\mathbf{q})'_i = [(d\mathbf{e}_n/dt)_{S'}]'_i = [A\ \mathbf{e}_n]'_i = A_{ij}(\mathbf{e}_n)'_j = A_{ij}R_{jn}$ Frame S' [wrong ]

[ second equation should be : $(\mathbf{q})'_i = [(d\mathbf{e}_n/dt)_{S'}]'_i = [A\mathbf{e}_n]'_i = A'_{ij}(\mathbf{e}_n)'_j = A'_{ij}R_{jn}$ ]

[ **STOP HERE for now on 3.1.17. This A versus A' problem is still not clear to me I am afraid and it needs still more work**. ]

Which one is it, and why is it? I more suspect the **q** equation.

Go back to this

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n - R(\mathbf{\Phi})\mathbf{e}'_n \equiv dQ\ \mathbf{e}'_n \qquad \text{(G.4.8)}$$

$$dQ \equiv [\ R_z(\psi+d\psi)\,R_x(\theta+d\theta)\,R_z(\varphi+d\varphi) - R_z(\psi)\,R_x(\theta)\,R_z(\varphi)\ ]\ . \qquad \text{(G.4.9)}$$

Comment: dQ is a difference of rotations, and therefore is NOT itself a rotation. I don't think it makes any sense to add rotation matrices. You multiply them, not add them.

Maybe write again

$$d\mathbf{e}_n = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n - R(\mathbf{\Phi})\mathbf{e}'_n \equiv dQ\ \mathbf{e}'_n \qquad \text{(G.4.8)}$$

$$dQ \equiv [\ R_z(\psi+d\psi)\,R_x(\theta+d\theta)\,R_z(\varphi+d\varphi) - R_z(\psi)\,R_x(\theta)\,R_z(\varphi)\ ]\ . \qquad \text{(G.4.9)}$$

Should I write this as

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n - R(\mathbf{\Phi})\mathbf{e}'_n \equiv (dQ)_{S'}\ \mathbf{e}'_n$$

$$(d\mathbf{e}_n/dt)_{S'} = (dQ/dt)_{S'}\ \mathbf{e}'_n$$

The question really is this: Why can't you evaluate the above equation in Frame S components?

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [(dQ/dt)_{S'}\ \mathbf{e}'_n]_i = [(dQ/dt)_{S'}]_{ij}(\mathbf{e}'_n)_j$$

$$\{(d\mathbf{e}_n/dt)_{S'}\} \bullet \mathbf{e}_i = \{(dQ/dt)_{S'}\ \mathbf{e}'_n\} \bullet \mathbf{e}_i$$

**Let's try a little Dirac notation** to learn about the object $(dQ/dt)_{S'}\ \mathbf{e}'_n$ . Do full parallel,

$1 = |\mathbf{e'}_m\rangle\langle\mathbf{e'}_m|$ $\qquad$ $1 = |\mathbf{e}_j\rangle\langle\mathbf{e}_j|$

---

$R_{ij} = [R^{ss}]_{ij}$

$\mathbf{e}_n = R\ \mathbf{e'}_n$

$|\mathbf{e}_n\rangle = \mathcal{R}\ |\mathbf{e'}_n\rangle$

$(\mathbf{e}_n)_i = [\ R\ \mathbf{e'}_n]_i = R_{ij}\ (\mathbf{e'}_n)_j$ $\qquad$ naively $\qquad$ so $\quad R_{ij} = [R^{ss}]_{ij}$

$\langle\mathbf{e}_i|\mathbf{e}_n\rangle = \langle\mathbf{e}_i|\mathcal{R}\ |\mathbf{e'}_n\rangle = \langle\mathbf{e}_i|\mathcal{R}\ |\mathbf{e}_j\rangle\langle\mathbf{e}_j|\mathbf{e'}_n\rangle\ = [R^{ss}]_{ij}(\mathbf{e'}_n)_j$

$(\mathbf{e}_n)'_i = [R\ \mathbf{e'}_n]'_i\ = R'_{ij}\ (\mathbf{e'}_n)'_j$ $\qquad$ naively $\qquad$ so $\quad R'_{ij} = [R^{s's'}]_{ij}$

$\langle\mathbf{e'}_i|\mathbf{e}_n\rangle = \langle\mathbf{e'}_i|\mathcal{R}\ |\mathbf{e'}_n\rangle = \langle\mathbf{e'}_i|\mathcal{R}\ |\mathbf{e'}_j\rangle\langle\mathbf{e'}_j|\mathbf{e'}_n\rangle\ = [R^{s's'}]_{ij}(\mathbf{e'}_n)'_j$

---

$\mathbf{e'}_n = R^{-1}\ \mathbf{e}_n$

$|\mathbf{e'}_n\rangle = \mathcal{R}^{-1}\ |\mathbf{e}_n\rangle$

$(\mathbf{e'}_n)_i = [R^{-1}\mathbf{e}_n]_i = R^{-1}{}_{ij}\ (\mathbf{e}_n)_j$ $\qquad$ naively $\qquad$ so $\quad R^{-1}{}_{ij} = [R^{-1ss}]_{ij}$

$\langle\mathbf{e}_i|\mathbf{e'}_n\rangle = \langle\mathbf{e}_i|\mathcal{R}^{-1}\ |\mathbf{e}_n\rangle = \langle\mathbf{e}_i|\mathcal{R}^{-1}\ |\mathbf{e}_j\rangle\langle\mathbf{e}_j|\mathbf{e}_n\rangle\ = [R^{-1ss}]_{ij}(\mathbf{e}_n)_j$ $\qquad$ Frame S

$(\mathbf{e'}_n)'_i = [R^{-1}\mathbf{e}_n]'_i = R'^{-1}{}_{ij}\ (\mathbf{e}_n)'_j$ $\qquad$ naively $\qquad$ so $\quad R'^{-1}{}_{ij}\ = [R^{-1s's}]_{ij}$

$\langle\mathbf{e'}_i|\mathbf{e'}_n\rangle = \langle\mathbf{e'}_i|\mathcal{R}^{-1}\ |\mathbf{e}_n\rangle = \langle\mathbf{e'}_i|\mathcal{R}^{-1}\ |\mathbf{e'}_j\rangle\langle\mathbf{e'}_j|\mathbf{e}_n\rangle\ = [R^{-1s's'}]_{ij}(\mathbf{e}_n)'_j$ $\qquad$ Frame S'

---

$(d\mathbf{e}_n)_{S'} = R(\Phi+d\Phi)\mathbf{e'}_n - R(\Phi)\mathbf{e'}_n \quad \equiv dQ\ \mathbf{e'}_n$

$|d\mathbf{e}_n\rangle\ = [\ \mathcal{R}_1-\mathcal{R}_2]\ |\mathbf{e'}_n\rangle\ =\ d\mathcal{Q}\ |\mathbf{e'}_n\rangle$

$[(d\mathbf{e}_n)_{S'}]_i = [dQ\ \ \mathbf{e'}_n]_i\ = dQ_{ij}\ (\mathbf{e'}_n)_j$ $\qquad$ naively $\qquad$ so $\qquad dQ_{ij} = Q^{ss}{}_{ij}$

$\langle\mathbf{e}_i|d\mathbf{e}_n\rangle\ = \langle\mathbf{e}_i|\ d\mathcal{Q}\ |\mathbf{e'}_n\rangle = \langle\mathbf{e}_i|\ d\mathcal{Q}\ \ |\mathbf{e}_j\rangle\langle\mathbf{e}_j|\mathbf{e'}_n\rangle\ = Q^{ss}{}_{ij}\ (\mathbf{e'}_n)_j$

$[(d\mathbf{e}_n)_{S'}]'_i = [dQ\ \mathbf{e}'_n]'_i = dQ'_{ij}\,(\mathbf{e}'_n)'_j$ naively so $dQ'_{ij} = Q^{s's'}{}_{ij}$

$\langle \mathbf{e}'_i | d\mathbf{e}_n \rangle = \langle \mathbf{e}'_i | \boldsymbol{dQ} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i | \boldsymbol{dQ} | \mathbf{e}'_j \rangle \langle \mathbf{e}'_j | \mathbf{e}'_n \rangle = Q^{s's'}{}_{ij}\,(\mathbf{e}'_n)'_j$

---

$(d\mathbf{e}_n)_S = dQ\ \mathbf{e}'_n = dQ\ R^{-1}\ \mathbf{e}_n$

$|d\mathbf{e}_n\rangle = \boldsymbol{dQ}\,|\mathbf{e}'_n\rangle = \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1}\,|\mathbf{e}_n\rangle$

$[(d\mathbf{e}_n)_S]_i = [dQ\ R^{-1}\ \mathbf{e}_n]_i = [dQ\ R^{-1}]_{ij}\,(\mathbf{e}_n)_j$ so $[dQ\ R^{-1}]_{ij} = [(dQ\ R^{-1})^{ss}]_{ij}$

$\langle \mathbf{e}_i | d\mathbf{e}_n \rangle = \langle \mathbf{e}_i | \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1} | \mathbf{e}_n \rangle = \langle \mathbf{e}_i | \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1} | \mathbf{e}_j \rangle \langle \mathbf{e}_j | \mathbf{e}_n \rangle = [(dQ\ R^{-1})^{ss}]_{ij}\,(\mathbf{e}_n)_j$

$[(d\mathbf{e}_n)_S]'_i = [dQ\ R^{-1}\ \mathbf{e}_n]'_i = [dQ\ R^{-1}]'_{ij}\,(\mathbf{e}_n)'_j$ so $[dQ\ R^{-1}]'_{ij} = [(dQ\ R^{-1})^{s's'}]_{ij}$

$\langle \mathbf{e}'_i | d\mathbf{e}_n \rangle = \langle \mathbf{e}'_i | \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1} | \mathbf{e}_n \rangle = \langle \mathbf{e}'_i | \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1} | \mathbf{e}'_j \rangle \langle \mathbf{e}'_j | \mathbf{e}_n \rangle = [(dQ\ R^{-1})^{s's'}]_{ij}\ (\mathbf{e}_n)'_j$

---

Now define

$\boldsymbol{\mathcal{A}} = \boldsymbol{dQ}\,\boldsymbol{\mathcal{R}}^{-1}$

Then I have shown above that

$[(d\mathbf{e}_n)_S]_i = [A^{ss}]_{ij}\,(\mathbf{e}_n)_j$ Frame S components

$[(d\mathbf{e}_n)_S]'_i = [A^{s's'}]_{ij}(\mathbf{e}_n)'_j$ Frame S' components

<u>Major Conclusion #1:</u> The two A matrices are NOT the same, and I can deduce how they are related it I want to.

Nubbins question: So WHICH of these matrices is equal to the stuff I have?

Here is what I want to show:

$[dQ^{s's'}]_{ij} = [\ R_z(\psi+d\psi)\,R_x(\theta+d\theta)\,R_z(\varphi+d\varphi) - R_z(\psi)\,R_x(\theta)\,R_z(\varphi)\ ]$

$[A^{s's'}]_{ij} = -[\ (\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\ ]_{ij}$

Go back to this

$$\mathbf{e}_n = R(\mathbf{\Phi})\, \mathbf{e'}_n \qquad (G.4.4)$$

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'}_n - R(\mathbf{\Phi})\mathbf{e'}_n \equiv dQ\, \mathbf{e'}_n \qquad (G.4.8)$$

$$dQ \equiv [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\, ] \ . \qquad (G.4.9)$$

I really have to restate this as follows:

$$(\mathbf{e}_n)'_i = [R^{s's'}]_{ij}\, (\mathbf{e'}_n)'_j \qquad \text{all in Frame S'}$$

where

$$[R^{s's'}]_{ij} = [R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]_{ij}$$

So I am assuming that the matrix $R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$ is specific to the Frame S' components!! I am assuming that this is the matrix WHEN I am working in Frame S' components.

Question on Section 1.1.

First I say

$$\mathbf{e}_n = R\, \mathbf{e'}_n \qquad n = 1,2,3 \qquad \text{or} \qquad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e'}_n)_j \qquad (1.1.1)$$

In the left equation, there is no commitment to component type, so R is just an operator, as it is in the Dirac notation. On the right I have committed to Frame S components, and I say as much in the text.

Now I am mystified by what I say in (1.1.2) :

$$\mathbf{e}_n = (R^{-1})_{nm}\, \mathbf{e'}_m \qquad \text{or} \qquad \mathbf{e'}_n = R_{nm}\, \mathbf{e}_m \qquad (1.1.2)$$

Here we have linear combinations of basis vectors. But there is not commitment to component types. So how can I have a specific matrix (that shown in 1.1.1) when I am not yet committed to component type?

Am I claiming that the equation is true for either component type, where R is the Frame S matrix?

I better examine the proof to see what is going on here. First I say

*Step 1:* The orthonormal Cartesian basis vectors have these properties

$$\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e'}_n \bullet \mathbf{e'}_k \qquad \text{and} \qquad (\mathbf{e}_n)_k = \delta_{n,k} \qquad (1.1.3)$$

where $(\mathbf{e}_n)_k = \mathbf{e}_k \bullet \mathbf{e}_n$ denotes the $k^{th}$ component of $\mathbf{e}_n$ in Frame S. The basis vectors $\mathbf{e}_n$ are axis-aligned in Frame S, while the $\mathbf{e'}_n$ are axis-aligned in Frame S'. The dot product is $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \equiv a_i b_i$ .

I see nothing wrong here, but I could add that $(\mathbf{e'}_n)'_k = \delta_{n,k}$ as well. But I have not yet really defined the prime notation, but pretend that the reader has read ahead on that. So step 1 is OK. Next,

*Step 2:* Note that

$$(\mathbf{e'}_n)_m = \mathbf{e}_m \bullet \mathbf{e'}_n \overset{(1.1.1)}{=} \mathbf{e}_m \bullet [R^{-1}\mathbf{e}_n] \overset{(1.1.3)}{=} (\mathbf{e}_m)_i\,(R)^{-1}_{ij}(\mathbf{e}_n)_j \overset{(1.1.3)}{=} \delta_{m,i}\,(R)^{-1}_{ij}\,\delta_{n,j}$$

Frame S basis

$$= (R^{-1})_{mn} = (R^T)_{mn} = R_{nm} \qquad (1.1.4)$$

What happens if I instead use the other basis. Try a similar equation

*Step 2 Similar:* Note that

$$(\mathbf{e'}_n)_m = \mathbf{e}_m \bullet \mathbf{e'}_n \overset{(1.1.1)}{=} \mathbf{e}_m \bullet [R^{-1}\mathbf{e}_n] \overset{(1.1.3)}{=} (\mathbf{e}_m)'_i\,(R')^{-1}_{ij}(\mathbf{e}_n)'_j \overset{(1.1.3)}{=} ????$$

Frame S' basis

Well I do know that $(\mathbf{e}_m)'_i = \mathbf{e}_m \bullet \mathbf{e'}_i = R_{im}$. So I could rewrite the above as

*Step 2 Similar:* Note that

$$(\mathbf{e'}_n)_m = \mathbf{e}_m \bullet \mathbf{e'}_n \overset{(1.1.1)}{=} \mathbf{e}_m \bullet [R^{-1}\mathbf{e}_n] \overset{(1.1.3)}{=} (\mathbf{e}_m)'_i\,(R')^{-1}_{ij}(\mathbf{e}_n)'_j \overset{(1.1.3)}{=} R_{im}\,(R')^{-1}_{ij}\,R_{jn}$$

Frame S' basis

$$= R^T_{mi}\,(R')^{-1}_{ij}\,R_{jn} = (R^T R'^{-1} R)_{mn}$$

But then comparing Step 2 and Step 2 Similar you could conclude that

$(\mathbf{e'}_n)_m = R_{nm} = (R^T R'^{-1} R)_{mn}$ // Frame S components

$R^T = R^T R'^{-1} R$ in the Frame S basis at least, probably any basis

Now left multiply by $(R^T)^{-1}$ to get

$1 = R'^{-1} R \quad \Rightarrow \quad R' = R$

That is way too simple a conclusion, so I see another day of Nubbins stuff on this.

My understanding of Section 1.1 is collapsing, girders flying everywhere, the opening gambit of frames doc! It is the crucial underpinning of everything.

Look in Dirac notation for help.

$R' = R$ ?

$\langle \mathbf{e'}_i | \mathcal{R} | \mathbf{e'}_j \rangle = \langle \mathbf{e}_i | \mathcal{R} | \mathbf{e}_j \rangle$ ?

Well,

$$\langle \mathbf{e'}_i | \mathcal{R} | \mathbf{e'}_j \rangle = \langle \mathbf{e'}_i | \mathbf{e}_n \rangle \langle \mathbf{e}_n | \mathcal{R} | \mathbf{e}_m \rangle \langle \mathbf{e}_m | \mathbf{e'}_j \rangle \quad ?$$

$$R'_{ij} = R_{in} R_{nm} R_{jm} = R_{in} R_{nm} R^{T}_{mj} = (RRR^{T})_{ij} = R_{ij}$$

there it is again! I just showed that $R^{ss}$ and $R^{s's'}$ are different, but now they are the same and my paradox with ω comes back to life?