Paradox2 v3 PhL 3.9.17

I want to summarize how I got the right answer for (ω)' in doc paradox2 v2.

Expression #1 : The first step is this:

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{s'}\rangle \equiv |\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle - |\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle - |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= |\mathbf{e}'_i\rangle\, [R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - |\mathbf{e}'_i\rangle [R(\mathbf{\Phi})]'_{in}$$

$$= (\, [R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - [R(\mathbf{\Phi})]'_{in}\,)\ |\mathbf{e}'_i\rangle$$

$$= (\, [R(\mathbf{\Phi}+d\mathbf{\Phi})]_{in} - [R(\mathbf{\Phi})]_{in}\,)\ |\mathbf{e}'_i\rangle \qquad \text{// lincom}$$

$$\equiv (dR)_{in}\,|\mathbf{e}'_i\rangle \qquad dR = R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \quad \text{(matrices)} \qquad (1)$$

I show in a separate line item that you can remove the primes from either R :

$$[R(\mathbf{\Phi})]_{mn} = \langle\mathbf{e}_m(\mathbf{\Phi})|\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= \langle\mathbf{e}_m(\mathbf{\Phi})\,|\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\,\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_j\rangle\langle\mathbf{e}'_j|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= [R(\mathbf{\Phi})]'_{im}\,[R(\mathbf{\Phi})]'_{ij}[R(\mathbf{\Phi})]'_{jn} = [\, R^T(\mathbf{\Phi})R(\mathbf{\Phi})R(\mathbf{\Phi})]'_{mn} = [R(\mathbf{\Phi})]'_{mn} \qquad (2)$$

Notice right off the bat that everything is "lin comb" oriented here. There are no "operators".

The second step to replace $|\mathbf{e}'_i\rangle$ as follows

$$|\mathbf{e}'_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})\,|\,\mathbf{e}'_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle[R(\mathbf{\Phi})]_{ij} \qquad (3)$$

Inserting (3) into (1) then gives

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{s'}\rangle = (dR)_{in}\,[R(\mathbf{\Phi})]_{ij}\,|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$= [(dR^T)R]_{nj}\ |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

But

$$(dR^T)R = [\, R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi})]\, R(\mathbf{\Phi}) = R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1$$

So at this point we have

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{s'}\rangle = [\, R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1]_{nj}\ |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

Third step is to close this with $\langle \mathbf{e'}_i |$ to get

$$\langle \mathbf{e'}_i | (d\mathbf{e}_n(\mathbf{\Phi}))_{S'} \rangle = (R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1)_{nj} \langle \mathbf{e'}_i | \mathbf{e}_j(\mathbf{\Phi}) \rangle$$

or

$$[(d\mathbf{e}_n)_{S'}]'_i = (R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1)_{nj} R(\mathbf{\Phi})_{ij} = (R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1)_{nj} R^{-1}(\mathbf{\Phi})_{ji}$$

$$= (R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi}))_{ni} = (R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}))_{in} = (dR)_{in}$$

So we seem to have done a lot of work to get this simple result

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = (dR/dt)_{in}$$

I could probably shorten that up a bit.

Expression #2 :

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n . \qquad (1.7.4) \qquad \text{(G.7.1)}$$

and this is just a shorthand for

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = - \varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k$$

and this gives the correct ω in Frame S'

or

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = - \varepsilon_{ijk}(\omega)'_j R_{kn}$$

Setting the two expressions equal

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = (dR/dt)_{in}$$

Now multiply both sides by R^{-1}_{nm} and sum on n to get

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} R^{-1}_{nm} = (dR/dt)_{in} R^{-1}_{nm}$$

or

$$- \varepsilon_{ijk}(\omega)'_j \delta_{km} = [(dR/dt) R^{-1}]_{im}$$

or

$$- \varepsilon_{ijm}(\omega)'_j = [(dR/dt) R^{-1}]_{im}$$

or

$$- \varepsilon_{ikm}(\omega)'_k = [(dR/dt) R^{-1}]_{im}$$

or

$$- \varepsilon_{ikj}(\omega)'_k = [(dR/dt) R^{-1}]_{ij}$$

or

$$\varepsilon_{kij}(\omega)'_k = [(dR/dt) R^{-1}]_{ij} \equiv A_{ij}$$

which I know is the correct answer.

Go off and compute A_{ij}

I do this in Appendix G.7 and the result is

$$A_{ij} = -\left[\left(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \right)\varepsilon_{1ij} + \left(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi \right)\varepsilon_{2ij} + \left(\dot{\psi} + \dot{\varphi}\cos\theta \right)\varepsilon_{3ij} \right] . \qquad (G.7.16)$$

and this then yields the Goldstein results.

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NOW, instead of taking Frame S' components, try the above process but take Frame S components.

Expression #1

We just copy the previous result

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = [\, R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1]_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

and this time we close with $\langle \mathbf{e}_i |$ top get

$$\langle \mathbf{e}_i |(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \langle \mathbf{e}_i |\mathbf{e}_j(\mathbf{\Phi})\rangle$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \delta_{ij}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{ni}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left[\left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - R^{-1}(\mathbf{\Phi}) \right)R(\mathbf{\Phi}) \right]_{ni}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left[\left(R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^{-1}(\mathbf{\Phi}) \right)R(\mathbf{\Phi}) \right]_{ni}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left[\left(R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi}) \right)R(\mathbf{\Phi}) \right]_{ni}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left[R^T(\mathbf{\Phi})\left(R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \right) \right]^T_{ni}$$

or

$$[(d\mathbf{e}_n)_{S'}]_i = \left[R^T(\mathbf{\Phi})(dR) \right]_{in}$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]_i = \left[R^T(\mathbf{\Phi})(dR/dt) \right]_{in} = \left[(dR/dt)^T R(\mathbf{\Phi}) \right]_{ni}$$

Expression #2

Now I am not exactly sure what to do. Here is the naive approach:

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

so

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\varepsilon_{ijk}(\omega)_j(\mathbf{e}_n)_k$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\varepsilon_{ijk}(\omega)_j\delta_{nk}$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\varepsilon_{ijn}(\omega)_j$$

Setting the two expressions equal

$$-\varepsilon_{ijn}(\omega)_j = [\, R^T(\mathbf{\Phi})(dR/dt)\,]_{in}$$

$$= R^T(\mathbf{\Phi})_{im}\,(dR/dt)_{ma}\;R^{-1}(\mathbf{\Phi})_{ab}\,R(\mathbf{\Phi})_{bn}$$

$$= [\; R^T(\mathbf{\Phi})(dR/dt)R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi})]_{in}$$

$$= [\, R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi})]_{in}$$

or

$$-\varepsilon_{ikn}(\omega)_k = [\, R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi})]_{in}$$

or

$$-\varepsilon_{ikj}(\omega)_k = [\, R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi})]_{ij}$$

or

$$\varepsilon_{kij}(\omega)_k = [\,(R^{-1}{}_{ij}\,J_j]_{ij} \equiv B_{ij}$$

Well, this does give a different result, though I don't know if it is the right result.

Go off and compute B_{ij}

We know that

$$A = -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\,]$$

Idea: Write $A = a_kJ_k$ where we know the a_i as above. Then using Theorem 1,

Theorem 1: $R^{-1}\mathbf{J}R = R\mathbf{J}$ (G.4.1)

we find,

$$B = R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi}) = a_kR^{-1}(\mathbf{\Phi})J_k\,R(\mathbf{\Phi}) = a_k\,R_{kj}\,J_j$$

$$= a_kQ_k \quad \text{where} \quad Q_k \equiv iR_{kj}\,J_j$$

We compute the vector Q_k as follows

```
Q := evalm(Rz(psi) &* Rx(theta) &* Rz(phi) &* matrix(3,1,[J1,J2,J3]));
```

$$Q := \begin{bmatrix} (\cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi))\,J2 + \sin(\psi)\sin(\theta)\,J3 \\ (\sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi))\,J2 - \cos(\psi)\sin(\theta)\,J3 \\ \sin(\theta)\sin(\phi)\,J1 + \sin(\theta)\cos(\phi)\,J2 + \cos(\theta)\,J3 \end{bmatrix}$$

Meanwhile, the vector a_k is

$a_1 = -i(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)$

$a_2 = -i(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$

$a_3 = -i(\dot{\psi} + \dot{\varphi}\cos\theta)$

I enter these quantities into Maple

```
a1 := -I*(td*cos(psi)+phid*sin(theta)*sin(psi));
```

$$a1 := -I\,(td\cos(\psi) + phid\sin(\theta)\sin(\psi))$$

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a2 := -I*(td*sin(psi)-phid*sin(theta)*cos(psi));
```

$$a2 := -I\,(td\sin(\psi) - phid\sin(\theta)\cos(\psi))$$

```
a3 := -I*(psid+phid*cos(theta));
```

$$a3 := -I\,(psid + phid\cos(\theta))$$

Then I compute $B = a_k Q_k = \mathbf{A} \bullet \mathbf{Q}$:

```
evalm(matrix(1,3,[a1,a2,a3]) &* Q):B :=simplify(%);
```

$$B := [-I\,psid\cos(\theta)\,J3 - I\,phid\,J3 - I\,td\,J1\cos(\phi) + I\,td\,J2\sin(\phi) - I\,psid\sin(\theta)\sin(\phi)\,J1 - I\,psid\sin(\theta)\cos(\phi)\,J2]$$

```
collect(B[1,1],[J1,J2,J3]);
```

$$(-I\,td\cos(\phi) - I\,psid\sin(\theta)\sin(\phi))\,J1 + (I\,td\sin(\phi) - I\,psid\sin(\theta)\cos(\phi))\,J2 + (-I\,psid\cos(\theta) - I\,phid)\,J3$$

We then transcribe the last result

$B = (-\dot{\psi}\sin\theta\sin\varphi - \dot{\theta}\cos\varphi)(iJ_1) + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)(iJ_2) + (-\dot{\psi}\cos\theta - \dot{\varphi})(iJ_3)$

so

$B_{ij} = (-\dot{\psi}\sin\theta\sin\varphi - \dot{\theta}\cos\varphi)\varepsilon_{1ij} + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)\varepsilon_{2ij} + (-\dot{\psi}\cos\theta - \dot{\varphi})\varepsilon_{3ij}$

and

$B_{23} = -\dot{\psi}\sin\theta\sin\varphi - \dot{\theta}\cos\varphi$

$B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$

$B_{12} = -\dot{\psi}\cos\theta - \dot{\varphi}$

I then have from above that

$$\varepsilon_{kij}(\omega)_k = B_{ij}$$

so then

$\varepsilon_{123}\,\omega_1 = B_{23}$ $\qquad$ $\omega_1 = -\dot{\psi}\sin\theta\sin\varphi - \dot{\theta}\cos\varphi$

$\varepsilon_{231}\,\omega_2 = B_{31}$ $\qquad$ $\omega_2 = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$

$\varepsilon_{312}\,\omega_3 = B_{12}$ $\qquad$ $\omega_3 = -\dot{\psi}\cos\theta - \dot{\varphi}$

If I now negate the three angles the results are

$\omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$ $\qquad$ $= \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi$

$\omega_2 = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$ $\qquad$ $= \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$

$\omega_3 = \dot{\psi}\cos\theta + \dot{\varphi}$ $\qquad$ $= \dot{\varphi} + \dot{\psi}\cos\theta$

My result computed in Appendix G was (after sign changes)

$$(\omega)_x = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi$$

$$(\omega)_y = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$

$$(\omega)_z = \dot{\varphi} + \dot{\psi}\cos\theta \ . \qquad \textit{(G.6.16)} \qquad \text{(G.7.20)}$$

and hurray, exact agreement!!!!!