Archive old Section G_7 PhL 3.11.17

G.7 Computation of ω (Method 2)

In this section our approach to computing **ω** for the Euler Angle rotation is to find an equation which involves **ω** and solve it for **ω**! More or less at random, we choose (1.7.4),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

Unlike our Method 1 computation of **ω** in the previous section, here we shall have no need for Goldstein's geometric Figure (G.5.5) or the various variables like ξ and ζ . We will, however, need various algebraic results developed in Section G.3.

Recall from (G.5.19) that the Goldstein Euler angle rotation is given by

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \,. \qquad \textit{(G.5.19)} \qquad \text{(G.7.2)}$$

In order to avoid a hundred minus signs, we shall *temporarily negate* all three angles, then when we are done, we will undo this negation. We therefore temporarily take R to be,

$$R(\psi,\theta,\varphi) = R_z(\psi)R_x(\theta)R_z(\varphi) \equiv R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi}\bullet\mathbf{J}) \,. \qquad \text{// temp} \qquad \text{(G.7.3)}$$

Comment: To find **Φ** we could write out the matrix $R_z(\psi)R_x(\theta)R_z(\varphi)$,

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R := evalm(Rz(psi) &* Rx(theta) &* Rz(phi));
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$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

and decompose it into its symmetric and antisymmetric components S and A. *In theory* we could then compute Φ (=θ) and $\hat{\boldsymbol{\Phi}}$ (=$\hat{\mathbf{n}}$) from (G.2.14) and come up with an explicit expression for **Φ** = $\Phi\hat{\boldsymbol{\Phi}}$**.** The reader is just reminded that this is mechanically possible, but luckily we have no need for the result (which "ain't purdy"). We could also compute d**Φ** from the following

$$R(\psi+d\psi,\theta+d\theta,\varphi+d\varphi) = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = \exp(-i\,[\boldsymbol{\Phi}+d\boldsymbol{\Phi}]\bullet\mathbf{J})$$

but again there is no need to do this. Note that **Φ** and d**Φ** will generally not be in the same direction.

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n(\boldsymbol{\Phi}) = R(\boldsymbol{\Phi})\mathbf{e}'_n \,. \qquad \text{(G.7.4)}$$

We are a Worker/Observer sitting in Frame S' and we want to compute the change we see in the Frame S basis vector $\mathbf{e}_n$ due to Euler angle changes dψ, dθ and dφ. Then,

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n - R(\mathbf{\Phi})\mathbf{e}'_n \qquad // \ \mathbf{e}'_n \text{ is a constant in Frame S'}$$

$$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi)\, \mathbf{e}'_n - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\, \mathbf{e}'_n$$

$$\equiv dR\, \mathbf{e}'_n \qquad (G.7.5)$$

// $(d\mathbf{e}_n)_{S'} = (dR)^T_{ni}\, \mathbf{e}'_i = (dR)_{in}\, \mathbf{e}'_i$ // apply sep to each rot in dR
// this agrees with paradox2 v3 doc

where

$$dR \equiv [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,] \ . \qquad (G.7.6)$$

Comment: Equation (G.7.5) is a vector equation. Because we assume that $d\mathbf{e}'_n = 0$ in writing (G.7.5), that equation is valid only in Frame S' (and not in Frame S). A vector equation is only a shorthand for its component equations, so we are not allowed to take the Frame S components of (G.7.5). When we take the Frame S' components, the rotation matrices are at first of the form $R(\mathbf{\Phi})'_{ij}$, but as shown in (1.1.35) this is the same as $R(\mathbf{\Phi})_{ij}$ with no prime. The upshot is that the matrices like $R_x(\theta)$ appearing in (G.7.6) are our standard rotation matrices of (E.2.1). We then have dR' = dR, but we shall continue to write dR' (and A' below) as a reminder that everything is only valid when evaluated in Frame S'.

Replace $\mathbf{e}'_n$ in (G.7.5) using (G.7.4) to get

$$\mathbf{e}'_n = R^{-1}(\mathbf{\Phi})\, \mathbf{e}_n \qquad \leftrightarrow \qquad |\mathbf{e}'_n\rangle = [R(\mathbf{\Phi})]_{nj}\, |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

ok to here

$$(d\mathbf{e}_n)_{S'} = dR\, \mathbf{e}'_n = dR\, [R^{-1}(\mathbf{\Phi})\mathbf{e}_n\,] = [dR\, R^{-1}(\mathbf{\Phi})\,]\, \mathbf{e}_n \ . \qquad (G.7.7)$$

$$= |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})|\, dR\, R^{-1}(\mathbf{\Phi})\, |\mathbf{e}_n\rangle = [dR\, R^{-1}(\mathbf{\Phi})]_{jn}\ |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$= [R\, dR^T\,]_{nj}\, |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

this disagrees with paradox2 v3 which shows $[(dR^T)R]_{nj}\ |\mathbf{e}_j(\mathbf{\Phi})\rangle$ at this point!! More detail:

$$(d\mathbf{e}_n)_{S'} = |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})|\, [\, R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})]\, R^{-1}(\mathbf{\Phi})\, |\mathbf{e}_n\rangle$$

$$= |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})|\, [\, R(\mathbf{\Phi}+d\mathbf{\Phi})R^T(\mathbf{\Phi}) - 1]\ |\mathbf{e}_n(\mathbf{\Phi})\rangle$$

In paradox 2 v2 I use these facts

$$[R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} = \langle\mathbf{e}'_i|\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle = \langle\mathbf{e}'_i|\, \mathcal{R}(\mathbf{\Phi}+d\mathbf{\Phi})|\mathbf{e}'_n\rangle$$
$$[R(\mathbf{\Phi})]'_{in} = \langle\mathbf{e}'_i|\mathbf{e}_n(\mathbf{\Phi})\rangle = \langle\mathbf{e}'_i|\, \mathcal{R}(\mathbf{\Phi})|\mathbf{e}'_n\rangle$$

So now redo the above

$$(d\mathbf{e}_n)_{S'} = dR\ \mathbf{e}'_n = dR\,[R^{-1}(\Phi)\mathbf{e}_n] = [(dR)\,R^{-1}(\Phi)]\,\mathbf{e}_n$$

$$= [(R(\Phi+d\Phi) - R(\Phi))\,R^{-1}(\Phi)]\,\mathbf{e}_n = [\,R(\Phi+d\Phi)\,R^{T}(\Phi) - 1]\,\mathbf{e}_n$$

$$= [\ |\mathbf{e}'_r\rangle\langle\mathbf{e}'_r|R(\Phi+d\Phi)\ |\mathbf{e}'_s\rangle\langle\mathbf{e}'_s|\,R^{T}(\Phi) - 1]\ |\mathbf{e}'_m\rangle\langle\mathbf{e}'_m|\,\mathbf{e}_n$$

$$= [\ |\mathbf{e}'_r\rangle R(\Phi+d\Phi)_{rs}\langle\mathbf{e}'_s|\,R^{T}(\Phi)\,|\mathbf{e}'_m\rangle - \ |\mathbf{e}'_m\rangle]\,\langle\mathbf{e}'_m|\,\mathbf{e}_n\rangle$$

$$= [\ |\mathbf{e}'_r\rangle R(\Phi+d\Phi)_{rs}\,R^{T}(\Phi)_{sm} - \ |\mathbf{e}'_m\rangle]\,R_{mn}(\Phi)$$

$$= [\ R(\Phi+d\Phi)_{rs}\,R^{T}(\Phi)_{sm}R_{mn}(\Phi)|\mathbf{e}'_r\rangle - R_{mn}(\Phi)\,|\mathbf{e}'_m\rangle]$$

$$= [\ R(\Phi+d\Phi)_{rs}\,R^{T}(\Phi)_{sm}R_{mn}(\Phi)|\mathbf{e}'_r\rangle - R_{rn}(\Phi)\,|\mathbf{e}'_r\rangle]$$

$$= [\ R(\Phi+d\Phi)_{rs}\,R^{T}(\Phi)_{sm}R_{mn}(\Phi) - R_{rn}(\Phi)]\ \ |\mathbf{e}'_r\rangle$$

$$= [\ [R(\Phi+d\Phi)R^{T}(\Phi)R(\Phi)]_{rn} - R_{rn}(\Phi)]\ \ |\mathbf{e}'_r\rangle$$

$$= [\ [R(\Phi+d\Phi)]_{rn} - R_{rn}(\Phi)]\ \ |\mathbf{e}'_r\rangle$$

$$= [dR]_{rn}\ |\mathbf{e}'_r\rangle = [dR]_{in}\ |\mathbf{e}'_i\rangle$$

ok to here

Now go back to the operator form

$$(d\mathbf{e}_n)_{S'} = dR\ \mathbf{e}'_n = [(dR)\,R^{-1}(\Phi)]\,\mathbf{e}_n$$

I don't know how to matricize this other than what I just did above. You might argue that $(dR)\,R^{-1}(\Phi)$ is just a rotation so you should be able to use the basis theorem. But dR contains rotations at different times!!! Thus you cannot just treat it as an operator at time t and use $|\mathbf{e}_r\rangle\langle\mathbf{e}_r|$ completeness to get the right matrix elements. This is a very subtle issue and the paradox2 method avoids this whole problem.

Next, divide by dt to get

$$(d\mathbf{e}_n/dt)_{S'} = [(dR/dt)R^{-1}(\Phi)]\,\mathbf{e}_n$$

$$\equiv A\,\mathbf{e}_n \qquad A \equiv [(dR/dt)\ R^{-1}(\Phi)] \qquad \text{(G.7.8a)}$$

$$= (1/dt)\ [dR\,R^{-1}(\Phi)]\,\mathbf{e}_n$$

$$= (1/dt)\ [(R(\Phi+d\Phi) - R(\Phi))\,R^{-1}(\Phi)]\,\mathbf{e}_n$$

$$= (1/dt)\ [(R(\mathbf{\Phi}+d\mathbf{\Phi})R^{T}(\mathbf{\Phi}) - 1\ ]\ \mathbf{e}_n$$

Insertion 3/10/17: Take components in Frame S' right here:

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = [A\ \mathbf{e}_n]'_i = A'_{ij}\ (\mathbf{e}_n)'_j = A'_{ij}\ R_{jn} \qquad \text{// note prime on A}$$

where A is defined as shown. Recall that we are trying to solve the following equation for **ω,**

$$(d\mathbf{e}_n/dt)_{S'} = -\ \boldsymbol{\omega} \times \mathbf{e}_n \qquad (G.7.1)$$

so we seem to be on the right track. We will set A $\mathbf{e}_n = -\ \boldsymbol{\omega} \times \mathbf{e}_n$ and solve for **ω**.

So far objects like the objects above like R(**Φ**), dR, $R_z(\psi)$ and A are really operators, not matrices. However, they become matrices when we rewrite (G.7.8a) using completeness of the basis states :

$$|\ (d\mathbf{e}_n/dt)_{S'}> \ = \ |\ A\ \mathbf{e}_n> = \mathcal{A}\ |\ \mathbf{e}_n> = \ |\mathbf{e}_i><\mathbf{e}_i|\ \mathcal{A}\ |\ \mathbf{e}_n> \ = \ |\mathbf{e}_i>\ A_{in} \ = A_{in}\ |\mathbf{e}_i> .$$

Then (G.7.8) becomes, in non-Dirac notation.

$$(d\mathbf{e}_n/dt)_{S'} \ = \ A_{jn}\ \mathbf{e}_j\ , \qquad (G.7.8b)$$

Insertion 3/10/17: Take components in Frame S' right here:

$$[(d\mathbf{e}_n/dt)_{S'}]'_i \ = \ A_{jn}\ (\mathbf{e}_j)'_i \ = \ A_{jn}R_{ij} \qquad \text{// note no prime on A}$$

Compare with last insertion:

$$A'_{ij}\ R_{jn} = A_{jn}R_{ij} \qquad \text{or} \qquad A'_{ij}\ R_{jn}R^{T}_{nm} = A_{jn}R_{ij}R^{T}_{nm}$$

$$\text{or} \quad A'_{ij}\delta_{jm} = R_{ij}A_{jn}R^{T}_{nm} \qquad \text{or} \qquad A'_{im} = [RAR^{T}]_{im}$$

At this point then everything is consistent.

We must now compute this matrix A, so we start with dR which we compute to first order in the differential angles,

$$dR = \ [\ R_z(\psi+d\psi)\ R_x(\theta+d\theta)\ R_z(\varphi+d\varphi) - \ R_z(\psi)\ R_x(\theta)\ R_z(\varphi)\ ]$$

$$= \ [\ R_z(\psi)R_z(d\psi)\ R_x(\theta)\ R_x(d\theta)R_z(\varphi)R_z(d\varphi) \ - \ R_z(\psi)\ R_x(\theta)\ R_z(\varphi)]$$

$$= \ [\ R_z(\psi)\{1-id\psi J_3\}\ R_x(\theta)\{1-id\theta J_1\}R_z(\varphi)\{1-id\varphi J_3\} \ - \ R_z(\psi)\ R_x(\theta)\ R_z(\varphi)]$$

$$= [-id\psi\ R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\ \ R_z(\psi)\ R_x(\theta)J_1R_z(\varphi) \ - id\varphi\ R_z(\psi)\ R_x(\theta)R_z(\varphi)J_3\ ] \ . \qquad (G.7.9)$$

We have used the first order expansion (1.5.6) for the small angle rotations. Dividing by dt gives,

$$(dR/dt)_{S'} = [\, -i\dot{\psi}\ R_z(\psi)J_3R_x(\theta)R_z(\varphi) - i\dot{\theta}\ R_z(\psi)R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\ R_z(\psi)R_x(\theta)R_z(\varphi)J_3\,] \,. \quad \text{(G.7.10)}$$

We now assemble the matrix A and process it with vigor, making use three times of the trivial sandwich rule (G.3.6) as shown in blue,

$$A \equiv (dR/dt)\ R^{-1}(\mathbf{\Phi}) \qquad \text{// where } R^{-1}(\mathbf{\Phi}) = R_z(-\varphi)R_x(-\theta)R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\, R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3R_z(-\varphi)\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3 - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1R_x(-\theta) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3 - i\dot{\theta}\ R_z(\psi)\, J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_z(-\psi) - i\dot{\theta}\ R_z(\psi)\, J_1\, R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\, R_z(-\psi)\ ]$$

$$= [\, -i\dot{\psi}\ J_3 - i\dot{\theta}\ R_z(\psi)\, J_1\, R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)\, R_z(-\psi)\ ]$$

$$= [\, -i\dot{\psi}\ J_3 - i\dot{\theta}\ M_1 - i\dot{\varphi}\, M_2\,] \qquad \text{(G.7.11)}$$

where

$$M_1 \equiv R_z(\psi)\, J_1R_z(-\psi) \qquad = [\, R_3(\psi)\, J_1R_3(-\psi)]$$

$$M_2 \equiv R_z(\psi)\, R_x(\theta)J_3R_x(-\theta)R_z(-\psi) \qquad = R_3(\psi)\, [\, R_1(\theta)J_3R_1(-\theta)\,]\, R_3(-\psi)\ . \qquad \text{(G.7.12)}$$

We now call upon our non-trivial sandwich formulas in (G.3.8) to simplify thing further :

$$M_1 = R_3(\psi)\, J_1R_3(-\psi) = \cos\psi J_1 + \sin\psi\, J_2 \qquad \text{// (G.3.8) line 5} \qquad \text{(G.7.13)}$$

$$M_2 = R_3(\psi)\, [\, R_1(\theta)J_3R_1(-\theta)\,]\, R_3(-\psi)$$

$$= R_3(\psi)\, [\, \cos\theta\, J_3 - \sin\theta\, J_2\,]\, R_3(-\psi) \qquad \text{// (G.3.8) line 2}$$

$$= \cos\theta\, [\, R_3(\psi)\, J_3\, R_3(-\psi)] - \sin\theta\, [R_3(\psi)J_2R_3(-\psi)]$$

$= \cos\theta\, J_3 - \sin\theta\,(\cos\psi J_2 - \sin\psi\, J_1)$ // (G.3.6) and (G.3.8) line 6

$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\, J_3$. (G.7.14)

Then,

$A = [\, -i\dot{\psi}\, J_3 - i\dot{\theta}\, M_1 - i\dot{\varphi}\, M_2\,]$

$= [\, -i\dot{\psi}\, J_3 - i\dot{\theta}\,(\cos\psi J_1 + \sin\psi\, J_2) - i\dot{\varphi}\,(\sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\, J_3)\,]$

$= [\, -i\dot{\psi}\, J_3 - i\dot{\theta}\cos\psi J_1 - i\dot{\theta}\sin\psi\, J_2 - i\dot{\varphi}\sin\theta\sin\psi J_1 + i\dot{\varphi}\sin\theta\cos\psi J_2 - i\dot{\varphi}\cos\theta\, J_3\,]$

$= (-i)[\, \dot{\psi}\, J_3 + \dot{\theta}\cos\psi J_1 + \dot{\theta}\sin\psi\, J_2 + \dot{\varphi}\sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2 + \dot{\varphi}\cos\theta\, J_3\,]$

$= (-i)[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi) J_1 + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi) J_2 + (\dot{\psi} + \dot{\varphi}\cos\theta) J_3\,]$

$= -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\,]$. (G.7.15)

Insertion 3/10/17. The above at this point is only an operator equation! A and the J_i are operators. Operators do not get primes or no primes. I want to evaluate this object

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = [A\,\mathbf{e}_n]'_i = (A)'_{ij}\,(\mathbf{e}_n)'_j = (A)'_{ij}\,R_{jn} = [(dR/dt)\;R^{-1}]'_{ij}R_{jn} = [(dR/dt)]_{in}$$

where $(A)'_{ij} = \langle \mathbf{e}'_i |\, \mathcal{A}\, | \mathbf{e}'_j \rangle$ so we will want to know $\langle \mathbf{e}'_i |\, i\mathcal{J}_k\, | \mathbf{e}'_j \rangle = (iJ_k)'_{ij}$. I need $(iJ_k)'_{ij} = \varepsilon_{kij}$ in order for things to work. This means that the Frame S' observer uses *** as his representation of the J's. Doing this, and looking at the other expression,

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k = -\varepsilon_{ijk}(\omega)'_j R_{kn} = -\varepsilon_{ikj}(\omega)'_k R_{jn} = \varepsilon_{ijk}(\omega)'_k R_{jn}$$

we then get the desired result that

$$A'_{ij} = \varepsilon_{ijk}(\omega)'_k$$

So in this approach, everything hinges on the Frame S' observer using $(iJ_k)'_{ij} = \varepsilon_{kij}$. I have now studied this issue in Paradox2 v2.doc and I conclude there that in fact $(J_i)_{mn} = (J_i)'_{mn}$. Here that implies that $A'_{ij} = A_{ij}$ so A is then not really a tensor. We then get the correct result above.

How then does the Frame S Plane work out ? 3/10/17.

$$(d\mathbf{e}_n/dt)_{S'} = A\mathbf{e}_n$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

Take components in Frame S to get

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [A\mathbf{e}_n]_i = A_{ij}(\mathbf{e}_n)_j$$

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\varepsilon_{ikj}(\omega)_k(\mathbf{e}_n)_j$$

This then gives the WRONG answer for the Frame S result!

We now take the ij element of this matrix using the fact (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$,

$$A_{ij} = -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\varepsilon_{1ij} + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot{\psi} + \dot{\varphi}\cos\theta)\varepsilon_{3ij}\,] \,. \qquad \text{(G.7.16)}$$

The tensor ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A is antisymmetric and thus has only three distinct matrix elements. They are:

$$A_{23} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\varepsilon_{123} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)$$

$$A_{31} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{231} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$A_{12} = -(\dot{\psi} + \dot{\varphi}\cos\theta)\,\varepsilon_{312} = -(\dot{\psi} + \dot{\varphi}\cos\theta)\,. \qquad \text{(G.7.17)}$$

The work is nearly done. In order to determine $\boldsymbol{\omega}$, we now want to compare,

$$(d\mathbf{e}_n/dt)_{S'} = A_{in}\,\mathbf{e}_i \qquad \textit{(G.7.8b)}$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n\,. \qquad \textit{(G.7.1)}$$

Since the left sides are the same, we take Frame S' components of the right sides and set them equal,

Comment: The first equation in the above pair is a true vector equation in the basis state Hilbert Space. It can be written $|(d\mathbf{e}_n/dt)_{S'}\rangle = A_{in}\,|\mathbf{e}_i\rangle$ in Dirac notation. One could evaluate the two sides of this vector equation by closing either with $\langle\mathbf{e}_k|$ for Frame S, or $\langle\mathbf{e'}_k|$ for Frame S' . However, the second equation of the above pair is not a Hilbert Space vector equation, although it is of course mathematically a vector equation. One cannot write $|(d\mathbf{e}_n/dt)_{S'}\rangle = -|\boldsymbol{\omega} \times \mathbf{e}_n\rangle$ because the object on the right has no meaning! Nor does $\boldsymbol{\omega} \times |\mathbf{e}_n\rangle$ have a meaning, a cross product between a math vector and a Hilbert Space vector. The second equation above is merely a *shorthand notation* for the following equation (components in Frame S')

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k \quad \Leftrightarrow \quad (d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

Therefore, the seeming vector equation $A_{in}\mathbf{e}_i = -\boldsymbol{\omega} \times \mathbf{e}_n$ has no meaning if one tries to evaluate both sides in Frame S. It is only valid when both sides are evaluated in Frame S'.

We now evaluate the equation $A_{in}\ \mathbf{e}_i = -\boldsymbol{\omega} \times \mathbf{e}_n$ in Frame S' :

$$A_{in}\ (\mathbf{e}_i)'_k = -\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k$$

$$R_{ki}A_{in} = -\varepsilon_{ijk}(\omega)'_j R_{kn}$$

Apply R^T_{nm} to both sides to get

$$R_{ki}A_{in}\,R^T_{nm} = -\varepsilon_{ijk}(\omega)'_j R_{kn}\,R^T_{nm}$$

or

$$R_{ki}A_{in}\,R^T_{nm} = -\varepsilon_{ijk}(\omega)'_j \delta_{km}$$

or

$$[RAR^{-1}]_{km} = -\varepsilon_{ijm}(\omega)'_j$$

$$A'_{km} = -\varepsilon_{ijm}(\omega)'_j$$

This only gives the right answer if I assume that my matrix A computed above has a prime on it! Then I need the crummy arm-waving argument that we have an Observer who is sitting in Frame S' and for whom we have $(iJ_k)'_{ij} = \varepsilon_{kij}$.

$$[A'\,\mathbf{e}_n]'_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]'_i \qquad \text{// where does LHS come from????}$$

or

$$A'_{ij}\,(\mathbf{e}_n)'_j = -\,\varepsilon_{ikj}\,(\omega)'_k(\mathbf{e}_n)'_j$$

or

$$A'_{ij} = -\,\varepsilon_{ikj}\,(\omega)'_k = \varepsilon_{ijk}\,(\omega)'_k$$

so

$$A'_{12} = \varepsilon_{123}(\omega)'_3 = (\omega)'_3$$
$$A'_{23} = \varepsilon_{231}(\omega)'_1 = (\omega)'_1$$
$$A'_{31} = \varepsilon_{312}(\omega)'_2 = (\omega)'_2\ . \qquad \text{(G.7.18)}$$

Then,

$$(\omega)'_1 = A'_{23} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi) = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi$$

$$(\omega)'_2 = A'_{31} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$(\omega)'_3 = A'_{12} = -(\dot\psi + \dot\varphi\cos\theta) = -\dot\varphi\cos\theta - \dot\psi$. // angles still negated

We now undo the temporary negation of the angles ψ,θ,φ enacted below (G.7.2). The velocities and all sines then negate. Our result for the components of **ω** in rotating Frame S' is then

$$(\omega)'_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot\varphi\cos\theta + \dot\psi \equiv \omega_{z'} \ . \qquad \text{(G.7.19)}$$

This is in agreement with our Method 1 calculation (G.6.11) and with Goldstein's result which we again quote from Goldstein page 134 (GPS page 174),

$$\omega_{x'} = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\omega_{y'} = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\omega_{z'} = \dot\phi\cos\theta + \dot\psi. \qquad (4.87)$$

These are the Frame S' components of **ω**. We could then obtain the Frame S components of **ω** from (G.6.12) that $\boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}'$, which can be written as in (G.6.14),

$(\omega)_i = [\, R_z(\varphi)R_x(\theta)R_z(\psi)]_{ij}\,(\omega)'_j$. // $\boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}'$, R from (G.7.2) *(G.6.14)*

This was done at the end of Section G.6 and the result was (G.6.16),

$$(\omega)_x = \dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi$$
$$(\omega)_y = \dot\theta\sin\varphi - \dot\psi\sin\theta\cos\varphi$$
$$(\omega)_z = \dot\varphi + \dot\psi\cos\theta \ . \qquad \textit{(G.6.16)} \qquad \text{(G.7.20)}$$

<u>Outline of the direct Method 2 derivation of the Frame S result (G.7.20)</u>

For equation (G.7.1) we start with (1.7.1) that $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ in place of $(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$, which adds an overall minus sign to the result. The new (G.7.4) becomes $\mathbf{e}'_n(\boldsymbol{\Phi}) = R^{-1}(\boldsymbol{\Phi})\mathbf{e}_n$ with $\mathbf{e}_n$ = constant in Frame S. Things go through as presented above, but since $R(\boldsymbol{\Phi}) \rightarrow R^{-1}(\boldsymbol{\Phi})$ and since $R(\boldsymbol{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi)$, one has $R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)\,R_z(-\psi)$. Thus, to convert the result (G.7.19) to the Frame S result, we have to make these changes: (1) $\varphi \rightarrow -\psi$, $\psi \rightarrow -\varphi$ and $\theta \rightarrow -\theta$; (2) add the overall minus sign just noted. We do that right here:

$$(\omega)'_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot\varphi\cos\theta + \dot\psi \equiv \omega_{z'} \ . \qquad \textit{(G.7.19)}$$

→

$$-(\omega)_1 = [-\dot{\psi}][-\sin\theta][-\sin\varphi] + [-\dot{\theta}]\cos\varphi \quad \Rightarrow \omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$$

$$-(\omega)_2 = [-\dot{\psi}][-\sin\theta]\cos\varphi - [-\dot{\theta}][-\sin\varphi] \quad \Rightarrow \omega_2 = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi$$

$$-(\omega)_3 = [-\dot{\psi}]\cos\theta + [-\dot{\varphi}] \quad \Rightarrow \omega_3 = \dot{\psi}\cos\theta + \dot{\varphi} \qquad \text{(G.7.21)}$$

and this does agree with (G.7.20).