G.7 Computation of ω (Method 2)

In this section our approach to computing **ω** for the Euler Angle rotation is to find an equation which involves **ω** and solve it for **ω**! More or less at random, we choose (1.7.4),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

Unlike our Method 1 computation of **ω** in the previous section, here we shall have no need for Goldstein's geometric Figure (G.5.5) or the various variables like ξ and ζ . We will, however, need various algebraic results developed in Section G.3.

Recall from (G.5.19) that the Goldstein Euler angle rotation is given by

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \,. \qquad \textit{(G.5.19)} \qquad \text{(G.7.2)}$$

In order to avoid a hundred minus signs, we shall *temporarily negate* all three angles, then when we are done, we will undo this negation. We therefore temporarily take R to be,

$$R(\psi,\theta,\varphi) = R_z(\psi)R_x(\theta)R_z(\varphi) \equiv R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi}\bullet\mathbf{J}) \,. \qquad \text{(G.7.3)}$$

Comment: To find **Φ** we could write out the matrix $R_z(\psi)R_x(\theta)R_z(\varphi)$,

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R := evalm(Rz(psi) &* Rx(theta) &* Rz(phi));
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$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

and decompose it into its symmetric and antisymmetric components S and A. *In theory* we could then compute Φ (=θ) and $\hat{\boldsymbol{\Phi}}$ (=$\hat{\mathbf{n}}$) from (G.2.14) and come up with an explicit expression for $\boldsymbol{\Phi} = \Phi\hat{\boldsymbol{\Phi}}$. The reader is just reminded that this is mechanically possible, but luckily we have no need for the result (which "ain't purdy"). We could also compute d**Φ** from the following

$$R(\psi+d\psi,\theta+d\theta,\varphi+d\varphi) = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = \exp(-i\,[\boldsymbol{\Phi}+d\boldsymbol{\Phi}]\bullet\mathbf{J})$$

but again there is no need to do this. Note that **Φ** and d**Φ** will generally not be in the same direction.

I will simplify this right now and say that ψ = dψ = θ = dθ = 0, so φ is the only variable.

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n(\varphi) = R(\varphi)\mathbf{e}'_n \,. \qquad \text{// as seen in Frame S'} \qquad \text{(G.7.4)}$$

We are a Worker/Observer sitting in Frame S' and we want to compute the change we see in the Frame S basis vector $\mathbf{e}_n$ due to Euler angle changes dψ, dθ and dφ. Then,

$$(d\mathbf{e}_n(\varphi))_{S'} \equiv \mathbf{e}_n(\varphi+d\varphi) - \mathbf{e}_n(\varphi)$$

$$= R_z(\varphi+d\varphi)\,\mathbf{e'}_n - R_z(\varphi)\,\mathbf{e'}_n \qquad \text{// note that } (d\mathbf{e'}_n(\varphi))_{S'} = 0$$

$$= [\,R_z(\varphi+d\varphi) - R_z(\varphi)]\;\mathbf{e'}_n$$

$$= dR\,\mathbf{e'}_n \qquad dR \equiv R_z(\varphi+d\varphi) - R_z(\varphi)$$

This equation makes sense in Frame S'. [What does that mean?]

In Frame S you would have

$$(d\mathbf{e}_n(\varphi))_S \equiv \mathbf{e}_n(\varphi+d\varphi) - \mathbf{e}_n(\varphi) = 0$$

because $\mathbf{e}_n$ is a constant in Frame S. So we really are tied down to Frame S' right off the bat here by writing the above stuff. So now ask this question. Consider,

$$(d\mathbf{e}_n(\varphi))_{S'} = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]\;\mathbf{e'}_n$$

Question: Is this a vector equation which you could evaluate in either Frame ??

If you tried to evaluate it in Frame S you would get

$$[(d\mathbf{e}_n(\varphi))_{S'}]_i = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]_{ij}\;(\mathbf{e'}_n)_j = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]_{ij}\;R(\varphi)^{-1}{}_{nj}$$

You would have to conclude then that $R_z(\varphi+d\varphi) = R_z(\varphi)$ since $[d\mathbf{e}_n(\varphi)]_i = 0$ in frame S [wrong!]. This is total garbage, so I think I have a conclusion

Fact: Then equation $d\mathbf{e}_n(\varphi) = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]\;\mathbf{e'}_n$ is valid ONLY in Frame S'.

[yes, so that is why we write the left side as $(d\mathbf{e}_n(\varphi))_{S'}$. Having done so, we can evaluate the resulting equation in any Frame.]

Question: What do you get when you evaluate this equation in Frame S' ?

$$[(d\mathbf{e}_n(\varphi))_{S'}]'_i = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]'_{ij}\;(\mathbf{e'}_n)'_j = [\,R_z(\varphi+d\varphi) - R_z(\varphi)]'_{ij}\,\delta_{nj}$$

$$= [\,R_z(\varphi+d\varphi) - R_z(\varphi)]'_{in}$$

$$= [\,R'_z(\varphi+d\varphi) - R'_z(\varphi)]_{in}$$

Now I guess I would claim that $R'_z(\varphi) = R_z(\varphi)\ R_z(\varphi)\ R_z(\varphi)^{-1} = R_z(\varphi)$ and similarly at $\varphi+d\varphi$ which is like at a time t + dt. Then I would claim

$$[(d\mathbf{e}_n(\varphi))_{S'}]'_i = [\ R_z(\varphi+d\varphi) - R_z(\varphi)]_{in}$$

and these are my standard rotation matrices!!!!

[So this is an arg why, when you take Frame S' components of $(d\mathbf{e}_n(\varphi))_{S'} = [\ R_z(\varphi+d\varphi) - R_z(\varphi)]\ \mathbf{e}'_n$ you end up with the "standard" rotation matrices, not some weird Frame S' versions. I *think* this argument is OK almost. Assumes things like $R'_z(\varphi+d\varphi) = R_z(\varphi+d\varphi)\ R_z(\varphi+d\varphi)\ R_z(\varphi+d\varphi)^{-1}$]

Weak point in the above. Suppose you use these facts

$$R'_z(\varphi) = R_z(\varphi)\ R_z(\varphi)\ R_z(\varphi)^{-1}$$
$$R'_z(\varphi+d\varphi) = R_z(\varphi+d\varphi)\ R_z(\varphi+d\varphi)\ R_z(\varphi+d\varphi)^{-1}$$

Then in the above you have the object.

$$[\ R_z(\varphi+d\varphi) - R_z(\varphi)]'_{ij}$$

Why are you allowed to distribute the prime onto the two terms? Note sure the above expression is well defined. Maybe in the limit of small dφ you are allowed to distribute. Something is fishy and ill defined here I am afraid. Go back to

$(d\mathbf{e}_n(\varphi))_{S'} = [\ R_z(\varphi+d\varphi) - R_z(\varphi)]\ \mathbf{e}'_n$ vector = vector

$$|\ (d\mathbf{e}_n(\varphi))_{S'}\rangle = (\boldsymbol{\mathcal{R}}_z(\varphi+d\varphi) - \boldsymbol{\mathcal{R}}_z(\varphi)\ )\ |\ \mathbf{e}'_n\rangle$$

$$\langle \mathbf{e}'_i\ |\ (d\mathbf{e}_n(\varphi))_{S'}\rangle = \langle \mathbf{e}'_i\ |(\boldsymbol{\mathcal{R}}_z(\varphi+d\varphi) - \boldsymbol{\mathcal{R}}_z(\varphi)\ )\ |\ \mathbf{e}'_n\rangle$$

$$= \langle \mathbf{e}'_i\ |\ \boldsymbol{\mathcal{R}}_z(\varphi+d\varphi)\ |\ \mathbf{e}'_n\rangle - \langle \mathbf{e}'_i\ |\ \boldsymbol{\mathcal{R}}_z(\varphi)\ |\ \mathbf{e}'_n\rangle$$

$$= [R_z(\varphi+d\varphi)]' - [R_z(\varphi)]' \qquad \text{// thinking t instead of } \varphi$$

This argues that distribution of the prime is OK. The basis $|\ \mathbf{e}'_n\rangle$ is the same at all time because I have assumed no time dependence like $|\ \mathbf{e}'_n(t)\rangle$ so it is a "fixed in time" basis like the snake skin V_C. Note that we regard $\boldsymbol{\mathcal{R}}_z(\varphi+d\varphi)$ and $\boldsymbol{\mathcal{R}}_z(\varphi)$ as two different operators in this fixed basis $|\ \mathbf{e}'_n\rangle$. Write

$$\varphi + d\varphi = \varphi + \dot{\varphi}\ dt = \varphi(t+dt)$$

so

$$= \langle \mathbf{e}'_i\ |\ \boldsymbol{\mathcal{R}}_z(\varphi(t+dt))\ |\ \mathbf{e}'_n\rangle - \langle \mathbf{e}'_i\ |\ \boldsymbol{\mathcal{R}}_z(\varphi(t))\ |\ \mathbf{e}'_n\rangle$$

$$= [R_z(\varphi+d\varphi)]' - [R_z(\varphi)]'$$

OK, now can you remove the primes?

$$[R_z(\varphi+d\varphi)]'_{in} = \langle e'_i \,|\, \mathcal{R}_z(\varphi(t+dt)) \,|\, e'_n \rangle = \langle e'_i \,|e_j\rangle\langle e_j|\, \mathcal{R}_z(\varphi(t+dt)) \,|e_k\rangle\langle e_k|\, e'_n \rangle$$

$$= R_{ij}(t)\; [R_z(\varphi+d\varphi)]_{jk}\; R^{-1}{}_{kn}(t)$$

or

$$[R_z(\varphi+d\varphi)]' = R\; R_z(\varphi+d\varphi)\; R^{-1}$$

So now this looks OK, it is always the same R(t) that does the sandwich. So we then really can say

$$[R_z(\varphi)]' = R\; R_z(\varphi)\; R^{-1}$$
$$[R_z(\varphi+d\varphi)]' = R\; R_z(\varphi+d\varphi)\; R^{-1}$$

with the same R. Thus we really can "distribute the prime". Dirac has provided an answer.

So back up and draw a conclusion please:

Conclusion: We have this vector equation: $(d\mathbf{e}_n(\varphi))_{S'} = [\, R_z(\varphi+d\varphi) - R_z(\varphi)]\; \mathbf{e}'_n$

When we evaluate components in Frame S' we find

$$[(d\mathbf{e}_n(\varphi))_{S'}]'_i = [\, R_z(\varphi+d\varphi) - R_z(\varphi)]'_{ij}(\mathbf{e}'_n)'_j = [\, R_z(\varphi+d\varphi) - R_z(\varphi)]'_{ij}\, \delta_{nj}$$

$$= [\, R_z(\varphi+d\varphi) - R_z(\varphi)]'_{in} = \{R\; [\, R_z(\varphi+d\varphi) - R_z(\varphi)]\, R^{-1}\}_{in}$$

$$= \{\; [\, RR_z(\varphi+d\varphi)R^{-1} - RR_z(\varphi)R^{-1}]\; \}_{in} = \{\; R'_z(\varphi+d\varphi) - R'_z(\varphi)\; \}_{in}$$

so we end up with

$$[(d\mathbf{e}_n(\varphi))_{S'}]'_i = [\, R'_z(\varphi+d\varphi) - R'_z(\varphi)\,]_{in}$$

But I have not claimed that THESE primes can just be removed.

The matrix dR' has a prime because it is a rank-2 tensor within Frame S'. The same comment applies to the matrix A' below.

Replace $\mathbf{e}'_n$ in (G.7.5) using (G.7.4) to get

$$(d\mathbf{e}_n)_{S'} = dR'\, \mathbf{e}'_n = dR'\, [R^{-1}(\mathbf{\Phi})\mathbf{e}_n\,] = [dR'\, R^{-1}(\mathbf{\Phi})\,]\, \mathbf{e}_n\,. \qquad \text{(G.7.7)}$$

Next, divide by dt to get

$$(d\mathbf{e}_n/dt)_{S'} = [(dR'/dt)_{S'} R^{-1}(\mathbf{\Phi})]\, \mathbf{e}_n$$

$$\equiv A'\, \mathbf{e}_n \qquad\qquad A' \equiv [(dR'/dt)_{S'} R^{-1}(\mathbf{\Phi})] \ . \qquad\qquad (G.7.8)$$

where A' is defined as shown. Recall that we are trying to solve the following equation for **ω,**

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n \qquad\qquad (G.7.1)$$

so we seem to be on the right track. We must compute this matrix A', so we start with dR' which we compute to first order in the differential angles,

$$dR' = [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,]$$

$$= [\, R_z(\psi) R_z(d\psi)\, R_x(\theta)\, R_x(d\theta) R_z(\varphi) R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$$

$$= [\, R_z(\psi)\{1-id\psi J_3\}\, R_x(\theta)\{1-id\theta J_1\} R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$$

$$= [-id\psi\, R_z(\psi)J_3 R_x(\theta) R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta) J_1 R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta) R_z(\varphi) J_3\,] \ . \qquad (G.7.9)$$

We have used the first order expansion (1.5.6) for the small angle rotations. Dividing by dt gives,

$$(dR'/dt)_{S'} = [\, -i\dot{\psi}\, R_z(\psi)J_3 R_x(\theta) R_z(\varphi) - i\dot{\theta}\, R_z(\psi) R_x(\theta) J_1 R_z(\varphi) - i\dot{\varphi}\, R_z(\psi) R_x(\theta) R_z(\varphi) J_3\,] \ . \qquad (G.7.10)$$

We now assemble the matrix A' and process it with vigor, making use three times of the trivial sandwich rule (G.3.6) as shown in blue,

$$A' \equiv (dR'/dt)_{S'} R^{-1}(\mathbf{\Phi}) \qquad \text{// where } R^{-1}(\mathbf{\Phi}) = R_z(\varphi) R_x(\theta) R_z(\psi) \text{ from (G.7.3)}$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3\, R_x(\theta) R_z(\varphi) - i\dot{\theta}\, R_z(\psi)\, R_x(\theta) J_1 R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) R_z(\varphi) J_3\,] R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\, R_z(\psi)\, R_x(\theta) J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) R_z(\varphi) J_3 R_z(-\varphi)\,] R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\, R_z(\psi)\, R_x(\theta) J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) J_3\,] R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3 - i\dot{\theta}\, R_z(\psi) R_x(\theta) J_1 R_x(-\theta) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) J_3 R_x(-\theta)\,]\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3 - i\dot{\theta}\, R_z(\psi)\, J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) J_3 R_x(-\theta)\,]\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\, R_z(\psi)J_3\, R_z(-\psi) - i\dot{\theta}\, R_z(\psi)\, J_1\, R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) J_3 R_x(-\theta)\, R_z(-\psi)\,]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ R_z(\psi)\ J_1\ R_z(-\psi)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ R_z(-\psi)\ ]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ M_1\ -\ i\dot{\varphi}\ M_2\ ] \qquad \text{(G.7.11)}$$

where

$$M_1 \equiv R_z(\psi)\ J_1 R_z(-\psi) \qquad = [\ R_3(\psi)\ J_1 R_3(-\psi)]$$

$$M_2 \equiv R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi) \qquad = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)\ . \qquad \text{(G.7.12)}$$

We now call upon our non-trivial sandwich formulas in (G.3.8) to simplify thing further :

$$M_1 = R_3(\psi)\ J_1 R_3(-\psi) = \cos\psi J_1 + \sin\psi\ J_2 \qquad \text{// (G.3.8) line 5} \qquad \text{(G.7.13)}$$

$$M_2 = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)$$

$$= R_3(\psi)\ [\ \cos\theta\ J_3 - \sin\theta\ J_2\ ]\ R_3(-\psi) \qquad \text{// (G.3.8) line 2}$$

$$= \cos\theta\ [\ R_3(\psi)\ J_3\ R_3(-\psi)]\ - \sin\theta\ [R_3(\psi)J_2R_3(-\psi)]$$

$$= \cos\theta\ J_3\ - \sin\theta\ (\cos\psi J_2 - \sin\psi\ J_1) \qquad \text{// (G.3.6) and (G.3.8) line 6}$$

$$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3\ . \qquad \text{(G.7.14)}$$

Then,

$$A' = [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ M_1\ -\ i\dot{\varphi}\ M_2\ ]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ (\cos\psi J_1 + \sin\psi\ J_2)\ -\ i\dot{\varphi}\ (\sin\theta c\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3)\ ]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ \cos\psi J_1\ -\ i\dot{\theta}\ \sin\psi\ J_2\ -\ i\dot{\varphi}\ \sin\theta\sin\psi J_1 + i\dot{\varphi}\sin\theta\cos\psi J_2 - i\dot{\varphi}\ \cos\theta\ J_3\ ]$$

$$= (-i)[\ \dot{\psi}\ J_3\ + \dot{\theta}\ \cos\psi J_1\ + \dot{\theta}\ \sin\psi\ J_2\ + \dot{\varphi}\ \sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2\ + \dot{\varphi}\ \cos\theta\ J_3\ ]$$

$$= (-i)[\ (\dot{\theta}\ \cos\psi + \dot{\varphi}\ \sin\theta\sin\psi)J_1\ + (\dot{\theta}\ \sin\psi - \dot{\varphi}\sin\theta\cos\psi)J_2 + (\dot{\psi} + \dot{\varphi}\ \cos\theta)J_3\ ]$$

$$= -[\ (\dot{\theta}\ \cos\psi + \dot{\varphi}\ \sin\theta\sin\psi)(iJ_1)\ + (\dot{\theta}\ \sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\ \cos\theta)(iJ_3)\ ]\ . \qquad \text{(G.7.15)}$$

We now take the ij element of this matrix using the fact (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$,

$$A'_{ij} = -[\ (\dot{\theta}\ \cos\psi + \dot{\varphi}\ \sin\theta\sin\psi)\varepsilon_{1ij}\ + (\dot{\theta}\ \sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot{\psi} + \dot{\varphi}\ \cos\theta)\varepsilon_{3ij}\ ]\ . \qquad \text{(G.7.16)}$$

The tensor ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A' is antisymmetric and thus has only three distinct matrix elements. They are:

$$A'_{23} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\varepsilon_{123} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)$$

$$A'_{31} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{231} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$A'_{12} = -(\dot{\psi} + \dot{\varphi}\cos\theta)\,\varepsilon_{312} = -(\dot{\psi} + \dot{\varphi}\cos\theta)\,. \qquad \text{(G.7.17)}$$

The work is nearly done. In order to determine **ω**, we now want to compare,

$$(d\mathbf{e}_n/dt)_{S'} = A'\,\mathbf{e}_n \qquad \textit{(G.7.8)}$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n\,. \qquad \textit{(G.7.1)}$$

Since the left sides are the same, we take Frame S' components of the right sides and set them equal,

$$[A'\,\mathbf{e}_n]'_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]'_i$$

or

$$A'_{ij}\,(\mathbf{e}_n)'_j = -\varepsilon_{ikj}\,(\omega)'_k(\mathbf{e}_n)'_j$$

or

$$A'_{ij} = -\varepsilon_{ikj}\,(\omega)'_k = \varepsilon_{ijk}\,(\omega)'_k$$

so

$$A'_{12} = \varepsilon_{123}(\omega)'_3 = (\omega)'_3$$
$$A'_{23} = \varepsilon_{231}(\omega)'_1 = (\omega)'_1$$
$$A'_{31} = \varepsilon_{312}(\omega)'_2 = (\omega)'_2\,. \qquad \text{(G.7.18)}$$

Then,

$$(\omega)'_1 = A'_{23} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi) = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi$$
$$(\omega)'_2 = A'_{31} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$(\omega)'_3 = A'_{12} = -(\dot{\psi} + \dot{\varphi}\cos\theta) = -\dot{\varphi}\cos\theta - \dot{\psi}\,. \qquad \text{// angles still negated}$$

We now undo the temporary negation of the angles ψ,θ,φ enacted below (G.7.2). The velocities and all sines then negate. Our result for the components of **ω** in rotating Frame S' is then

$$(\omega)'_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot{\varphi}\cos\theta\,\dot{\psi} \equiv \omega_{z'}\,. \qquad \text{(G.7.19)}$$

This is in agreement with our Method 1 calculation (G.6.11) and with Goldstein's result which we again quote from Goldstein page 134 (GPS page 174),

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}. \qquad (4.87)$$

These are the Frame S' components of $\boldsymbol{\omega}$. We could then obtain the Frame S components of $\boldsymbol{\omega}$ from (G.6.12) that $\boldsymbol{\omega} = \mathrm{R}^{-1}\boldsymbol{\omega}'$, which can be written as in (G.6.14),

$$(\omega)_i = [\, R_z(\varphi)R_x(\theta)R_z(\psi)]_{ij}\,(\omega)'_j \quad . \qquad // \ \boldsymbol{\omega} = \mathrm{R}^{-1}\boldsymbol{\omega}', \text{ R from (G.7.2)} \qquad \textit{(G.6.14)}$$

This was done at the end of Section G.6 and the result was (G.6.16),

$$(\omega)_x = \cos\varphi\,\dot{\theta} + \sin\theta\sin\varphi\,\dot{\psi}$$
$$(\omega)_y = \sin\varphi\,\dot{\theta} - \sin\theta\cos\varphi\,\dot{\psi}$$
$$(\omega)_z = \dot{\varphi} + \cos\theta\,\dot{\psi} \ . \qquad \textit{(G.6.16)} \qquad (G.7.20)$$