Dealing with Primes on the Rotation Matrices PhL 3.2.17

Rerun the starting clip of the movie:

$\mathbf{e}_n(\mathbf{\Phi}) = R(\mathbf{\Phi})\mathbf{e'}_n$

$(d\mathbf{e}_n(\mathbf{\Phi}))_{S'} \equiv \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) - \mathbf{e}_n(\mathbf{\Phi})$

$= R(\mathbf{\Phi}+d\mathbf{\Phi})\ \mathbf{e'}_n - R(\mathbf{\Phi})\ \mathbf{e'}_n$ // note that $(d\mathbf{e'}_n(\varphi))_{S'} = 0$

$= [\ R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})]\ \mathbf{e'}_n$

$= dR\ \mathbf{e'}_n$ $dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})$

At this point in the film, the R's are really operators [ok] , not matrices, so write this last as

$|\ (d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = (\mathcal{R}(\mathbf{\Phi}+d\mathbf{\Phi}) - \mathcal{R}(\mathbf{\Phi})\ )\ |\ \mathbf{e'}_n\rangle$

Use time instead of angle to write this as

$|\ (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'}\rangle = (\mathcal{R}(\mathbf{\Phi}(t+dt)) - \mathcal{R}(\mathbf{\Phi}(t))\ )\ |\ \mathbf{e'}_n\rangle$

Now take components in Frame S' to get

$\langle \mathbf{e'}_i\ |\ (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'}\rangle = \langle \mathbf{e'}_i\ |\ \mathcal{R}(\mathbf{\Phi}(t+dt)) - \mathcal{R}(\mathbf{\Phi}(t))\ |\ \mathbf{e'}_n\rangle$

$= \langle \mathbf{e'}_i\ |\ \mathcal{R}(\mathbf{\Phi}(t+dt))\ |\ \mathbf{e'}_n\rangle - \langle \mathbf{e'}_i\ |\ \mathcal{R}(\mathbf{\Phi}(t))\ |\ \mathbf{e'}_n\rangle$

$= [R(\mathbf{\Phi}(t+dt))]'_{in} - [R(\mathbf{\Phi}(t))]'_{in}$ [ok]

Now consider

$\langle \mathbf{e'}_i\ |\ \mathcal{R}(\mathbf{\Phi}(t+dt))\ |\ \mathbf{e'}_n\rangle = \langle \mathbf{e'}_i\ |\mathbf{e}_j\rangle\langle\mathbf{e}_j|\ \mathcal{R}(\mathbf{\Phi}(t+dt))\ |\mathbf{e}_k\rangle\langle\mathbf{e}_k|\ \mathbf{e'}_n\rangle$

$= R(\mathbf{\Phi}(t))_{ij}\ R(\mathbf{\Phi}(t+dt))_{jk}\ R^{-1}(\mathbf{\Phi}(t))_{kn}$

$= [\ R(\mathbf{\Phi}(t))\ R(\mathbf{\Phi}(t+dt))\ R^{-1}(\mathbf{\Phi}(t)]_{in}$

Correct but not very useful! In writing $1 = |\mathbf{e}_j\rangle\langle\mathbf{e}_j|$ I am inserting completeness at time t which reads $\mathbf{1} = |\mathbf{e}_n(\mathbf{\Phi})\rangle\langle\mathbf{e}_n(\mathbf{\Phi})|$ and then I claim $[R(\mathbf{\Phi}(t+dt))]_{jk} = \langle\mathbf{e}_j(\mathbf{\Phi})|\ \mathcal{R}(\mathbf{\Phi}(t+dt))\ |\mathbf{e}_k(\mathbf{\Phi})\rangle$. The operator and the states do not exist at the same time, so this is not really the matrix element of interest. The matrix of interest is rather $[R(\mathbf{\Phi}(t+dt))]_{jk} = \langle\mathbf{e}_j(\mathbf{\Phi}+d\mathbf{\Phi})|\ \mathcal{R}(\mathbf{\Phi}(t+dt))\ |\mathbf{e}_k(\mathbf{\Phi}+d\mathbf{\Phi})\rangle$.

I fail to show the dependence on Φ or time. The correct completeness to use here is $\mathbf{1} = |\, \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) >< \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) \,|$ saying the e_n are complete at time t+dt. Then

So this shows this matrix results:

$$[R(\mathbf{\Phi}(t+dt))]' = R(\mathbf{\Phi}(t))\, R(\mathbf{\Phi}(t+dt))\, R^{-1}(\mathbf{\Phi}(t))$$

$$[R(\mathbf{\Phi}(t))]' = R(\mathbf{\Phi}(t))\, R(\mathbf{\Phi}(t))\, R^{-1}(\mathbf{\Phi}(t)) = R(\mathbf{\Phi}(t))$$

So you CANNOT remove the prime on matrix $[R(\mathbf{\Phi}(t+dt))]'$. That is the main point so far. You could write

$$R(\mathbf{\Phi}(t+dt)) \approx R(\mathbf{\Phi}(t)) - i\,\mathbf{J} \bullet \mathbf{\Phi}(t)$$

Then you get

$$[R(\mathbf{\Phi}(t+dt))]' = R(\mathbf{\Phi}(t))\, R(\mathbf{\Phi}(t+dt))\, R^{-1}(\mathbf{\Phi}(t))$$

$$= R(\mathbf{\Phi}(t))\, [\, R(\mathbf{\Phi}(t)) - i\,\mathbf{J} \bullet \mathbf{\Phi}(t)]\, R^{-1}(\mathbf{\Phi}(t))$$

$$= R(\mathbf{\Phi}(t)) - i\,\mathbf{\Phi}(t) \bullet [R(\mathbf{\Phi}(t))\mathbf{J}R^{-1}(\mathbf{\Phi}(t))] \qquad \text{// all matrices}$$

So OK then the opening film sequence is this,

$$|\, (d\mathbf{e}_n(\mathbf{\Phi}))_{S'} > = (\mathcal{R}(\mathbf{\Phi}+d\mathbf{\Phi}) - \mathcal{R}(\mathbf{\Phi})\,)\,|\, \mathbf{e'}_n >$$

$$< \mathbf{e'}_i \,|\, (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'} > = [R(\mathbf{\Phi}(t+dt))]'_{in} - [R(\mathbf{\Phi}(t))]'_{in}$$

$$= \{\ [R(\mathbf{\Phi}(t+dt))]' - [R(\mathbf{\Phi}(t))]'\ \}_{in}$$

$$= \{\ [R(\mathbf{\Phi}(t+dt))]' - [R(\mathbf{\Phi}(t))]\ \}_{in}$$

$$= \{\, R(\mathbf{\Phi}(t)) - i\,\mathbf{\Phi}(t) \bullet [R(\mathbf{\Phi}(t))\mathbf{J}R^{-1}(\mathbf{\Phi}(t))] - [R(\mathbf{\Phi}(t))]\ \}_{in}$$

$$= \{\, - i\,\mathbf{\Phi}(t) \bullet [R(\mathbf{\Phi}(t))\mathbf{J}R^{-1}(\mathbf{\Phi}(t))]\ \}_{in}$$

$$= - i\,\mathbf{\Phi}(t) \bullet [R(\mathbf{\Phi}(t))\mathbf{J}R^{-1}(\mathbf{\Phi}(t))]_{in}$$

Here we are looking at the Frame S' evaluation of the opening scene equation. Suppose now that

$$R(\mathbf{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi) \qquad \text{// after doing our sign change in G.7}$$

Then you would say

$$\langle \mathbf{e'}_i \mid (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'} \rangle = - i\, \mathbf{\Phi}(t) \bullet [R_z(\psi)R_x(\theta)R_z(\varphi)\mathbf{J}R_z(-\varphi)R_x(-\theta)R_z(-\psi)]_{in}$$

What is $\mathbf{\Phi}(t)$ in this case? This path is a little clumsy because $\mathbf{\Phi}(t)$ is some very messy expression. The good part of this path is that you have Frame S standard matrices appearing.

In Section G.7 I take a different path. I say

$$R(\mathbf{\Phi}(t)) = R_z(\psi(t))R_x(\theta(t))R_z(\varphi(t))$$

$$R(\mathbf{\Phi}(t+dt)) = R_z(\psi(t+dt))R_x(\theta(t+dt))R_z(\varphi(t+dt))$$

I still have to regard these as operators at this point and not matrices. The movie open goes then,

$$\mathbf{e}_n(\mathbf{\Phi}) = R(\mathbf{\Phi})\mathbf{e'}_n$$

$$(d\mathbf{e}_n(\mathbf{\Phi}))_{S'} \equiv \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) - \mathbf{e}_n(\mathbf{\Phi})$$

$$= R(\mathbf{\Phi}+d\mathbf{\Phi})\,\mathbf{e'}_n - R(\mathbf{\Phi})\,\mathbf{e'}_n \qquad \text{// note that } (d\mathbf{e'}_n(\varphi))_{S'} = 0$$

$$= [\,R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})]\;\mathbf{e'}_n$$

$$= [\,R_z(\psi(t+dt))R_x(\theta(t+dt))R_z(\varphi(t+dt)) - R_z(\psi(t))R_x(\theta(t))R_z(\varphi(t))]\;\mathbf{e'}_n$$

$$= dR\,\mathbf{e'}_n \qquad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi})$$

and still all operators. I then show that

$$R_z(\psi(t+dt))R_x(\theta(t+dt))R_z(\varphi(t+dt)) - R_z(\psi(t))R_x(\theta(t))R_z(\varphi(t))$$

$$= [-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\; R_z(\psi)\,R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\,R_x(\theta)R_z(\varphi)J_3\,]$$

$$= dR \qquad \text{// operator}$$

where things are still all operators. I would then get (computing Frame S' components).

$$\langle \mathbf{e'}_i \mid (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'} \rangle = \langle \mathbf{e'}_i \mid \boldsymbol{d\mathcal{R}} \mid \mathbf{e'}_n \rangle$$

$$= \langle \mathbf{e'}_i | \; [-id\psi\, \boldsymbol{\mathcal{R}}_z(\psi)\boldsymbol{\mathcal{J}}_3\boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi) - id\theta\; \boldsymbol{\mathcal{R}}_z(\psi)\,\boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{J}}_1\boldsymbol{\mathcal{R}}_z(\varphi) - id\varphi\, \boldsymbol{\mathcal{R}}_z(\psi)\,\boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi)\boldsymbol{\mathcal{J}}_3\,] | \mathbf{e'}_n \rangle$$

$$= \{\; [-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\; R_z(\psi)\,R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\,R_x(\theta)R_z(\varphi)J_3\,]\;\}'_{in}$$

$$= \{dR\}'_{in}$$

and take note of the prime there at the right end. This is a matrix of some sort, but I have no idea how to compute the matrix elements of this matrix. Using Dirac, I think I could show that

$\{dR\}' = R(\mathbf{\Phi})\, dR\, R^{-1}(\mathbf{\Phi})$

$= R_z(\psi(t))R_x(\theta(t))R_z(\varphi(t)) *$

$[-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,] *$

$R_z(-\varphi(t))R_x(-\theta(t))R_z(-\psi(t))$

and THEN I think everything is written in standard matrices, nothing has a prime.

Now consider the next scene in the film which is this

$(d\mathbf{e}_n)_{S'} = [(dR')R^{-1}(\mathbf{\Phi})]\, \mathbf{e}_n = (dA')\, \mathbf{e}_n$

$dA' \equiv dR'\, R^{-1}(\mathbf{\Phi})$

This then produces this horrible mess

$dA' =$

$= R_z(\psi(t))R_x(\theta(t))R_z(\varphi(t)) *$

$[-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,] *$

$R_z(-\varphi(t))R_x(-\theta(t))R_z(-\psi(t))\, R_z(-\varphi(t))R_x(-\theta(t))R_z(-\psi(t))$

where everything is now written in standard rotation matrices.

Had I carried out Appendix G.7 correctly, THIS is what I would have for the Frame S' evaluation of the opening scene equation, when written in terms of standard matrices which I regard as Frame S matrices.

Now lets go back and instead take the Frame S components of things

$\langle \mathbf{e}_i \,|\, (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'} \rangle = \langle \mathbf{e}_i \,|\, d\boldsymbol{\mathcal{R}} \,|\mathbf{e}'_n\rangle$

$= \langle \mathbf{e}_i| \, [-id\psi\, \boldsymbol{\mathcal{R}}_z(\psi)\boldsymbol{\mathcal{J}}_3\boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi) - id\theta\, \boldsymbol{\mathcal{R}}_z(\psi)\, \boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{J}}_1\boldsymbol{\mathcal{R}}_z(\varphi) - id\varphi\, \boldsymbol{\mathcal{R}}_z(\psi)\, \boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi)\boldsymbol{\mathcal{J}}_3\,] |\mathbf{e}'_n\rangle$

$= \langle \mathbf{e}_i| \, [-id\psi\, \boldsymbol{\mathcal{R}}_z(\psi)\boldsymbol{\mathcal{J}}_3\boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi) - id\theta\, \boldsymbol{\mathcal{R}}_z(\psi)\, \boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{J}}_1\boldsymbol{\mathcal{R}}_z(\varphi) - id\varphi\, \boldsymbol{\mathcal{R}}_z(\psi)\, \boldsymbol{\mathcal{R}}_x(\theta)\boldsymbol{\mathcal{R}}_z(\varphi)\boldsymbol{\mathcal{J}}_3\,] |e_k\rangle\langle e_k|\mathbf{e}'_n\rangle$

$= \{ \, [-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,] \, \}_{ik}\langle e_k|\mathbf{e}'_n\rangle$

$= (dR)_{ik}\langle e_k|\mathbf{e}'_n\rangle$

Now

$$\langle e_k | \mathbf{e'}_n \rangle = R(\mathbf{\Phi}(t))_{nk}$$

so we then have

$$\langle \mathbf{e}_i \,|\, (d\mathbf{e}_n(\mathbf{\Phi}(t)))_{S'} \rangle = (dR)_{ik}\, R(\mathbf{\Phi}(t))_{nk} = [\,(dR)\, R^{-1}(\mathbf{\Phi}(t))]_{in}$$

Now consider the next scene in the film which is this

$$(d\mathbf{e}_n)_{S'} = [(dR)R^{-1}(\mathbf{\Phi})]\, \mathbf{e}_n = (dA)\, \mathbf{e}_n$$

$$dA \equiv dR\, R^{-1}(\mathbf{\Phi})$$

This then produces,

$$dA = [-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\, R^{-1}(\mathbf{\Phi})$$

$$= [-id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi)$$

Now THIS is what I have used in Appendix G.7 !!! So I must be evaluating components in Frame S and not in Frame S' as I thought!

But THIS in turn leads to the "wrong answer" because it produces an expression for $\boldsymbol{\omega}$ in Frame S, but that result then agrees with Goldstein's expression for $\boldsymbol{\omega}$ in Frame S' ! So we are back to

The Fundamental Paradox