This has been installed into Appendix G, do not edit here.

G.7 Computation of ω (Method 2)

In this section our approach to computing **ω** for the Euler Angle rotation is to find an equation which involves **ω** and solve it for **ω**! More or less at random, we choose (1.7.4),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

Unlike our Method 1 computation of **ω** in the previous section, here we shall have no need for Goldstein's geometric Figure (G.5.5) or the various variables like ξ and ζ . We will, however, need various algebraic results developed in Section G.3.

Recall from (G.5.19) that the Goldstein Euler angle rotation is given by

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \,. \qquad \textit{(G.5.19)} \qquad \text{(G.7.2)}$$

In order to avoid a hundred minus signs, we shall *temporarily negate* all three angles, then when we are done, we will undo this negation. We therefore temporarily take R to be,

$$R(\psi,\theta,\varphi) = R_z(\psi)R_x(\theta)R_z(\varphi) \equiv R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi}\bullet\mathbf{J}) \,. \qquad \text{(G.7.3)}$$

Comment: To find **Φ** we could write out the matrix $R_z(\psi)R_x(\theta)R_z(\varphi)$,

```
R := evalm(Rz(psi) &* Rx(theta) &* Rz(phi));
```

$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

and decompose it into its symmetric and antisymmetric components S and A. *In theory* we could then compute Φ (=θ) and $\hat{\boldsymbol{\Phi}}$ (=$\hat{\mathbf{n}}$) from (G.2.14) and come up with an explicit expression for $\boldsymbol{\Phi} = \Phi\hat{\boldsymbol{\Phi}}$. The reader is just reminded that this is mechanically possible, but luckily we have no need for the result (which "ain't purdy"). We could also compute d**Φ** from the following

$$R(\psi+d\psi,\theta+d\theta,\varphi+d\varphi) = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = \exp(-i\,[\boldsymbol{\Phi}+d\boldsymbol{\Phi}]\bullet\mathbf{J})$$

but again there is no need to do this. Note that **Φ** and d**Φ** will generally not be in the same direction.

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n = R\mathbf{e}'_n \,. \qquad \text{(G.7.4)}$$

We are a Worker/Observer sitting in Frame S' and we want to compute the change we see in the Frame S basis vector $\mathbf{e_n}$ due to Euler angle changes dψ, dθ and dφ. Then,

$$(d\mathbf{e_n})_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'_n} - R(\mathbf{\Phi})\mathbf{e'_n} \qquad // \ \mathbf{e'_n} \text{ is a constant in Frame S'}$$

$$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi)\, \mathbf{e'_n} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\, \mathbf{e'_n}$$

$$\equiv dR'\, \mathbf{e'_n} \qquad \text{(G.7.5)}$$

where

$$dR' \equiv [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,] \ . \qquad \text{(G.7.6)}$$

The matrix dR' has a prime because it is a rank-2 tensor within Frame S'. The same comment applies to the matrix A' below.

Replace $\mathbf{e'_n}$ in (G.7.5) using (G.7.4) to get

$$(d\mathbf{e_n})_{S'} = dR'\, \mathbf{e'_n} = dR'\, [R^{-1}(\mathbf{\Phi})\mathbf{e_n}\,] = [dR'\, R^{-1}(\mathbf{\Phi})\,]\, \mathbf{e_n} \ . \qquad \text{(G.7.7)}$$

Next, divide by dt to get

$$(d\mathbf{e_n}/dt)_{S'} = [(dR'/dt)_{S'} R^{-1}(\mathbf{\Phi})\,]\, \mathbf{e_n}$$

$$\equiv A'\, \mathbf{e_n} \qquad A' \equiv [(dR'/dt)_{S'} R^{-1}(\mathbf{\Phi})\,] \ . \qquad \text{(G.7.8)}$$

where A' is defined as shown. Recall that we are trying to solve the following equation for **ω,**

$$(d\mathbf{e_n}/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e_n} \qquad \textit{(G.7.1)}$$

so we seem to be on the right track. We must compute this matrix A', so we start with dR' which we compute to first order in the differential angles,

$$dR' = [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,]$$

$$= [\, R_z(\psi) R_z(d\psi)\, R_x(\theta)\, R_x(d\theta) R_z(\varphi) R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$$

$$= [\, R_z(\psi)\{1-id\psi J_3\}\, R_x(\theta)\{1-id\theta J_1\} R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$$

$$= [-id\psi\, R_z(\psi)J_3 R_x(\theta) R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta) J_1 R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta) R_z(\varphi) J_3\,] \ . \qquad \text{(G.7.9)}$$

We have used the first order expansion (1.5.6) for the small angle rotations. Dividing by dt gives,

$$(dR'/dt)_{S'} = [\, -i\dot{\psi}\, R_z(\psi)J_3 R_x(\theta) R_z(\varphi) - i\dot{\theta}\, R_z(\psi) R_x(\theta) J_1 R_z(\varphi) - i\dot{\varphi}\, R_z(\psi) R_x(\theta) R_z(\varphi) J_3\,] \ . \qquad \text{(G.7.10)}$$

We now assemble the matrix A' and process it with vigor, making use three times of the trivial sandwich rule (G.3.6) as shown in blue,

$$A' \equiv (dR'/dt)_S \cdot R^{-1}(\mathbf{\Phi}) \qquad \text{// where } R^{-1}(\mathbf{\Phi}) = R_z(\varphi)R_x(\theta)R_z(\psi) \text{ from (G.7.3)}$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) \ - \ i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) \ - \ i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) \ - \ i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)\color{blue}{R_z(\varphi)J_3R_z(-\varphi)}\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) \ - \ i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)\color{blue}{J_3}\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3 \ - \ i\dot{\theta}\ R_z(\psi)\, \color{blue}{R_x(\theta)J_1R_x(-\theta)} \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3 \ - \ i\dot{\theta}\ R_z(\psi)\, \color{blue}{J_1} \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ \color{blue}{R_z(\psi)J_3\, R_z(-\psi)} \ - \ i\dot{\theta}\ R_z(\psi)\, J_1\, R_z(-\psi) \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\, R_z(-\psi)\ ]$$

$$= [\, -i\dot{\psi}\ \color{blue}{J_3} \ - \ i\dot{\theta}\ R_z(\psi)\, J_1\, R_z(-\psi) \ - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\, R_z(-\psi)\ ]$$

$$= [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\ M_1 \ - i\dot{\varphi}\, M_2\,] \qquad \text{(G.7.11)}$$

where

$$M_1 \equiv R_z(\psi)\, J_1R_z(-\psi) \qquad = [\, R_3(\psi)\, J_1R_3(-\psi)]$$

$$M_2 \equiv R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)R_z(-\psi) \qquad = R_3(\psi)\, [\, R_1(\theta)J_3R_1(-\theta)\,]\, R_3(-\psi)\ . \qquad \text{(G.7.12)}$$

We now call upon our non-trivial sandwich formulas in (G.3.8) to simplify thing further :

$$M_1 = R_3(\psi)\, J_1R_3(-\psi) = \cos\psi J_1 + \sin\psi\, J_2 \qquad \text{// (G.3.8) line 5} \qquad \text{(G.7.13)}$$

$$M_2 = R_3(\psi)\, [\, R_1(\theta)J_3R_1(-\theta)\,]\, R_3(-\psi)$$

$$= R_3(\psi)\, [\, \cos\theta\, J_3 - \sin\theta\, J_2\,]\, R_3(-\psi) \qquad \text{// (G.3.8) line 2}$$

$$= \cos\theta\, [\, R_3(\psi)\, J_3\, R_3(-\psi)] \ - \sin\theta\, [R_3(\psi)J_2R_3(-\psi)]$$

$$= \cos\theta\, J_3 \ - \sin\theta\, (\cos\psi J_2 - \sin\psi\, J_1) \qquad \text{// (G.3.6) and (G.3.8) line 6}$$

$$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\, J_3\ . \qquad \text{(G.7.14)}$$

Then,

$$A' = [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\ M_1 \ - i\dot{\varphi}\, M_2\,]$$

$$= [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\,(\cos\psi J_1 + \sin\psi\, J_2) \ - i\dot{\varphi}\,(\sin\theta c\sin\psi J_1 - \sin\theta\cos\psi J_2 + \ \cos\theta\, J_3)\,]$$

$$= [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\,\cos\psi J_1 \ - \ i\dot{\theta}\,\sin\psi\, J_2 \ - i\dot{\varphi}\,\sin\theta\sin\psi J_1 + \ i\dot{\varphi}\sin\theta\cos\psi J_2 - i\dot{\varphi}\ \cos\theta\, J_3\,]$$

$$= (-i)[\, \dot{\psi}\ J_3 \ + \dot{\theta}\,\cos\psi J_1 \ + \dot{\theta}\,\sin\psi\, J_2 \ + \dot{\varphi}\,\sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2 \ + \dot{\varphi}\ \cos\theta\, J_3\,]$$

$$= (-i)[\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi) J_1 \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi) J_2 + (\dot{\psi} + \dot{\varphi}\,\cos\theta) J_3\,]$$

$$= -\,[\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)(iJ_1) \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\,\cos\theta)(iJ_3)\,] \ . \qquad (G.7.15)$$

We now take the ij element of this matrix using the fact (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$,

$$A'_{ij} = -\,[\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)\varepsilon_{1ij} \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot{\psi} + \dot{\varphi}\,\cos\theta)\varepsilon_{3ij}\,] \ . \qquad (G.7.16)$$

The tensor ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A' is antisymmetric and thus has only three distinct matrix elements. They are:

$$A'_{23} = -\,(\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)\varepsilon_{123} \ = \ -\,(\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)$$

$$A'_{31} = -\,(\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{231} = -\,(\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$A'_{12} = -\ (\dot{\psi} + \dot{\varphi}\,\cos\theta)\ \varepsilon_{312} \ = \ -\ (\dot{\psi} + \dot{\varphi}\,\cos\theta)\ . \qquad (G.7.17)$$

The work is nearly done. In order to determine **ω**, we now want to compare,

$$(d\mathbf{e}_n/dt)_{S'} \ = \ A'\,\mathbf{e}_n \qquad (G.7.8)$$

$$(d\mathbf{e}_n/dt)_{S'} \ = -\,\boldsymbol{\omega} \times \mathbf{e}_n \ . \qquad (G.7.1)$$

Since the left sides are the same, we take Frame S' components of the right sides and set them equal,

$$[A'\,\mathbf{e}_n]'_i = -\,[\boldsymbol{\omega} \times \mathbf{e}_n]'_i$$

or

$$A'_{ij}\,(\mathbf{e}_n)'_j \ = \ -\,\varepsilon_{ikj}\,(\omega)'_k(\mathbf{e}_n)'_j$$

or

$$A'_{ij} = -\varepsilon_{ikj}(\omega)'_k = \varepsilon_{ijk}(\omega)'_k$$

so

$$A'_{12} = \varepsilon_{123}(\omega)'_3 = (\omega)'_3$$
$$A'_{23} = \varepsilon_{231}(\omega)'_1 = (\omega)'_1$$
$$A'_{31} = \varepsilon_{312}(\omega)'_2 = (\omega)'_2 \ . \qquad \text{(G.7.18)}$$

Then,

$$(\omega)'_1 = A'_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi) = -\dot\varphi\sin\theta\sin\psi - \dot\theta\cos\psi$$
$$(\omega)'_2 = A'_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi) = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$(\omega)'_3 = A'_{12} = -(\dot\psi + \dot\varphi\cos\theta) = -\dot\varphi\cos\theta - \dot\psi \ . \quad \text{// angles still negated}$$

We now undo the temporary negation of the angles ψ,θ,φ enacted below (G.7.2). The velocities and all sines then negate. Our result for the components of **ω** in rotating Frame S' is then

$$(\omega)'_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot\varphi\cos\theta\ \dot\psi \equiv \omega_{z'} \ . \qquad \text{(G.7.19)}$$

This is in agreement with our Method 1 calculation (G.6.11) and with Goldstein's result which we again quote from Goldstein page 134 (GPS page 174),

$$\omega_{x'} = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\omega_{y'} = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\omega_{z'} = \dot\phi\cos\theta + \dot\psi. \qquad (4.87)$$

These are the Frame S' components of **ω**. We could then obtain the Frame S components of **ω** from (G.6.12) that $\boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}'$, which can be written as in (G.6.14),

$$(\omega)_i = [R_z(\varphi)R_x(\theta)R_z(\psi)]_{ij}(\omega)'_j \ . \quad \text{// } \boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}', \text{ R from (G.7.2)} \qquad \textit{(G.6.14)}$$

This was done at the end of Section G.6 and the result was (G.6.16),

$$(\omega)_x = \cos\varphi\,\dot\theta + \sin\theta\sin\varphi\,\dot\psi$$
$$(\omega)_y = \sin\varphi\,\dot\theta - \sin\theta\cos\varphi\,\dot\psi$$
$$(\omega)_z = \dot\varphi + \cos\theta\,\dot\psi \qquad \textit{(G.6.16)} \qquad \text{(G.7.20)}$$