Trying again on Question 7.

Review : Write $\mathbf{e}_n(t) = R(t)\mathbf{e}'_n(t)$ at time t which may be evaluated in any Frame. Then

$$d\mathbf{e}_n(t) \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$$

$$= R(t+dt)\mathbf{e}'_n(t+dt) - R(t)\mathbf{e}'_n(t)$$

$$\approx [\, R(t)\,(\partial_t \mathbf{e}'_n(t)) + (\partial_t R(t))\mathbf{e}'_n(t)\,]\, dt \quad . \qquad \text{// neglect order } (dt)^2 \text{ etc .}$$

so

$$(d\mathbf{e}_n(t)/dt) = R(t)\,[\partial_t \mathbf{e}'_n(t)] + [\partial_t R(t)]\,\mathbf{e}'_n(t) \quad .$$

$$[\partial_t \mathbf{e}_n(t)] = R(t)\,[\partial_t \mathbf{e}'_n(t)] + [\partial_t R(t)]\,\mathbf{e}'_n(t) \quad .$$

$$[\partial_t \mathbf{e}_n(t)]_X = R(t)\,[\partial_t \mathbf{e}'_n(t)]_X + [\partial_t R(t)]_X\,\mathbf{e}'_n(t) \qquad \text{X is some Frame.}$$

One can take components of this equation in any Frame. For example,

$$(\,[\partial_t \mathbf{e}_n(t)]_X\,) = R(t)\,([\partial_t \mathbf{e}'_n(t)]_X) + ([\partial_t R(t)]_X)\,\mathbf{e}'_n(t) \qquad \text{X is some Frame.}$$

This vector equation is valid in any frame. For example,

$$(\,[\partial_t \mathbf{e}_n(t)]_X\,)_i = \{R(t)\,([\partial_t \mathbf{e}'_n(t)]_X)\}_i + \{([\partial_t R(t)]_X)\,\mathbf{e}'_n(t)\}_i \qquad \text{Frame S}$$

$$(\,[\partial_t \mathbf{e}_n(t)]_X\,)'_i = \{R(t)\,([\partial_t \mathbf{e}'_n(t)]_X)\}'_i + \{([\partial_t R(t)]_X)\,\mathbf{e}'_n(t)\}'_i \qquad \text{Frame S'}$$

If we choose Frame X = Frame S', we get

$$(\,[\partial_t \mathbf{e}_n(t)]_{S'}\,)_i = \{R(t)\,([\partial_t \mathbf{e}'_n(t)]_{S'})\}_i + \{([\partial_t R(t)]_{S'})\,\mathbf{e}'_n(t)\}_i \qquad \text{Frame S}$$

$$(\,[\partial_t \mathbf{e}_n(t)]_{S'}\,)'_i = \{R(t)\,([\partial_t \mathbf{e}'_n(t)]_{S'})\}'_i + \{([\partial_t R(t)]_{S'})\,\mathbf{e}'_n(t)\}'_i \qquad \text{Frame S'}$$

Now we know that $[\partial_t \mathbf{e}'_n(t)]_{S'} = 0$ I guess because S' is moving but always e'_n are constant within S'. So

$$(\,[\partial_t \mathbf{e}_n(t)]_{S'}\,)_i = \{([\partial_t R(t)]_{S'})\,\mathbf{e}'_n(t)\}_i \qquad \text{Frame S}$$

$$(\,[\partial_t \mathbf{e}_n(t)]_{S'}\,)'_i = \{([\partial_t R(t)]_{S'})\,\mathbf{e}'_n(t)\}'_i \qquad \text{Frame S'}$$

It then seems that the equation continues to be evaluatable in both Frame's components. Again, you just dot with basis vectors! [agreed!]

OK, now let's write out the matrix products

$$([\partial_t \mathbf{e}_n(t)]_{S'})_i = ([\partial_t R(t)]_{S'})_{ij} (\mathbf{e}'_n(t))_j \qquad \text{Frame S}$$

$$([\partial_t \mathbf{e}_n(t)]_{S'})'_i = ([\partial_t R(t)]_{S'})'_{ij} (\mathbf{e}'_n(t)')_j \qquad \text{Frame S'}$$

Now define

$$([\partial_t R(t)]_{S'} = F_{S'}(t)$$

Then have

$$([\partial_t \mathbf{e}_n(t)]_{S'})_i = [F_{S'}(t)]_{ij} (\mathbf{e}'_n(t))_j \qquad \text{Frame S}$$

$$([\partial_t \mathbf{e}_n(t)]_{S'})'_i = [F_{S'}(t)]'_{ij} (\mathbf{e}'_n(t)')_j \qquad \text{Frame S'}$$

How are the two matrices related? I presume like so

$$[F_{S'}(t)]' = R(t)\, [F_{S'}(t)] R^{-1}(t)$$

Now take a closer look at

$$F_{S'}(t) = [\partial_t R(t)]_{S'} = [\{R(t+dt) - R(t)\}/dt\,]_{S'}$$

Take a special case where $R(t) = R_z(\varphi(t))$. Then

$$F_{S'}(t) = [\{R(t+dt) - R(t)\}/dt\,]_{S'} = [\,\{R_z(\varphi(t+dt)) - R_z(\varphi(t))\}/dt\,]_{S'}$$

Are these standard rotation matrices or not???

$$[F_{S'}(t)]'_{ij} = \{\,[\,\{R_z(\varphi(t+dt)) - R_z(\varphi(t))\}/dt\,]_{S'}\,\}'_{ij}$$

$$= \{dR'_z(\varphi(t+dt)/dt\}_{ij}$$

Plan B. Start over with,

$$d\mathbf{e}_n(t) \equiv \mathbf{e}_n(t+dt) - \mathbf{e}_n(t)$$

$$= R(t+dt)\mathbf{e}'_n(t+dt) - R(t)\mathbf{e}'_n(t)$$

$$\approx [\, R(t)\, (\partial_t \mathbf{e}'_n(t)) + (\partial_t R(t))\mathbf{e}'_n(t)\,]\, dt \;. \qquad \text{// neglect order } (dt)^2 \text{ etc .}$$

Then evaluations:

$$(d\mathbf{e}_n(t))_i = [\, R(t)_{ij}\, (\partial_t \mathbf{e}'_n(t))_j + (\partial_t R(t))_{ij} (\mathbf{e}'_n(t))_j\,]\, dt$$

$$(d\mathbf{e}_n(t))'_i = [\, R'(t)_{ij}\, (\partial_t \mathbf{e}'_n(t))'_j + (\partial_t R(t))'_{ij} (\mathbf{e}'_n(t))'_j \,]\, dt$$

So you have to face up to talking about these two objects

$$(\partial_t R(t))_{ij} dt = [\, R(t+dt) - R(t)]_{ij} = R(t+dt) - R(t)$$

$$(\partial_t R(t))'_{ij} dt = [\, R(t+dt) - R(t)]'_{ij} = R'(t+dt) - R'(t) = R(t+dt) - R(t)$$

Could these really be the same?

Plan C.

$$A \equiv (dR/dt)\ R^{-1}(\mathbf{\Phi})$$
$$A' \equiv (dR'/dt)\ R'^{-1}(\mathbf{\Phi})$$

$$Adt \equiv dR\ R^{-1}(\mathbf{\Phi})$$
$$A'dt \equiv dR'\ R'^{-1}(\mathbf{\Phi})$$

Then how about

$$[A'dt] = R(\mathbf{\Phi})\, [Adt]\, R^{-1}(\mathbf{\Phi}) = R(\mathbf{\Phi})\, [dR\ R^{-1}(\mathbf{\Phi})]\, R^{-1}(\mathbf{\Phi})$$

$$= R(\mathbf{\Phi}) dR\ \{R^{-1}(\mathbf{\Phi})\, R(\mathbf{\Phi})\}\, R^{-1}(\mathbf{\Phi}) R^{-1}(\mathbf{\Phi})$$

$$= [R(\mathbf{\Phi}) dR\, R^{-1}(\mathbf{\Phi})]\, [\, R(\mathbf{\Phi})\, R^{-1}(\mathbf{\Phi}) R^{-1}(\mathbf{\Phi})]$$

$$= [R(\mathbf{\Phi}) dR\, R^{-1}(\mathbf{\Phi})]\, R^{-1}(\mathbf{\Phi})$$

$$= dR'\ R^{-1}(\mathbf{\Phi})$$