This is the Title PhL 1.11.15

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n(\mathbf{\Phi}) = R(\mathbf{\Phi})\mathbf{e}'_n \quad \text{or} \quad |\mathbf{e}_n(\mathbf{\Phi})\rangle = |R(\mathbf{\Phi})\mathbf{e}'_n\rangle = \mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_n\rangle \qquad \text{(G.7.4)}$$

We show the Section 1.1 Dirac notation on the right and it is especially helpful in understanding what follows.

Since we have in mind that $\mathbf{\Phi} = \mathbf{\Phi}(t)$, one sees that the Frame S basis vectors change in time as viewed from Frame S'. In contrast, the basis vectors $\mathbf{e}'_n$ are static. The basis vectors $|\mathbf{e}_n(\mathbf{\Phi})\rangle$ are complete at time t so we can write, in analogy with (1.1.20),

$$\mathbf{1} = |\mathbf{e}_n(\mathbf{\Phi})\rangle\langle\mathbf{e}_n(\mathbf{\Phi})| \qquad \text{completeness of the } \mathbf{e}_n \text{ at time t} \qquad \text{(a)}$$

$$\mathbf{1} = |\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle\langle\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})| \qquad \text{completeness of the } \mathbf{e}_n \text{ at time t +dt} \qquad \text{(b)}$$

$$\mathbf{1} = |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| \qquad \text{completeness of the } \mathbf{e}'_n \text{ at any time} \qquad \text{(c)} \qquad \text{(G.7.5)}$$

In (b) the rotation vector has changed from $\mathbf{\Phi}$ to some $\mathbf{\Phi}+d\mathbf{\Phi}$ as time moved from t to d+dt. A key point is that the $\mathbf{e}_n$ basis vectors are complete at any point in time. The rotation operator $\mathcal{R}(\mathbf{\Phi})$ similarly is real orthogonal at any time, analogous to (1.1.37),

$$\mathcal{R}^{\mathrm{T}}(\mathbf{\Phi}) = \mathcal{R}^{-1}(\mathbf{\Phi}) \text{ and similarly for matrices } \quad R^{\mathrm{T}}(\mathbf{\Phi}) = R^{-1}(\mathbf{\Phi}) \qquad \text{(G.7.6)}$$

Now we close the Dirac equation in (G.7.4) on the left with $\langle\mathbf{e}'_m|$ to get

$$\langle\mathbf{e}'_m|\,\mathbf{e}_n(\mathbf{\Phi})\rangle = \langle\mathbf{e}'_m|\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_n\rangle = [R(\mathbf{\Phi})]'_{mn} \qquad \text{(G.7.7)}$$

On the other hand

$$[R(\mathbf{\Phi})]_{mn} = \langle\mathbf{e}_m(\mathbf{\Phi})|\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= \langle\mathbf{e}_m(\mathbf{\Phi})\,|\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\,\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_j\rangle\langle\mathbf{e}'_j|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= [R(\mathbf{\Phi})]'_{im}\,[R(\mathbf{\Phi})]'_{ij}[R(\mathbf{\Phi})]'_{jn} = [\,R^{\mathrm{T}}(\mathbf{\Phi})R(\mathbf{\Phi})R(\mathbf{\Phi})]'_{mn} = [R(\mathbf{\Phi})]'_{mn} \qquad \text{(G.7.8)}$$

We thus find that $[R(\mathbf{\Phi})]_{mn} = [R(\mathbf{\Phi})]'_{mn}$ as in (1.1.34). This is also true for $\mathbf{\Phi}+d\mathbf{\Phi}$.

Question added 3/10/17: What is the implication for a generator J_k ? Consider

$$[R(\mathbf{\Phi})]_{mn} = [R(\mathbf{\Phi})]'_{mn}$$

$[\exp(-i\boldsymbol{\Phi} \bullet \mathbf{J})]_{mn} = [\exp(-i\boldsymbol{\Phi} \bullet \mathbf{J})]'_{mn}$

$[\ 1 - i\boldsymbol{\Phi} \bullet \mathbf{J})]_{mn} = [\ 1 - i\boldsymbol{\Phi} \bullet \mathbf{J})]'_{mn}$ // if we assume Φ's components all small

$\Rightarrow\ (\mathbf{J})_{mn} = (\mathbf{J})'_{mn}$

Question: Does $[R(\boldsymbol{\Phi})]_{mn} = [R(\boldsymbol{\Phi})]'_{mn}$ say that $[R(\boldsymbol{\Phi})]_{mn}$ is a scalar? I think the answer is that it is a rank-2 tensor because it obeys the rank-2 tensor rule $A' = RAR^{-1}$ and it just happens that the values are the same in the two frames, so it is not a scalar. It is a very special rank-2 tensor being the basis change matrix in an all-Cartesian world.

So each generator matrix is the same on Frame S and in Frame S'. On the other hand, one would expect that

$(\mathbf{J})' = R\,(\mathbf{J})\,R^{-1}$ since this is the rule for any tensor.

In (G.4.1) I claim that

$R\mathbf{J}R^{-1} = R^{-1}\mathbf{J}$

Question: Is J really a basis object and does it back-rotate like a basis vector?

I now seem to have three facts which completely conflict with each other!!

$(\mathbf{J})_{mn} = (\mathbf{J})'_{mn}$ // "proved" just above

$(\mathbf{J})' = [R\ \mathbf{J}\ R^{-1}]$ // rule *if* each J_k is a rank-2 tensor

$R^{-1}\mathbf{J} = R\mathbf{J}R^{-1}$ // Theorem 1 of (G.4.1)

Notice the prime locations in the first two lines. Not the same.

In the vector world we have $(\mathbf{V})' = R\mathbf{V}$ in passive view for a kinematic vector, but $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$ for a basis vector. How is this statement extended to the rank-2 tensor world? Look again at vectors

$\mathbf{V} = V_i\ \mathbf{e}_i = (V)'_i\ \mathbf{e}'_i$

Now install $\mathbf{e}'_i = R^{-1}\mathbf{e}_i$ or $\mathbf{e}'_i = R_{im}\,\mathbf{e}_m$ (Basis Theorem) to get

$\mathbf{V} = (V)'_i\ R_{im}\,\mathbf{e}_m = V_m\,\mathbf{e}_m$

so conclude that

$V_m = (V)'_i R_{im}$ or $V_m\,R^{-1}{}_{mn} = (V)'_i R_{im} R^{-1}{}_{mn}$ or $R_{nm}V_m = (V)'_i\delta_{in} = (V)'_n$

and we end up with

$$(V)'_n = R_{nm}V_m \qquad \text{or} \quad (\mathbf{V})' = R\mathbf{V}$$

Now let's try something analogous for rank-2 tensors. Start with

$$\mathcal{T} = (T)_{ij}\,|e_i\rangle\langle e_j| = (T)'_{ij}\,|e'_i\rangle\langle e'_j|$$

$$\mathcal{T} = (T)'_{ij}\,|e'_i\rangle\langle e'_j| = (T)'_{ij}\,|R^{-1}e_i\rangle\langle R^{-1}e_j| = (T)'_{ij}\mathcal{R}^{-1}\,|e_i\rangle\langle e_j|\,\mathcal{R}$$

$$= (T)'_{ij}\,|e_n\rangle\langle e_n|\mathcal{R}^{-1}\,|e_i\rangle\langle e_j|\,\mathcal{R}\,|e_m\rangle\langle e_m|$$

$$= (T)'_{ij}\,|e_n\rangle R^{T}_{ni}R_{jm}\langle e_m| = R^{T}_{ni}(T)'_{ij}R_{jm}\,|e_n\rangle\langle e_m|$$

On the other hand we also have

$$\mathcal{T} = (T)_{nm}\,|e_n\rangle\langle e_m|$$

So comparison shows that

$$(T)_{nm} = R^{T}_{ni}(T)'_{ij}R_{jm} = [R^{T}\,(T)'\,R]_{nm}$$

and thus

$$(T)' = R(T)R^{-1}$$

so everything seems to work right here. So *if* the components of J were kinematic tensors, we would claim that

$$(J_k)' = R(J_k)R^{-1} \qquad \text{or} \qquad (\mathbf{J})' = R(\mathbf{J})R^{-1}$$

So having done a few confidence-building exercises, we go back to our **Triple Paradox** which it is safer to write out without vector notation

1 $(J_i)_{mn} = (J_i)'_{mn}$ // proved from $[R(\mathbf{\Phi})]_{mn} = [R(\mathbf{\Phi})]'_{mn}$

2 $(J_i)' = R\,J_i\,R^{-1}$ // rule *if* each J_i were a rank-2 tensor (all matrices)

3 $(R^{-1}{}_{ij}\,J_j) = RJ_iR^{-1}$ // Theorem 1 of (G.4.1) (all matrices)

Lines 1 and 2 conflict for sure because we have

1 $(J_i)_{mn} = (J_i)'_{mn}$

2 $(J_i)' = R\,J_i\,R^{-1}$

I will verify that line 1 and line 2 conflict with a specific counterexample. Suppose $R = R_z(\theta)$ and take J_1. Then line 2 claims that

$$(J_1)'_{mn} = R_z(\theta)_{mi}(J_1)_{ij}\, R_z(-\theta)_{jn} = [\, R_z(\theta)(J_1)R_z(-\theta)]_{mn}$$

$$= [\, R_3(\theta)J_1R_3(-\theta)]_{mn} = \cos\theta(J_1)_{mn} + \sin\theta\,(J_2)_{mn} \qquad \text{// from (G.3.8)}$$

So if line 2 were true we would have

$$(J_1)'_{mn} = \cos\theta(J_1)_{mn} + \sin\theta\,(J_2)_{mn}$$

But line 1 says

$$(J_1)'_{mn} = (J_1)_{mn}$$

so lines 1 and 2 conflict for dead sure.

Let's look now at line 1 versus line 3 for this same example. But line 3 makes no reference to $(\mathbf{J})'_{mn}$ so I cannot really compare these two lines. But I can see if I think line 3 is valid. Consider line 3:

$$R^{-1}\mathbf{J} = R\mathbf{J}R^{-1}$$

This is a bit ambiguous (!) so you have to write out the intended meaning as shown in (G.4.1),

$$R_z(-\theta)_{1j}J_j = R_z(\theta)J_1R_z(-\theta)$$

The math for the RHS is already done above, and it gives $\cos\theta J_1 + \sin\theta\, J_2$. The left side is

$$R_z(-\theta)_{11}J_1 + R_z(-\theta)_{12}J_2 + R_z(-\theta)_{13}J_3 = \cos\theta J_1 + \sin\theta\, J_2 + 0\, J_3 = \text{agrees}$$

where

$$R_z(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So I have just verified Theorem 1 for a specific case, not a bad thing to do.

Ouch! I have just found a <u>counterexample to my Theorem 1</u> of (G.4.1), that is not good!

How about lines 2 and 3: They claim that

2 $(J_i)' = R\, J_i\, R^{-1}$ // rule *if* each J_i were a rank-2 tensor (all matrices)

3 $R^{-1}_{ij}\, J_j = R J_i R^{-1}$ // Theorem 1 of (G.4.1) (all matrices)

I don't know how to check this because I don't know what $(J_1)'_{mn}$ is! I don't really have a definition of the numbers $(J_1)'_{mn}$, but I do have a definition of $(J_1)_{mn}$.

Proposed resolution of the Triple Paradox.

1. Line 1 is valid, and one should regard the J_i as matrices of constants, the same in all frames. This then makes my Appendix G.7 operator approach work!

2. Line 2 is wrong, and the matrix J_i does not transform as a kinematic tensor.

3. Line 3 is valid and does not conflict with line 1, since line 3 does not mention $(J)'_{mn}$.

Question: How does this relate to Theorem 2 which says

$R\, R_n(\theta)\, R^{-1} = R_{n'}(\theta)$ where $\mathbf{n}' = R\mathbf{n}$

In this theorem R is an arbitrary rotation. If we take $R = R_n(\theta)$ it says $R_n(\theta) = R_{n'}(\theta)$ but this is true because in this case $\mathbf{n}' = R_n(\theta)\mathbf{n} = \mathbf{n}$. So no conflicts here.

Note: We are NOT saying that $RJ_iR^{-1} = J'_i = J_i$!!! The object RJ_iR^{-1} is non-trivial because J_i does not commute with a general R. What we know is that $RJ_iR^{-1} = R^{-1}_{ij}\, J_j$.

But now here is another paradox. Consider that we find

$$A = -\left[\left(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \right)(iJ_1) + \left(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi\right)(iJ_2) + \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)(iJ_3) \right]$$

as an operator statement. Is this a tensor or not? According to the resolution above, it must NOT be a tensor because taking matrix elements and then using $(J_i)_{mn} = (J_i)'_{mn}$, we get $A'_{ij} = A_{ij}$. This then fouls up the Section G.7 Frame S evaluation of ω !

Possible resolution:

1	$(J_i)_{mn} = (J_i)'_{mn}$	and	$(A)'_{ij} = (A)_{ij}$	both valid
2	$(J'_i) = R\, J_i\, R^{-1}$	and	$(A')_{ij} = [R\, A\, R^{-1}]_{ij}$	both valid

We now examine (from Frame S') a small change in the basis vector $\mathbf{e}_n$

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle \equiv |\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle - |\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= |\mathbf{e'}_i\rangle\langle\mathbf{e'}_i|\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle - |\mathbf{e'}_i\rangle\langle\mathbf{e'}_i|\mathbf{e}_n(\mathbf{\Phi})\rangle$$

$$= |\mathbf{e'}_i\rangle\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - |\mathbf{e'}_i\rangle[R(\mathbf{\Phi})]'_{in}$$

$$= (\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{in} - [R(\mathbf{\Phi})]'_{in}\,)\;|\mathbf{e'}_i\rangle$$

$$= (\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]_{in} - [R(\mathbf{\Phi})]_{in}\,)\;|\mathbf{e'}_i\rangle \qquad \text{// lincom}$$

$$\equiv (dR)_{in}\,|\mathbf{e'}_i\rangle \qquad dR = R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \quad \text{(matrices)}$$

The next step is to replace $|\mathbf{e'}_i\rangle$ as follows

$$|\mathbf{e'}_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle\langle\mathbf{e}_j(\mathbf{\Phi})\,|\,\mathbf{e'}_i\rangle = |\mathbf{e}_j(\mathbf{\Phi})\rangle[R(\mathbf{\Phi})]_{ij} \qquad \text{// lincom}$$

Then

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = (dR)_{in}[R(\mathbf{\Phi})]_{ij}\,|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

or

$$(d\mathbf{e}_n(\mathbf{\Phi}))_{S'} = (dR)_{in}[R(\mathbf{\Phi})]_{ij}\mathbf{e}_j(\mathbf{\Phi})$$

Divide by dt to get

$$(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'} = (dR/dt)_{in}[R(\mathbf{\Phi})]_{ij}\mathbf{e}_j(\mathbf{\Phi})$$

$$= [R^{-1}(\mathbf{\Phi})]_{ji}(dR/dt)_{in}\,\mathbf{e}_j(\mathbf{\Phi})$$

$$= [R^{-1}(\mathbf{\Phi})\,(dR/dt)]_{jn}\;\mathbf{e}_j(\mathbf{\Phi})$$

ok to here 3.8.17

Write this last result out please without the dt,

$$(d\mathbf{e}_n(\mathbf{\Phi}))_{S'} = [R^{-1}(\mathbf{\Phi})]_{ji}(dR)_{in}\,\mathbf{e}_j(\mathbf{\Phi})$$

$$= [R^{-1}(\mathbf{\Phi})]_{ji}(\,[R(\mathbf{\Phi}+d\mathbf{\Phi})]_{in} - [R(\mathbf{\Phi})]_{in}\,)\,\mathbf{e}_j(\mathbf{\Phi})$$

$$= (\;[R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi}+d\mathbf{\Phi})]_{jn} - \delta_{jn}\,)\,\mathbf{e}_j(\mathbf{\Phi})$$

$$= (\;[R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi}+d\mathbf{\Phi})] - 1\,)_{jn}\;\;\mathbf{e}_j(\mathbf{\Phi}) \qquad (*)$$

or

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = (\;[R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi}+d\mathbf{\Phi})] - 1\,)_{jn}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}\rangle = (\;[R^{T}(\mathbf{\Phi})R(\mathbf{\Phi}+d\mathbf{\Phi})] - 1\,)_{jn}\;|\mathbf{e}_j(\mathbf{\Phi})\rangle$$

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}> = \left(\left[R^T(\mathbf{\Phi})R(\mathbf{\Phi}+d\mathbf{\Phi})\right] - 1 \right)^T_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})>$$

$$|(d\mathbf{e}_n(\mathbf{\Phi}))_{S'}> = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})>$$

This is a vector equation which you should be able to evaluate in Frame S or in Frame S'. [I think that might be true, you really do have a true HS vector on each side.] Maybe write this as

$$|(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'}>dt = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})>$$

Now the other equation of interest is this

$$|(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'}> = -\,\boldsymbol{\omega} \times |\mathbf{e}_n(\mathbf{\Phi})>$$

or

$$|(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'}>dt = -\,dt\,\boldsymbol{\omega} \times |\mathbf{e}_n(\mathbf{\Phi})>$$ // no meaning as vector equation

So we should be able to equation the vector right hand sides to get

$$\left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})> = -\,dt\,\boldsymbol{\omega} \times |\mathbf{e}_n(\mathbf{\Phi})>$$

Is the object $\boldsymbol{\omega} \times |\mathbf{e}_n(\mathbf{\Phi})>$ well defined? What does it mean? It is a math vector crossed with a Hilbert Space vector. I suspect this is NOT well defined. Go back to

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n$$

Where does this come from?

Review: Go back to frames doc Section 1.6. We have $d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a}$. This is a vector equation, yes. There is only one Space at this point in that section. Then $(d\mathbf{a}/dt) = \boldsymbol{\omega} \times \mathbf{a}$ same comment. Then I jump to $(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n$ and this is interpreted as $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$. Although this is a vector equation, I now today think it is valid only in Frame S! This is really a shorthand notation for

$$[(d\mathbf{e}'_n/dt)_S]_i = \varepsilon_{ijk}\,\omega_j\,(\mathbf{e}'_n)_k$$

Note: This is not the case I treat in App G.7, but I will get to that case below.

Then I should be able to construct a vector equation as follows,

$$(d\mathbf{e}'_n/dt)_S = \left[\varepsilon_{ijk}\,\omega_j\,(\mathbf{e}'_n)_k\right] \mathbf{e}_i$$

$$|(d\mathbf{e}'_n/dt)_S> = \left[\varepsilon_{ijk}\,\omega_j\,(\mathbf{e}'_n)_k\right] |\mathbf{e}_i>$$ // this is OK Dirac notation

Now suppose I close that equation from the left with $<\mathbf{e}'_m|$. That would give

$$<\mathbf{e}'_m|(d\mathbf{e}'_n/dt)_S> = \left[\varepsilon_{ijk}\,\omega_j\,(\mathbf{e}'_n)_k\right] <\mathbf{e}'_m|\mathbf{e}_i>$$

or

$$[(d\mathbf{e}'_n/dt)_S]'_m = [\varepsilon_{ijk}\,\omega_j\,(\mathbf{e}'_n)_k]\,(\mathbf{e}_i)'_m$$

Notice that $(\omega)'_j$ does not appear in this equation! Writing things out we get

$$[(d\mathbf{e}'_n/dt)_S]'_m = \varepsilon_{ijk}\,\omega_j\,R_{nk}R_{mi}$$

or

$$[(d\mathbf{e}'_c/dt)_S]'_b = \omega_J\,\varepsilon_{jJk}\,R_{ck}R_{bi} = \omega_J\varepsilon_{jJk}R^T_{jb}R^T_{kc} = -\,\omega_J\varepsilon_{Jjk}R^T_{jb}R^T_{kc}$$

My App A result is

$$R_{Ja}\varepsilon_{abc} = \varepsilon_{Jjk}R_{jb}R_{kc} = \varepsilon_{Jjk}R_{jb}R_{kc}$$

Using this I then get

$$[(d\mathbf{e}'_c/dt)_S]'_b = -\,\omega_J\varepsilon_{Jjk}R^T_{jb}R^T_{kc} = -\,\omega_J R^T_{Ja}\varepsilon_{abc} = -\,\omega_J\varepsilon_{abc}R_{aJ} = -\,\varepsilon_{abc}R_{aJ}\omega_J$$

$$= -\,\varepsilon_{abc}[R\omega]_a = -\,\varepsilon_{abc}(\omega')_a$$

Not sure of this result, but lets go do things the other way now. Start with

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

and this is just a shorthand for

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k$$ and this gives the correct ω in Frame S'

Now let's turn this into a vector equation by writing

$$[(d\mathbf{e}_n/dt)_{S'}]'_i\,\mathbf{e}'_i = -\,\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k\,\mathbf{e}'_i$$

or

$$|\,(d\mathbf{e}_n/dt)_{S'}\rangle = -\,\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k\,|\mathbf{e}'_i\rangle$$ // seems an OK vector equation

or

$$|\,(d\mathbf{e}_n/dt)_{S'}\rangle = -\,\varepsilon_{ajk}(\omega)'_j(\mathbf{e}_n)'_k\,|\mathbf{e}'_a\rangle$$ // seems an OK vector equation

Now take components of this equation in Frame S',

$$\langle\mathbf{e}'_i\,|\,(d\mathbf{e}_n/dt)_{S'}\rangle = -\,\varepsilon_{ajk}(\omega)'_j(\mathbf{e}_n)'_k\,\langle\mathbf{e}'_i\,|\mathbf{e}'_a\rangle$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ajk}(\omega)'_j(\mathbf{e}_n)'_k\,\delta_{ia}$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -\,\varepsilon_{ijk}(\omega)'_j R_{kn}$$

Now recall from above that

$$|(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'}>dt = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} \; |\mathbf{e}_j(\mathbf{\Phi})>$$

Suppose we close this with $<\mathbf{e'}_i|$ on the left. That gives,

$$<\mathbf{e'}_i|(d\mathbf{e}_n(\mathbf{\Phi})/dt)_{S'}>dt = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} <\mathbf{e'}_i|\mathbf{e}_j(\mathbf{\Phi})>$$

or

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} R_{ij} / dt$$

Setting the two right hand sides equal we seem then to get

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} R_{ij} / dt$$

or

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi})R(\mathbf{\Phi}) - 1 \right)_{nj} R^T_{ji} / dt$$

or

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = \left(R^T(\mathbf{\Phi}+d\mathbf{\Phi}) - R^T(\mathbf{\Phi}) \right)_{ni} / dt$$

or

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = \left(R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \right)_{in} / dt$$

or

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} = (dR)_{in} / dt$$

Now multiply both sides by R^{-1}_{nm} and sum on n to get

$$- \varepsilon_{ijk}(\omega)'_j R_{kn} R^{-1}_{nm} = (dR)_{in} R^{-1}_{nm} / dt$$

or

$$- \varepsilon_{ijk}(\omega)'_j \delta_{km} = (dR * R^{-1})_{im} / dt = [(dR/dt) * R^{-1}]_{im} \equiv A_{im}$$

or

$$- \varepsilon_{ijm}(\omega)'_j = A_{im}$$

or

$$\varepsilon_{jim}(\omega)'_j = A_{im}$$

or

$$\varepsilon_{kim}(\omega)'_k = A_{im}$$

or

$$\varepsilon_{kij}(\omega)'_k = A_{ij}$$

THIS IS THE DESIRED RESULT!!!!!!!

I did this all with "lin comb" stuff instead of "operator stuff". Right now this is the ONLY place I am getting the right answer to this problem!!!!

Now take components in Frame S instead of Frame S' and see what happens,

$$\langle \mathbf{e}_m \,|\, (d\mathbf{e}_n/dt)_{S'} \rangle = -\varepsilon_{ijk}(\omega)'_j (\mathbf{e}_n)'_k \langle \mathbf{e}_m \,|\mathbf{e}'_i\rangle$$

or

$$[\,(d\mathbf{e}_n/dt)_{S'}\,]_m = -\varepsilon_{ijk}(\omega)'_j R_{kn} R_{im} = -(\omega)'_j \varepsilon_{ijk} R_{kn} R_{im} = -(\omega)'_j \varepsilon_{jki} R_{kn} R_{im}$$

$$= +(\omega)'_j \varepsilon_{jik} R_{im} R_{kn} = +(\omega)'_c \varepsilon_{cjk} R_{jm} R_{kn} = (\omega)'_c R_{ca} \varepsilon_{amn} = \varepsilon_{amn} R_{ca} (\omega)'_c$$

where in the last step I have used App A,

$$R_{ca}\varepsilon_{amn} = \varepsilon_{cjk} R_{jm} R_{kn} \ .$$

So, if I were to insist on doing things in Frame S components, I have just shown that

$$[\,(d\mathbf{e}_n/dt)_{S'}\,]_m = \varepsilon_{amn} R_{ca} (\omega)'_c$$

Now the vector equation I end up with in G.7 is this

$$(d\mathbf{e}_n/dt)_{S'} = A_{in}\, \mathbf{e}_i \qquad \textit{(G.7.8b)}$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \ . \qquad \textit{(G.7.1)}$$

If I take Frame S' components of both, using the above, I get

$$[(d\mathbf{e}_n/dt)_{S'}]_m = A_{in} (\mathbf{e}_i)_m = A_{in}\delta_{im} = A_{mn}$$

$$[\,(d\mathbf{e}_n/dt)_{S'}\,]_m = \varepsilon_{amn} R_{ca} (\omega)'_c$$

I would then end up with

$$\varepsilon_{amn} R_{ca} (\omega)'_c = A_{mn}$$

Then for example, I get

$$\varepsilon_{123} R_{c1} (\omega)'_c = A_{23} \qquad \text{or} \qquad R_{c1} (\omega)'_c = A_{23} \qquad \text{or } R^T{}_{1c} (\omega)'_c = A_{23}$$

$$\text{or} \quad [R^T \omega]'_1 = A_{23} \qquad \text{or} \qquad [R^{-1}\boldsymbol{\omega}]'_1 = A_{23}$$

But this does not seem to give the right result!

There are no Hilbert Space vectors in this equation so no place for the Dirac notation here. Could you say

$$|(d\mathbf{e}'_n/dt)_S\rangle = |\boldsymbol{\omega} \times \mathbf{e}'_n\rangle \qquad ??$$

The LHS I think is an OK vector in HS, but what is the meaning of the RHS?? Try this instead

$$|(d\mathbf{e}'_n/dt)_S\rangle = \boldsymbol{\omega} \times |\mathbf{e}'_n\rangle \qquad ??$$

I claim today this has NO MEANING. You are cross producting vectors in two different spaces.

3.9.17: Let's try a little harder on the meaning question. Start with

$$[(d\mathbf{e}'_n/dt)_S]_i = \varepsilon_{ijk}\, \omega_j\, (\mathbf{e}'_n)_k$$

Conclusions:

1. The equation $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ *appears* to be a "vector equation" which you could evaluate in either Frame S or Frame S'. This is a vector equation in a pure math sense, but not in the Hilbert Space sense of our basis vectors. There is no meaning to objects $|\boldsymbol{\omega} \times \mathbf{e}'_n\rangle$ or $\boldsymbol{\omega} \times |\mathbf{e}'_n\rangle$ for the RHS in the Hilbert Space which is spanned by the basis vectors. These are cross products between a math vector and a HS vector and such an operation is not defined, though maybe we can provide a definition below.

2. The equation $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ is nothing more than a shorthand for this equation

$$[(d\mathbf{e}'_n/dt)_S]_i = \varepsilon_{ijk}\, \omega_j\, (\mathbf{e}'_n)_k$$

in which no Hilbert Space vectors appear. You *can* write this as

$$\mathbf{e}_i \bullet [(d\mathbf{e}'_n/dt)_S] = \varepsilon_{ijk}\, \omega_j\, \mathbf{e}_k \bullet \mathbf{e}'_n$$

If you could express the RHS as $\mathbf{e}_i \bullet \mathbf{Q}$, then you could write $|(d\mathbf{e}'_n/dt)_S\rangle = |\mathbf{Q}\rangle$ and then you would have a true HS vector equation. So lets try to do this

$$\mathbf{e}_i \bullet \mathbf{Q} = \varepsilon_{ijk}\, \omega_j\, \mathbf{e}_k \bullet \mathbf{e}'_n \qquad ??$$

The k index on the right is tied to the ε so it just does not work.

****

Now that Dirac has helped out a bit, let's try things without Dirac. We go once again to the opening scene of our movie, showing the Basis Theorem linear combination sums on the right

$$\mathbf{e}_n(\mathbf{\Phi}) = R(\mathbf{\Phi})\mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}_n(\mathbf{\Phi}) = [R^{-1}(\mathbf{\Phi})]_{nm}\ \mathbf{e}'_m \qquad //\mathbf{e}'_n = [R(\mathbf{\Phi})]_{nm}\,\mathbf{e}_m(\mathbf{\Phi})$$

$$\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) = [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})]_{nm}\ \mathbf{e}'_m$$

Then

$$(d\mathbf{e}_n)_{S'} \equiv \mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi}) - \mathbf{e}_n(\mathbf{\Phi}) = ([R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})]_{nm} - [R^{-1}(\mathbf{\Phi})]_{nm})\mathbf{e}'_m$$

$$= ([R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})]_{nm} - [R^{-1}(\mathbf{\Phi})]_{nm}) [R(\mathbf{\Phi})]_{mk}\ \mathbf{e}_k(\mathbf{\Phi})$$

$$\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi}) \}_{nk}\ \mathbf{e}_k(\mathbf{\Phi})$$

$$\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}\ \mathbf{e}_k(\mathbf{\Phi}) \qquad \text{// agrees with (*) above}$$

Rewrite as

$$dt\ (d\mathbf{e}_n/dt)_{S'} = [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nk}\ \mathbf{e}_k(\mathbf{\Phi})$$

ok to here 3.8.17

Evaluate this first in Frame S:

$$dt\ [\ (d\mathbf{e}_n/dt)_{S'}]_m = dt\ (d\mathbf{e}_n/dt)_{S'} \bullet \mathbf{e}_m = [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nm} \equiv D_{nm}$$

Now evaluate instead in Frame S' :

$$dt\ [\ (d\mathbf{e}_n/dt)_{S'}]'_m = dt\ (d\mathbf{e}_n/dt)_{S'} \bullet \mathbf{e}'_m = [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nk}\ \mathbf{e}_k(\mathbf{\Phi}) \bullet \mathbf{e}'_m$$

$$= [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nk}\ R(\mathbf{\Phi})_{mk}$$

$$= [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nk}\ R^{-1}(\mathbf{\Phi})_{km}$$

$$= D_{nk}\ R^{-1}(\mathbf{\Phi})_{km}$$

Comment: I think D_{nk} here is just a "matrix of numbers" which has the same value in Frame S and Frame S'.

The two results are different, and are related by a transformation matrix. As it should be.

ok to here 3.8.17

Now consider

$$dt(d\mathbf{e}_n/dt)_{S'} = -\,dt\ \boldsymbol{\omega} \times \mathbf{e}_n(\mathbf{\Phi})$$

Simplify notation and compare

$dt\,(d\mathbf{e}_n/dt)_{S'} = [R^{-1}(\Phi+d\Phi)R^{-1}(\Phi) - 1]_{nk}\ \mathbf{e}_k \equiv A_{nk}\,\mathbf{e}_k$

$dt(d\mathbf{e}_n/dt)_{S'} = -\,dt\ \boldsymbol{\omega} \times \mathbf{e}_n$

Pause: Can I show that A_{nk} is antisymmetric?

$$A = [R^{-1}(\Phi+d\Phi)R^{-1}(\Phi) - 1] = [R^{T}(\Phi+d\Phi)R^{T}(\Phi) - 1]$$

$$A^{T} = [R(\Phi)R(\Phi+d\Phi) - 1]$$

One would have to show that

$$1 - R(\Phi)R(\Phi+d\Phi) = R^{T}(\Phi+d\Phi)R^{T}(\Phi) - 1$$

Apply $R^{-1}(\Phi)$ from the left to get

$$R^{-1}(\Phi) - R^{-1}(\Phi)R(\Phi)R(\Phi+d\Phi) = R^{-1}(\Phi)R^{T}(\Phi+d\Phi)R^{T}(\Phi) - R^{-1}(\Phi)$$

or

$$R^{-1}(\Phi) - R(\Phi+d\Phi) = R^{-1}(\Phi)R^{T}(\Phi+d\Phi)R^{-1}(\Phi) - R^{-1}(\Phi)$$

Now apply $R(\Phi)$ from the right to get

$$R^{-1}(\Phi)\,R(\Phi) - R(\Phi+d\Phi)\,R(\Phi) = R^{-1}(\Phi)R^{T}(\Phi+d\Phi)R^{-1}(\Phi)\,R(\Phi) - R^{-1}(\Phi)\,R(\Phi)$$

or

$$1 - R(\Phi+d\Phi)\,R(\Phi) = R^{-1}(\Phi)R^{T}(\Phi+d\Phi) - 1$$

or

$$1 - R(\Phi+d\Phi)\,R(\Phi) = R^{T}(\Phi)R^{T}(\Phi+d\Phi) - 1$$

I give up on showing antisymmetry for the moment.

Comparing we get

$$-\,dt\ \boldsymbol{\omega} \times \mathbf{e}_n = A_{nk}\,\mathbf{e}_k$$

ok to here 3.8.17

Evaluate in Frame S: // I now think this is not meaningful to do

$$-\,dt\,[\boldsymbol{\omega} \times \mathbf{e}_n]_i = A_{nk}\,(\mathbf{e}_k)_i$$

$$-\,dt\,\varepsilon_{ijk}\omega_j\,(\mathbf{e}_n)_k = A_{nk}\,(\mathbf{e}_k)_i$$

$$-\,dt\,\varepsilon_{ijk}\omega_j\,\delta_{nk} = A_{nk}\,\delta_{ki}$$

$$- dt\ \varepsilon_{ijn}\omega_j = A_{ni}$$

$$- dt\ \varepsilon_{nij}\omega_j = A_{ni}$$

$$- dt\ \varepsilon_{nik}\omega_k = A_{ni}$$

$$- dt\ \varepsilon_{ijk}\omega_k = A_{ij}$$

This does imply that A_{ij} should be antisymmetric, but not obvious

$$- dt\ \varepsilon_{12k}\omega_k = A_{12} = -dt\ \omega_3$$

$$- dt\ \varepsilon_{13k}\omega_k = A_{13} = +dt\ \omega_2$$

$$- dt\ \varepsilon_{23k}\omega_k = A_{23} = -dt\ \omega_1$$

So we get our results

$$dt\ \omega_1 = -A_{23}$$
$$dt\ \omega_2 = A_{13}$$
$$dt\ \omega_3 = -A_{12}$$

ok to here 3.8.17

Go back

$$- dt\ \boldsymbol{\omega} \times \mathbf{e}_n = A_{nk}\,\mathbf{e}_k$$

Evaluate in Frame S':

$$- dt\ [\boldsymbol{\omega} \times \mathbf{e}_n]'_i = A_{nk}\,(\mathbf{e}_k)'_i$$ // why no prime on A ??

$$- dt\ \varepsilon_{ijk}\omega'_j\,(\mathbf{e}_n)'_k = A_{nk}\,(\mathbf{e}_k)'_i$$

$$- dt\ \varepsilon_{ijk}\omega'_j\,R_{kn} = A_{nk}R_{ik}$$

$$- dt\ \varepsilon_{ijk}\omega'_j\,R_{kn}R^{-1}_{ns} = R^{-1}_{ns}A_{nk}R_{ik}$$

$$- dt\ \varepsilon_{ijk}\omega'_j\,\delta_{ks} = R_{sn}A_{nk}R^{-1}_{ki}$$

$$- dt\ \varepsilon_{ijs}\omega'_j = [RAR^{-1}]_{si}$$

The result is then DIFFERENT!

$$A = [R^{-1}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R^{-1}(\boldsymbol{\Phi}) - 1]$$

$$RAR^{-1} = R[R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]R^{-1}$$

$$= [\, R(\mathbf{\Phi})\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1\,]$$

This is surely wrong.

OK, enough for this shot. At least I have some "ideas".

****

Question. Define this as

$$Q_{nk} \equiv [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]_{nk} \qquad Q = R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1$$

Then

$$Q^{T} = [\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1]^{T} = [\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi})]^{T} - 1]$$

$$= [\, R(\mathbf{\Phi})\, R(\mathbf{\Phi}+d\mathbf{\Phi}) - 1\,] \qquad \text{// cannot tell if antisymmetric or not}$$

Divide by dt to get

$$(d\mathbf{e}_n/dt)_{S'} = \frac{\{\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1\,\}_{nk}}{dt}\ \mathbf{e}_k(\mathbf{\Phi})$$

Compare to

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n(\mathbf{\Phi})$$

This seems to suggest the following vector equation,

$$-\,\boldsymbol{\omega} \times \mathbf{e}_n(\mathbf{\Phi}) = \frac{\{\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1\,\}_{nk}}{dt}\ \mathbf{e}_k(\mathbf{\Phi})$$

Let's try tacking the Frame S component of each side

$$-\,[\boldsymbol{\omega} \times \mathbf{e}_n(\mathbf{\Phi})]_i = \frac{\{\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1\,\}_{nk}}{dt}\ [\mathbf{e}_k(\mathbf{\Phi})]_i$$

$$-\,\varepsilon_{ijk}\omega_j[\mathbf{e}_n(\mathbf{\Phi})]_k = \frac{\{\, R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1\,\}_{nk}}{dt}\ [\mathbf{e}_k(\mathbf{\Phi})]_i$$

$$- \varepsilon_{ijk}\omega_j\delta_{nk} = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ \delta_{ki}$$

$$- \varepsilon_{ijn}\omega_j = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{ni}}{dt} \equiv C_{ni}$$

Then we find

$$- \varepsilon_{1j2}\omega_j = - \varepsilon_{132}\omega_3 = \omega_3 = C_{12}$$

$$- \varepsilon_{1j3}\omega_j = - \varepsilon_{123}\omega_2 = - \omega_2 = C_{13}$$

$$- \varepsilon_{2j3}\omega_j = - \varepsilon_{213}\omega_1 = \omega_1 = C_{23}$$

And then I have an answer to the problem in Frame S:

$$\omega_1 = C_{23}$$
$$\omega_2 = - C_{13}$$
$$\omega_3 = C_{12}$$

Now instead evaluate in Frame S'

$$- [\boldsymbol{\omega} \times \mathbf{e}_n(\mathbf{\Phi})]'_i = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ [\mathbf{e}_k(\mathbf{\Phi})]'_i$$

$$- \varepsilon_{ijk}\,\omega'_j\,(\mathbf{e}_n(\mathbf{\Phi}))'_k = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ [\mathbf{e}_k(\mathbf{\Phi})]'_i$$

$$- \varepsilon_{ijk}\,\omega'_j\,R(\Phi)_{kn} = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ R(\Phi)_{ik}$$

Apply $R^{-1}(\Phi)_{ns}$ to both sides of this equation

$$- \varepsilon_{ijk}\,\omega'_j\,R(\Phi)_{kn}R^{-1}(\Phi)_{ns} = R^{-1}(\Phi)_{ns}\frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ R(\Phi)_{ik}$$

or

$$- \varepsilon_{ijk}\,\omega'_j\,\delta_{ks} = R(\Phi)_{sn}\frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ R(\Phi)_{ik}$$

$$- \varepsilon_{ijs}\,\omega'_j = R(\Phi)_{sn}\frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}\ R^{-1}(\Phi)_{ki}$$

$$- \varepsilon_{ijs}\,\omega'_j = \frac{\{ R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})R^{-1}(\mathbf{\Phi}) - 1 \}_{nk}}{dt}$$

****

I am now reviewing once again the existing Section G.7 I quickly get to this point,

$$(d\mathbf{e}_n/dt)_{S'} = [(dR/dt)R^{-1}(\mathbf{\Phi})]\, \mathbf{e}_n$$

$$\equiv A\, \mathbf{e}_n \qquad\qquad A \equiv [(dR/dt)\ R^{-1}(\mathbf{\Phi})] \qquad\qquad (G.7.8)$$

At this point, A is really an operator. In Dirac we have

$$|\,(d\mathbf{e}_n/dt)_{S'}> = |A\ \mathbf{e}_n> = \boldsymbol{A}\,|\mathbf{e}_n> \;=\; |\mathbf{e}_i><\mathbf{e}_i|\boldsymbol{A}\,|\mathbf{e}_n> \;=\; A_{in}\ |\mathbf{e}_i>$$

OK, so now we can say

$$(d\mathbf{e}_n/dt)_{S'} \;=\; A_{in}\, \mathbf{e}_i$$

and now we have an actual matrix A_{in} sitting there. I show that

$$A \;=\; -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) \;+ (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)\,]$$

and then

$$A_{ij} = -[\,(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)\varepsilon_{1ij} \;+ (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot{\psi} + \dot{\varphi}\cos\theta)\varepsilon_{3ij}\,] \,. \qquad (G.7.16)$$