A rotation matrix problem PhL 2.11.17

Statement of the Problem: Suppose you have the following

$\mathbf{e}'_i = R(\mathbf{\Phi})\mathbf{e}_i$ where $R(\mathbf{\Phi}) = R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$ = rotation by Goldstein Euler angles

Here $\mathbf{e}_i$ are the usual Cartesian unit vectors of Frame S and $\mathbf{e}'_i$ are some unit vectors of Frame S'.

Following the path of frames doc, we know that

$d\mathbf{e}'_n = d\mathbf{\Phi} \times \mathbf{e}_n$

or

$(\partial_S \mathbf{e}'_n) = \boldsymbol{\omega} \times \mathbf{e}'_n$ where $\boldsymbol{\omega} = d\mathbf{\Phi}/dt$

This is all shown in frames doc for vector **a**, and we are here applying it to the basis vectors themselves.

The Problem is to find an expression for $\boldsymbol{\omega}$ which involves $\dot{\psi}$ and $\dot{\theta}$ and $\dot{\varphi}$. Hopefully this expression will look something like Goldstein page 134 (4-103).

Note that $\mathbf{\Phi} \neq (\psi,\theta,\varphi)$ or anything simple like that.

A separate Problem would be to compute the function $\mathbf{\Phi}(\psi,\theta,\varphi)$ which I am not doing right now.

I am not sure how to start, so I will try several Plans.

Plan A. Write

$d\mathbf{e}'_n = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}_n - R(\mathbf{\Phi})\mathbf{e}_n$

$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi)\, \mathbf{e}_n - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\, \mathbf{e}_n$

Not sure if this makes sense, but pursue for a while. I can rewrite the Euler stuff as

$= [\, R_z(\psi)R_z(d\psi)\, R_x(\theta)\, R_x(d\theta)R_z(\varphi)R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]\, \mathbf{e}_n$

$= [\, R_z(\psi)\{1-id\psi J_3\}\, R_x(\theta)\{1-id\theta J_1\}R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]\, \mathbf{e}_n$

$\approx [\, -id\psi\, R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\, \mathbf{e}_n$

Then I could say

$(d\mathbf{e}'_n/dt) =$

$[\, -i\dot{\psi}\, R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) - i\dot{\theta}\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\, \mathbf{e}_n$

The [...] thing is a matrix I could have Maple compute. But we are not yet in the proper form because I want to have $\mathbf{e'}_n$ on the right in the final formula. I can write

$$\mathbf{e}_n = R(-\boldsymbol{\Phi})\mathbf{e'}_n \quad \text{where} \quad R(-\boldsymbol{\Phi}) = R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi) \quad = \text{the inverse rotation}$$

Then I have

$$(d\mathbf{e'}_n/dt) = [\, -i\dot{\psi}\; R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) \;-\; i\dot{\theta}\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) \;-\; i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]$$

$$R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi)\mathbf{e'}_n$$

and now at least it is in the right general form. Let's now examine the three terms' matrices

term1

$$R_z(\psi)J_3\, R_x(\theta)R_z(\varphi)\, R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi) \;=\; R_z(\psi)J_3\, R_z(-\psi) = J_3$$

term2

$$R_z(\psi)\, R_x(\theta)J_1R_z(\varphi)\, R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi) \;= R_z(\psi)\, R_x(\theta)J_1R_x(-\theta)\, R_z(-\psi)$$

$$= R_z(\psi)J_1R_z(-\psi)$$

term 3

$$R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3R_z(-\varphi)\, R_x(-\theta)\, R_z(-\psi) = R_z(\psi)\, R_x(\theta)J_3\, R_x(-\theta)\, R_z(-\psi)$$

So things have at least simplified a bit and we have

$$(d\mathbf{e'}_n/dt) = [\, -i\dot{\psi}\, J_3 - i\dot{\theta}R_z(\psi)J_1R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3\, R_x(-\theta)\, R_z(-\psi)]\, \mathbf{e'}_n$$
$$(d\mathbf{e'}_n/dt) = \boldsymbol{\omega} \times \mathbf{e'}_n$$

or

$$(d\mathbf{e'}_n/dt) = A\mathbf{e'}_n \qquad A = [\, -i\dot{\psi}\, J_3 - i\dot{\theta}R_z(\psi)J_1R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3\, R_x(-\theta)\, R_z(-\psi)]$$
$$(d\mathbf{e'}_n/dt) = \boldsymbol{\omega} \times \mathbf{e'}_n$$

where I show the target on the second line. It sure seems far-fetched but I think is correct. I have basically this statement

$$A\,\mathbf{e'}_n = \boldsymbol{\omega} \times \mathbf{e'}_n$$

Suppose I just look at the three components of the equations

$$[A\,\mathbf{e'}_n]_i = [\boldsymbol{\omega} \times \mathbf{e'}_n]_i$$

$A_{ik}(\mathbf{e'}_n)_k = \varepsilon_{ijk}\,\omega_j(\mathbf{e'}_n)_k$ ok

It must then be true that

$A_{ik} = \varepsilon_{ijk}\,\omega_j = \varepsilon_{kij}\omega_j = -\varepsilon_{ikj}\omega_j$ ok

or

$A_{ij} = -\varepsilon_{ijk}\omega_k$

Specifically this says

$A_{12} = -\varepsilon_{123}\omega_3 = -\omega_3$

$A_{23} = -\varepsilon_{231}\omega_1 = -\omega_1$

$A_{31} = -\varepsilon_{312}\omega_2 = -\omega_2$

So we conclude that

$-\omega_1 = A_{23}$
$-\omega_2 = A_{31}$
$-\omega_3 = A_{12}$.

But for this to have a chance of being valid, we must show that A is antisymmetric! Let's look at the three terms' matrices

term 1 J_3 this matrix IS antisymmetric as are all the generator matrices

term 2 $[R_z(\psi)J_1R_z(-\psi)]$

$[R_z(\psi)J_1R_z(-\psi)]^T = R_z(-\psi)^T\,J_1{}^T\,R_z(\psi)^T = -R_z(\psi)\,J_1R_z(-\psi) = -$ term 2 OK

term 3

$[\,R_z(\psi)\,R_x(\theta)J_3\,R_x(-\theta)\,R_z(-\psi)\,]^T = -R_z(\psi)R_x(\theta)J_3\,R_x(-\theta)R_z(-\psi) = -$ term 3

This is promising. Each matrix term really is antisymmetric.

So the task now is to compute

$A = [\,-i\dot{\psi}\,J_3 - i\dot{\theta}R_z(\psi)J_1R_z(-\psi) - i\dot{\varphi}\,R_z(\psi)\,R_x(\theta)J_3\,R_x(-\theta)\,R_z(-\psi)]$

I could have Maple create this matrix and then just read off the results.

Now let's try to get a full result here. I think I have formulas for things

$$R_z(\psi)J_2R_z(-\psi) = ...$$

Yes I have all these sandwich rules as in

$$\exp(-i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;J_k\;\exp(+i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = J_k\cos\theta + \varepsilon_{kms}J_m n_s\sin\theta + n_k(\hat{\mathbf{n}}\bullet\mathbf{J})(1-\cos\theta) \qquad (1)$$

And I also have this interesting result

$$R_n(\theta) = \exp(-i\theta\mathbf{n}\bullet\mathbf{J}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (C_\theta - 1)\begin{pmatrix} 1-n_1^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^2 \end{pmatrix} + S_\theta\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

Let's apply the above rules to my necessary special cases.

Go back to

$$\exp(-i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;J_k\;\exp(+i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = J_k\cos\theta + \varepsilon_{kms}J_m n_s\sin\theta + n_k(\hat{\mathbf{n}}\bullet\mathbf{J})(1-\cos\theta) \qquad (1)$$

as a vector statement

$$\exp(-i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J})\;\mathbf{J}\;\exp(+i\theta\,\hat{\mathbf{n}}\bullet\mathbf{J}) = \mathbf{J}\cos\theta + \mathbf{J}\times\hat{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}(\hat{\mathbf{n}}\bullet\mathbf{J})(1-\cos\theta) \qquad (1)$$

Suppose $n_s = \delta_{s,r}$ pointing in the r direction. Then the above reads

$$\exp(-i\theta J_r)\;J_k\;\exp(+i\theta J_r) = J_k\cos\theta + \varepsilon_{kmr}J_m\sin\theta + \delta_{k,r}J_r(1-\cos\theta)$$

or

$$R_r(\theta)\,J_kR_r(-\theta) = \cos\theta\,J_k - \varepsilon_{krm}\sin\theta\,J_m + \delta_{k,r}(1-\cos\theta)J_r$$

When k = r we know that $R_r(\theta)\,J_rR_r(-\theta) = J_r$ since everything commutes.

$$R_r(\theta)\,J_rR_r(-\theta) = J_r\,.$$

When $k \neq r$ we know $\delta_{k,r} = 0$ so

$$R_r(\theta)\,J_kR_r(-\theta) = \cos\theta\,J_k - \varepsilon_{krm}\sin\theta\,J_m \qquad k \neq r$$

We can quickly build a complete table of these cases

$$R_1(\theta)\,J_2R_1(-\theta) = \cos\theta\,J_2 - \varepsilon_{21m}\sin\theta\,J_m = \cos\theta\,J_2 + \sin\theta\,J_3$$
$$R_1(\theta)\,J_3R_1(-\theta) = \cos\theta\,J_3 - \varepsilon_{31m}\sin\theta\,J_m = \cos\theta\,J_3 - \sin\theta\,J_2$$

$$R_2(\theta)\, J_1 R_2(-\theta) = \cos\theta\, J_1 - \varepsilon_{12m} \sin\theta\, J_m = \cos\theta\, J_1 - \sin\theta\, J_3$$
$$R_2(\theta)\, J_3 R_2(-\theta) = \cos\theta\, J_3 - \varepsilon_{32m} \sin\theta\, J_m = \cos\theta\, J_3 + \sin\theta\, J_1$$

$$R_3(\theta)\, J_1 R_3(-\theta) = \cos\theta\, J_1 - \varepsilon_{13m} \sin\theta\, J_m = \cos\theta J_1 + \sin\theta\, J_2$$
$$R_3(\theta)\, J_2 R_3(-\theta) = \cos\theta\, J_2 - \varepsilon_{23m} \sin\theta\, J_m = \cos\theta J_2 - \sin\theta\, J_1$$

Now consider the matrices which appear in A above. In the second term we have,

$$R_z(\psi) J_1 R_z(-\psi) = R_3(\psi) J_1 R_3(-\psi) = \cos\psi J_1 + \sin\psi\, J_2$$

In the third term,

$$R_z(\psi)\, R_x(\theta) J_3\, R_x(-\theta)\, R_z(-\psi) = R_3(\psi)\, R_1(\theta) J_3\, R_1(-\theta)\, R_3(-\psi)$$

$$= R_3(\psi)[\ \cos\theta\, J_3 - \sin\theta\, J_2\ ]\ R_3(-\psi)$$

$$= \cos\theta\, R_3(\psi) J_3 R_3(-\psi) - \sin\theta\, R_3(\psi) J_2\, R_3(-\psi)$$

$$= \cos\theta\, J_3 - \sin\theta\, [\ \cos\psi J_2 - \sin\psi\, J_1\ ]$$

$$= \sin\theta\sin\psi\, J_1 - \sin\theta\cos\psi\, J_2 + \cos\theta\, J_3$$

Then we may write

$$A = [\ -i\dot{\psi}\, J_3 - i\dot{\theta} R_z(\psi) J_2 R_z(-\psi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta) J_3\, R_x(-\theta)\, R_z(-\psi)]$$

$$= -i\dot{\psi}\, J_3 - i\dot{\theta}\ (\cos\psi J_1 + \sin\psi\, J_2) - i\dot{\varphi}\ (\sin\theta\sin\psi\, J_1 - \sin\theta\cos\psi\, J_2 + \cos\theta\, J_3)$$

$$= -\, i\ [\ \dot{\psi}\, J_3 + \dot{\theta}\ (\cos\psi J_1 + \sin\psi\, J_2) + \dot{\varphi}\ (\sin\theta\sin\psi\, J_1 - \sin\theta\cos\psi\, J_2 + \cos\theta\, J_3)$$

$$= -\, i\ [\ (\cos\psi\, \dot{\theta} + \sin\theta\sin\psi\, \dot{\varphi}\ )\, J_1 + (\sin\psi\, \dot{\theta} - \sin\theta\cos\psi\, \dot{\varphi}) J_2 + (\dot{\psi} + \cos\theta\dot{\varphi}\ ) J_3\ ]$$

Therefore

$$-A = (\cos\psi\, \dot{\theta} + \sin\theta\sin\psi\, \dot{\varphi}\ )\ (iJ_1) + (\sin\psi\, \dot{\theta} - \sin\theta\cos\psi\, \dot{\varphi})(iJ_2) + (\dot{\psi} + \cos\theta\dot{\varphi}\ )(iJ_3)$$

Taking the ij matrix element

$$-A_{ij} = (\cos\psi\, \dot{\theta} + \sin\theta\sin\psi\, \dot{\varphi}\ )\ (iJ_1)_{ij} + (\sin\psi\, \dot{\theta} - \sin\theta\cos\psi\, \dot{\varphi})(iJ_2)_{ij} + (\dot{\psi} + \cos\theta\dot{\varphi}\ )(iJ_3)_{ij}$$

$$= (\cos\psi\, \dot{\theta} + \sin\theta\sin\psi\, \dot{\varphi}\ )\varepsilon_{1ij} + (\sin\psi\, \dot{\theta} - \sin\theta\cos\psi\, \dot{\varphi})\varepsilon_{2ij} + (\dot{\psi} + \cos\theta\dot{\varphi}\ )\varepsilon_{3ij}$$

Then from relations stated earlier,

$-\omega_1 = A_{23} = -(\cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\varphi})$

$-\omega_2 = A_{31} = -(\sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\varphi})$

$-\omega_3 = A_{12} = -(\dot{\psi} + \cos\theta\dot{\varphi})$

And we have then found our ω vector:

$\boldsymbol{\omega} = (\cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\varphi},\ \sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\varphi},\ \dot{\psi} + \cos\theta\dot{\varphi})$

$= (\cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\varphi})\,\mathbf{e}_1 + (\sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\varphi})\mathbf{e}_2 + (\dot{\psi} + \cos\theta\dot{\varphi})\mathbf{e}_3$

or

$\omega_x = \cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\varphi}$

$\omega_y = \sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\varphi}$

$\omega_z = \dot{\psi} + \cos\theta\,\dot{\varphi}$

This can be compared with GPS 3rd edition page 174 which I have (don't know my printing)

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}. \qquad (4.87)$$

We disagree on the sign of ω_y which does seem very odd. Do I recall an errata on this? I checked, and I don't see any errata on this equation apart from adding commas.

Here is what I have done in the above. I start with this Euler rotation

$\mathbf{e}'_i = R(\boldsymbol{\Phi})\mathbf{e}_i$ where $R(\boldsymbol{\Phi}) = R_z(\psi)\,R_x(\theta)\,R_z(\varphi)$ = rotation by Goldstein Euler angles

I then write

$(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$

and I solve for $\boldsymbol{\omega}$. This then gives the $\boldsymbol{\omega}$ which appears in my Fig 1.

Does Maple Confirms My result ?

Since I have this GPS discrepancy, I will try a Maple version and see what it days. Maple gives A:

$$\begin{bmatrix} 0 & -psid - phid\cos(\theta) & thetad\sin(\psi) - phid\cos(\psi)\sin(\theta) \\ psid + phid\cos(\theta) & 0 & -thetad\cos(\psi) - phid\sin(\psi)\sin(\theta) \\ -thetad\sin(\psi) + phid\cos(\psi)\sin(\theta) & thetad\cos(\psi) + phid\sin(\psi)\sin(\theta) & 0 \end{bmatrix}$$

This says, reading off

$A_{12} = -\dot{\psi} - \dot{\phi}\cos\theta$

$A_{13} = \dot{\theta}\sin\psi - \dot{\phi}\cos\psi\sin\theta$

$A_{23} = -\dot{\theta}\cos\psi - \dot{\phi}\sin\psi\sin\theta$ all three ok

$-\omega_1 = A_{23}$

$-\omega_2 = A_{31}$

$-\omega_3 = A_{12}$ all three ok

and therefore Maple is saying

$-\omega_1 = -\dot{\theta}\cos\psi - \dot{\phi}\sin\psi\sin\theta$

$-\omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta$

$-\omega_3 = -\dot{\psi} - \dot{\phi}\cos\theta$ all three ok

Now here is what I got above using the non-Maple method:

$\omega_x = \cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\phi}$

$\omega_y = \sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\phi}$

$\omega_z = \dot{\psi} + \cos\theta\,\dot{\phi}$

So the Maple method and my Hand method give exactly the same results.

Question: I have shown by both MY methods that

$\omega_x = \cos\psi\,\dot{\theta} + \sin\theta\sin\psi\,\dot{\phi}$

$\omega_y = \sin\psi\,\dot{\theta} - \sin\theta\cos\psi\,\dot{\phi}$

$\omega_z = \dot{\psi} + \cos\theta\,\dot{\phi}$

where

$\boldsymbol{\omega} = \omega_x \mathbf{e}_1 + \omega_y \mathbf{e}_2 + \omega_z \mathbf{e}_3$

and so I am showing the Frame S components of $\boldsymbol{\omega}$ where Frame S is my inertial frame.

What are the Frame S' components?

These are the ones Goldstein seems to be talking about.

$\boldsymbol{\omega} = (\omega)'_x \mathbf{e}'_1 + (\omega)'_y \mathbf{e}'_2 + (\omega)'_z \mathbf{e}'_3$?

Well, consider

$(\omega)'_x = \boldsymbol{\omega} \bullet \mathbf{e}'_1 = \omega_x \mathbf{e}_1 \bullet \mathbf{e}'_1 + \omega_y \mathbf{e}_2 \bullet \mathbf{e}'_1 + \omega_z \mathbf{e}_3 \bullet \mathbf{e}'_1$

$= \omega_x \mathbf{e}_1 \bullet \mathbf{R}\mathbf{e}_1 + \ldots.$

$= \omega_x (\mathbf{e}_1)_i (\mathbf{R}\mathbf{e}_1)_i + \ldots.$

$= \omega_x (\mathbf{e}_1)_i R_{ij} (\mathbf{e}_1)_j + \ldots.$

$= \omega_x \delta_{1i} R_{ij} \delta_{1j} + \ldots.$

$= \omega_x R_{11} + \ldots$

But I don't know what $R(\boldsymbol{\Phi}) = R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$ unless I multiply it out. This method seems painful, there must be a more direct route.

Go back to this point:

$(d\mathbf{e}'_n/dt) = [\, -i\dot{\psi} J_3 - i\dot{\theta} R_z(\psi) J_1 R_z(-\psi) - i\dot{\varphi} R_z(\psi) R_x(\theta) J_3 R_x(-\theta) R_z(-\psi)] \mathbf{e}'_n = A\mathbf{e}'_n$

$(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n$

The matrix A is a 3x3 matrix of elements which are functions of the generator matrices and the rates. Can I think of this as a matrix of numbers like π ? I think yes. Then:

Suppose I take the Frame S' components of both sides at this point. Then

$[\boldsymbol{\omega} \times \mathbf{e}'_n]'_i = \varepsilon_{ijk} (\omega')_j (\mathbf{e}'_n)'_k = \varepsilon_{ijk} (\omega')_j \delta_{n,k} = \varepsilon_{ijn} (\omega')_j$

$[A\mathbf{e}'_n]'_i = A_{ij} (\mathbf{e}'_n)'_j = A_{in}$ A = matrix of constants???

Then I get the same thing I got before

$$A_{in} = \varepsilon_{ijn}(\omega')_j = -\varepsilon_{inj}(\omega')_j$$

or

$$A_{ij} = \varepsilon_{ikj}(\omega')_k = -\varepsilon_{ijk}(\omega')_k$$

Compare this to the earlier result which was

$$A_{ij} = -\varepsilon_{ijk}\omega_k$$

This seems to suggest that the components are the same in both frames ???

I think this is the wrong step (been here before)

$$[A\mathbf{e}'_n]'_i = A_{ij}(\mathbf{e}'_n)'_j = A_{in}$$

In general suppose you have

$$\mathbf{A} = M\mathbf{B} \qquad \text{in Frame S}$$

How do you write this in Frame S' ? Let $\mathbf{A}' = R\mathbf{A}$ and $\mathbf{B}' = R\mathbf{B}$ so then

$$\mathbf{A} = M\mathbf{B} \Rightarrow R\mathbf{A} = RM\mathbf{B} = RMR^{-1}R\mathbf{B} \quad \Rightarrow \mathbf{A}' = (RMR^{-1})\mathbf{B}' = M'\mathbf{B}'$$

So my real equation then is

$$[A\mathbf{e}'_n]'_i = A'_{ij}(\mathbf{e}'_n)'_j = A'_{in} = [RAR^{-1}]_{in} \qquad R = R_z(\psi)\,R_x(\theta)\,R_z(\varphi)$$

and that removed the paradox at least. How about

$$\boldsymbol{\omega}' = R\boldsymbol{\omega}$$

I have argued in (1.3.5) that when this is true for a vector, we have $(\omega)'_n = (\omega')_n$. Now

$$\boldsymbol{\omega}' = R\boldsymbol{\omega} \Rightarrow (\omega')_i = (R\boldsymbol{\omega})_i = R_{ij}(\omega)_j$$

and then I get

$$(\omega)'_i = (\omega')_i = R_{ij}(\omega)_j$$

and this is an explicit method for computing the desired components, but it just seems very messy.

Plan B. Find a more direct path. Suppose I start out *differently*:

$$\mathbf{e}_i = R(\boldsymbol{\Phi})\mathbf{e}'_i \quad \text{where } R(\boldsymbol{\Phi}) = R_z(\psi)\,R_x(\theta)\,R_z(\varphi) = \text{rotation by Goldstein Euler angles}$$

where $\mathbf{e}_i$ are still the Frame S axis-aligned basis vectors, and Frame S is still the inertial frame. Then.

$$d\mathbf{e}_n = d\mathbf{\Phi} \times \mathbf{e}_n$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \qquad (1.7.4)$$

I think I can track the previous path in this framework. I will now have, as seen from Frame S',

$$d\mathbf{e}_n = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'}_n - R(\mathbf{\Phi})\mathbf{e'}_n$$

$$= [\, -id\psi\ R_z(\psi)J_3 R_x(\theta)R_z(\varphi) - id\theta\ R_z(\psi) R_x(\theta)J_1R_z(\varphi) - id\varphi\ R_z(\psi) R_x(\theta)R_z(\varphi)J_3 \,]\ \mathbf{e'}_n$$

Then I install $\mathbf{e'}_i = R^{-1}(\mathbf{\Phi})\mathbf{e}_i$ and I will get the same result as before (with this new $R(\mathbf{\Phi})$ relation)

$$(d\mathbf{e}_n/dt)_{S'} = [\, -i\dot{\psi}\ R_z(\psi)J_3 R_x(\theta)R_z(\varphi) - i\dot{\theta}\ R_z(\psi) R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\ R_z(\psi) R_x(\theta)R_z(\varphi)J_3 \,]$$

$$R_z(-\varphi) R_x(-\theta) R_z(-\psi)\mathbf{e}_n$$

$$= [\, -i\dot{\psi}\ J_3 - i\dot{\theta}R_z(\psi)J_1R_z(-\psi) - i\dot{\varphi}\ R_z(\psi) R_x(\theta)J_3 R_x(-\theta) R_z(-\psi)]\ \mathbf{e}_n$$

$$= [A\mathbf{e}_n] \qquad \text{exactly the same A as before}$$

Then we have

$$(d\mathbf{e}_n/dt)_{S'} = [A\mathbf{e}_n]$$

$$(d\mathbf{e}_n/dt)_{S'} = -[\boldsymbol{\omega} \times \mathbf{e}_n]$$

So I have this equation of interest

$$[A\mathbf{e}_n] = -[\boldsymbol{\omega} \times \mathbf{e}_n]$$

or

$$A\ R(\mathbf{\Phi})\mathbf{e'}_i = -\boldsymbol{\omega} \times [R(\mathbf{\Phi})\mathbf{e'}_i]$$

$$R(\mathbf{\Phi})^{-1}A\ R(\mathbf{\Phi})\mathbf{e'}_i = R(\mathbf{\Phi})^{-1}\,[\, -\boldsymbol{\omega} \times [R(\mathbf{\Phi})\mathbf{e'}_i]\,]$$

How about a simple theorem:

Theorem: Imagine Frame S and Frame S' with the same origins. Suppose in Frame S we see some rotation vector $\boldsymbol{\omega}$ with certain Frame S components ω_i. The Frame S' basis vectors are ALL rotating

about some $\hat{\boldsymbol{\omega}}$ axis with rate ω. If we jump onto Frame S', we will see the Frame S basis vectors rotating with the same $\hat{\boldsymbol{\omega}}$ axis but with rate -ω. They are just rotating backwards. That is why we have

$$(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

Well, this does not tell how to compare the components in the two frames, but OK. Go back again and recall that

$$A = [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) \;-\; i\dot{\theta}\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) \;-\; i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\, R^{-1}(\boldsymbol{\Phi})$$

Then perhaps we can say

$$A' = R\, A\, R^{-1} \qquad ???$$

Plan C. Think in terms of $\boldsymbol{\omega}' = R\boldsymbol{\omega}$ and so

$$(\omega)'_i = (\omega')_i = R_{ij}(\omega)_j \qquad R = R(\boldsymbol{\Phi}) = R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

Have Maple directly compute the components $(\omega)'_i$ and just see what they look like before doing any more threading attempts. Use

$$\omega_x = \cos\psi\, \dot{\theta} + \sin\theta\sin\psi\, \dot{\varphi}$$
$$\omega_y = \sin\psi\, \dot{\theta} - \sin\theta\cos\psi\, \dot{\varphi}$$
$$\omega_z = \dot{\psi} + \cos\theta\, \dot{\varphi}$$

And what about the bugaboo : " if the basis vectors rotate one way, then general vectors rotate the other way". Is this a myth or is there some truth to the statement which I have encountered many times? And how would that apply here? Questions for manana to get things rolling again.