Question 7: Rotating Vectors Questions PhL 3.1.17

Consider equation (1.5.8) which says,

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \qquad (1.5.8)$$

1. Is this a vector equation which can be evaluated in different Frames?

I am pretty sure the answer is YES. In my graphical derivation, you could view my picture from any direction, or from any frame of reference.

2. What about this equation?

$$\mathbf{a} = (d\mathbf{v}/dt) = \boldsymbol{\omega} \times \mathbf{v} \quad \text{where } \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt \quad . \qquad (1.6.1)$$

Again I draw a picture in (1.6.2) showing cone motion. I draw this say in Frame S. Then it says

$$\mathbf{a} = [(d\mathbf{v}/dt)_S] = \boldsymbol{\omega} \times \mathbf{v}$$

It still seems you could compute

$$a_i = [(d\mathbf{v}/dt)_S]_i = [\boldsymbol{\omega} \times \mathbf{v}]_i$$

$$a'_i = [(d\mathbf{v}/dt)_S]'_i = [\boldsymbol{\omega} \times \mathbf{v}]'_i$$

The vector **a** and the vector **ω**x**v** certainly have components in any frame you want.
So I guess again the answer is YES. Notice that we have all kinematic vectors here.

Another way to answer: taking components in Frame S" just means dotting with $\mathbf{e}''_n$. You cannot stop someone from dotting a vector equation with a vector!

3. What about this equation?

$$(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n \quad . \qquad (1.7.1)$$

or better

$$\mathbf{c} \equiv (d\mathbf{e}'_n)_S = \boldsymbol{\omega} \times \mathbf{e}'_n \, dt$$

Now we do not have all Kinematic Vectors in this equation, we have a mix. So more suspicious.

In Frame S there is some vector **c** and there exists some vector quantity **ω** x $\mathbf{e}'_n$ dt . You can dot both sides into either $\mathbf{e}_i$ or $\mathbf{e}'_i$ and you don't get zero. So

$$c_i = [(d\mathbf{e}'_n)_S]_i = [\boldsymbol{\omega} \times \mathbf{e}'_n]_i \, dt$$

$c'_i = [(d\mathbf{e}'_n)_S]'_i = [\boldsymbol{\omega} \times \mathbf{e}'_n]'_i \, dt$

In particular, I have no reason to say that $[(d\mathbf{e}'_n)_S]'_i = \mathbf{c} \bullet \mathbf{e}'_i = 0$.

Conclusion: You can evaluate $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ in either Frame S or in Frame S'.

4. What about these equations?

$\mathbf{e}_n(\varphi) = R(\varphi)\mathbf{e}'_n$

$d\mathbf{c} \equiv (d\mathbf{e}_n(\varphi))_{S'} \equiv \mathbf{e}_n(\varphi+d\varphi) - \mathbf{e}_n(\varphi)$

$= R(\varphi+d\varphi)\mathbf{e}'_n - R(\varphi)\mathbf{e}'_n$

$= [\, R(\varphi+d\varphi)\mathbf{e} - R(\varphi)]\mathbf{e}'_n$

Maybe use time t as a parameter

$\mathbf{e}_n(t) = R(t)\mathbf{e}'_n$

$d\mathbf{c} \equiv (d\mathbf{e}_n(t))_{S'} \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$

$= R(t+dt)\mathbf{e}'_n - R(t)\mathbf{e}'_n$

$= [\, R(t+dt) - R(t)]\mathbf{e}'_n$ // assumes Frame S'

If you wanted to be general, you should have written

$\mathbf{e}_n(t) = R(t)\mathbf{e}'_n(t)$

$d\mathbf{c} \equiv (d\mathbf{e}_n(t))_{S'} \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$

$= R(t+dt)\mathbf{e}'_n(t+dt) - R(t)\mathbf{e}'_n$

Then

$\mathbf{e}'_n(t+dt) = \mathbf{e}'_n(t) + [\partial_t \mathbf{e}'_n(t)]dt$

$R(t+dt) = R(t) + \partial_t R(t)dt$

and then

$d\mathbf{c} = (d\mathbf{e}_n(t))_{S'} \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$

$= [R(t+dt)]\,[\mathbf{e}'_n(t+dt)] - R(t)\mathbf{e}'_n(t)$

$= [R(t) + \partial_t R(t)dt]\,[\mathbf{e}'_n(t) + [\partial_t \mathbf{e}'_n(t)]dt] - R(t)\mathbf{e}'_n(t)$

$= [\,R(t)\,\mathbf{e}'_n(t) - R(t)\mathbf{e}'_n(t)] + [R(t)[\partial_t \mathbf{e}'_n(t)] + [\partial_t R(t)]\mathbf{e}'_n(t)]\,dt + O(t^2)$

$\approx [\,R(t)(\partial_t \mathbf{e}'_n(t)) + (\partial_t R(t))\mathbf{e}'_n(t)\,]\,dt$

and there are two terms! In general you do NOT get $d\mathbf{c} = [\,R(t+dt) - R(t)]\mathbf{e}'_n$.

Let's evaluate the above equation in both frames

$d\mathbf{c} = (d\mathbf{e}_n(t))_{S'} = [\,R(t)\partial_t \mathbf{e}'_n(t)) + (\partial_t R(t))\mathbf{e}'_n(t)\,]\,dt$

$[d\mathbf{c}] = [(d\mathbf{e}_n(t))_{S'}] = R(t)[\partial_t \mathbf{e}'_n(t)]dt + [\partial_t R(t)]\,\mathbf{e}'_n(t)\,dt$

$[d\mathbf{c}]_i = [(d\mathbf{e}_n(t))_{S'}]_i = R(t)_{ij}[\partial_t \mathbf{e}'_n(t)]_j dt + [\partial_t R(t)]_{ij}\,(\mathbf{e}'_n(t))_j\,dt$

$[d\mathbf{c}]'_i = [(d\mathbf{e}_n(t))_{S'}]'_i = R'(t)_{ij}[\partial_t \mathbf{e}'_n(t)]'_j dt + [\partial_t R(t)]'_{ij}\,(\mathbf{e}'_n(t))'_j\,dt$

Is it OK to evaluate the above equations in the two frames as I have done?

I think the answer is YES, because the equation is valid in both frames.

Now back to this:

Conclusion:

$d\mathbf{c} \equiv (d\mathbf{e}_n(t))_{S'} \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$ valid in both Frame S and Frame S'

$d\mathbf{c} \equiv (d\mathbf{e}_n(t))_{S'} = [\,R(t+dt) - R(t)]\mathbf{e}'_n$ valid ONLY in Frame S'

4a. Can you "evaluate" the second equation above in both Frames?

NO! Yes it is a vector equation, but a vector equation is just a shorthand for an equation in components. If you say that a certain vector equation is not valid in Frame S, what you mean is that when that vector equation is written in Frame S components, it is invalid. So this is certainly something to ponder.

Correction to the above.

The equation

$(d\mathbf{e}_n(t))_{S'} = [\,R(t+dt) - R(t)]\mathbf{e}'_n$

has a Frame S' descriptor on the derivative. The equation says vector = vector. You can dot this equation with any vector you like. Thus, you can evaluate the equation in Frame S or in Frame S'. It is wrong to say this equation is valid ONLY in Frame S' . It is true that the equation $(d\mathbf{e_n}(t))_{\mathbf{X}} = [\, R(t+dt) - R(t)]\mathbf{e'_n}$ is valid only when X = S', but once you X = S', you can evaluate the equation in any frame you like.