Euler Angle Picture Paradox 1 PhL 2.16.17

This doc is fully reviewed and I am happy with all the conclusions shown in red.

Despite 45 years of study, I am unable to understand this simple Goldstein picture. I see the picture, but I don't understand the picture. I don't understand what the drawings MEAN. I read Goldstein's words on page 107 for these pictures and they seem to make sense in that he is just rotating coordinate AXES.

Interpretation A: These pictures are showing a mapping of points on the white disk in the top picture. A point on this disk has coordinates (x,y,0). This point ends up on the tilted grey disk in the last picture and our point has coordinates (x',y',z') on this final disk.

As examples

the point $\hat{\mathbf{x}}$ in the first picture ends up at point $\hat{\mathbf{x}}'$ in the last picture.
the point $\hat{\mathbf{y}}$ in the first picture ends up at point $\hat{\mathbf{y}}'$ in the last picture.
the point $\hat{\mathbf{z}}$ in the first picture ends up at point $\hat{\mathbf{z}}'$ in the last picture.

I am unsure whether the above interpretation is correct. But I think it is correct at least for all the basis vectors which are shown in the pictures. I will continue just thinking about those basis vectors.

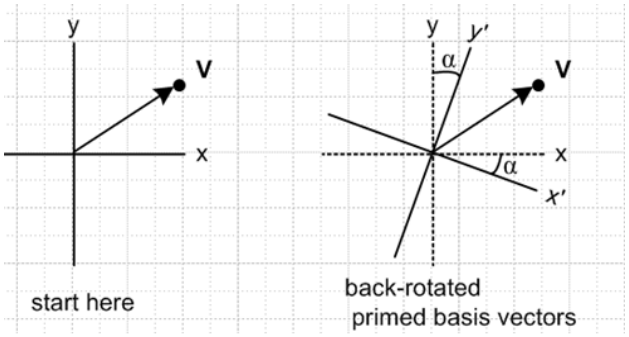
I think the above interpretation is wrong. In my latest picture I show a Kinematic Vector r, and this vector **r** is the same in all three Goldstein pictures. This point (particle location) is not being mapped at all in this Goldstein picture set. It is true that r has different coordinates in terms of the different basis vector sets. But the picture is really about Basis Vectors! It shows 4 different sets of basis vectors, and each set has a set of axes associated with it. The mappings like

$$(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_z(\varphi)\,(\hat{x},\hat{y},\hat{z}) \qquad \text{// each equation is like } \mathbf{e'_n} = R^{-1}\mathbf{e_n}$$

$$(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_\xi(\theta)\,R_z(\varphi)\,(\hat{x},\hat{y},\hat{z})$$

$$(\hat{x}',\hat{y}',\hat{z}') = R_{\zeta'}(\psi)\,(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_{\zeta'}(\psi)\,R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_{\zeta'}(\psi)\,R_\xi(\theta)\,R_z(\varphi)\,(\hat{x},\hat{y},\hat{z}) \quad . \qquad (G.5.7)$$

are all of the template form $\mathbf{e'_n} = R^{-1}\mathbf{e_n}$ and that is what the Goldstein picture is all about. The picture is intrinsically a Passive View picture and it is showing the basis vectors being back-rotated. I think the Fig (G.5.5) of Goldstein is like three applications of back-rotating basis vector sets, each analogous to :



The vector **V** stays put here, just as my **r** stays put in G.5.5. You don't really talk about active **r**' = R**r** as you would for a rotating apparatus in the Active View. So you are not mapping a point on the white disk in the top picture to a point on the tilted disk on the lower right. You could draw an Active Mode picture all with only the top picture showing how r becomes **r**' = R**r**.

Above I say

" A point on this disk has coordinates (x,y,0). This point "ends up" on the tilted grey disk in the last picture and our point has coordinates (x',y',z') on this final disk."

This is an Active View statement, a point "ends up" somewhere. What it really means is that the point (x,y,0) on the disk in Frame S "ends up" having *components* (x',y',z') in Frame S', but the point itself does not actively move anywhere. This is why I added the **r** vector to G.5.5, to make this fact clearer.

I show in G.5 that $\mathbf{e}'_i = R^{-1}\mathbf{e}_i$ and therefore **r**' = R**r** as usual where $R = R_z(-\psi)R_x(-\theta)R_z(-\varphi)$. So this R transform takes ANY point in x-space to its corresponding location in x'-space.

Suppose I take the vector $\mathbf{r} = r\hat{\mathbf{x}}$. I am claiming here that

$$\mathbf{r}' = R\mathbf{a} = r\ \{R_z(-\psi)R_x(-\theta)R_z(-\varphi)\hat{\mathbf{x}}\}$$

It is not very obvious that this is right, but I know that

$$R_{\zeta'}(\psi)\,R_\xi(\theta)\,R_z(\varphi) \;=\; R_z(\varphi)R_x(\theta)R_z(\psi)$$

and therefore

$$R = R_z(-\varphi)\,R_\xi(-\theta)\,R_{\zeta'}(-\psi)$$

and then I would have

$$\mathbf{r}' = R\mathbf{a} = r\ R_z(-\varphi)\,R_\xi(-\theta)\,R_{\zeta'}(-\psi)\hat{\mathbf{x}}$$

Boy! I just don't see this!

Plan A

Some of this is now in Section G.5

Can I trace the steps showing how this happens with each of these vectors????

First Picture:

$\hat{\boldsymbol{\xi}} = R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{x}}$ $\qquad\qquad$ $R_{\mathbf{z}}(\varphi) = \exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J})$

$\hat{\boldsymbol{\eta}} = R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{y}}$

$\hat{\boldsymbol{\zeta}} = R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{z}} = \hat{\mathbf{z}}$

Second Picture:

$\hat{\boldsymbol{\xi}}' = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}}$

$\hat{\boldsymbol{\eta}}' = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\hat{\boldsymbol{\eta}}$

$\hat{\boldsymbol{\zeta}}' = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\hat{\boldsymbol{\zeta}}$

Third Picture

$\hat{\mathbf{x}}' = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\hat{\boldsymbol{\xi}}' = \exp(-i\theta\hat{\boldsymbol{\zeta}}')\hat{\boldsymbol{\xi}}$

$\hat{\mathbf{y}}' = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\hat{\boldsymbol{\eta}}'$

$\hat{\mathbf{z}}' = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\,\hat{\boldsymbol{\zeta}}' = \hat{\boldsymbol{\zeta}}'$

So I think I end up with this general rule for how the three basis vectors transform

$$\mathbf{e}'_{\mathbf{i}} = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,R_{\mathbf{z}}(\varphi)\,\mathbf{e}_{\mathbf{i}}$$

This is NOT what I was at first expecting, and I am not sure what happens next!!

From the same pictures I can say this for sure:

$$\hat{\mathbf{z}}' = R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta)\,\hat{\mathbf{z}}$$

This then is also true,

$$\hat{\mathbf{z}}' = R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)\,\hat{\mathbf{z}}$$

and then I would conjecture that the general rule is

$$\mathbf{e}'_{\mathbf{i}} = R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)\;\mathbf{e}_{\mathbf{i}}$$

so we seem to have the implication that

$$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ R_z(\varphi) \;=\; R_z(\varphi)\ R_x(\theta)R_z(\psi)\ .$$ [turned out to be correct!]

From the last picture I think I can write

$$\hat{\xi} \;=\; \hat{\xi}' = \cos\psi\ \hat{\mathbf{x}}' - \sin\psi\ \ \hat{\mathbf{y}}'$$

$$\hat{\zeta}' = \hat{\mathbf{z}}'$$

$$\hat{\mathbf{z}} \;=\; \text{could do it but it is a messy picture to draw in 3D}$$

Somehow I think I am "doing this the hard way" ! Even with these expressions, you still have messy non-commuting things like

$$\mathbf{e}'_i \;=\; \exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta[\ \cos\psi\ \hat{\mathbf{x}}' - \sin\psi\ \ \hat{\mathbf{y}}']\bullet\mathbf{J})\ \exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J})\ \mathbf{e}_i$$

So go back to

$$\mathbf{e}'_i \;=\; \exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ \exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J})\ \mathbf{e}_i$$

Play with middle matrix

$$\hat{\xi}\bullet\mathbf{J} \;=\; [R_z(\varphi)\ \hat{\mathbf{x}}]\bullet\mathbf{J} \;=[\ \cos\varphi\ \hat{\mathbf{x}} + \sin\varphi\ \hat{\mathbf{y}}\ ]\bullet\mathbf{J} \;=\cos\varphi\ J_1 + \sin\varphi\ J_2$$

so

$$\exp(-i\theta\hat{\xi}\bullet\mathbf{J}) = \exp(-i\theta[\ \cos\varphi\ J_1 + \sin\varphi\ J_2])$$

Same idea, you end up with an ugly non-commuting monster. [all cleared up in section G.5]

Plan B

Here I don't yet know about Appendix G.5 Theorem 3 so am flailing around.

Suppose I start with last picture and work backwards.

Third picture

$$\hat{\mathbf{x}}' \;= R_{z'}(-\psi)\ \hat{\xi}$$
$$\hat{\mathbf{y}}' \;=\; R_{z'}(-\psi)\ \hat{\eta}$$
$$\hat{\zeta}' = R_{z'}(-\psi)\hat{\mathbf{z}}' = \hat{\mathbf{z}}'$$

Second picture

rotations still about weird axes, so not any simpler!

G says to apply BCD to $\hat{\mathbf{z}}$ to get $\hat{\mathbf{z}}'$ I *think* on page 134. In MY matrix notation that would say

$$R_z(-\psi)R_x(-\theta)R_z(-\varphi)\,\hat{\mathbf{z}} = \hat{\mathbf{z}}'$$

In general then I think he is claiming that

$$\mathbf{e}'_n = R_z(-\psi)R_x(-\theta)R_z(-\varphi)\,\mathbf{e}_i \qquad \text{// what G implies, but my } R_i \text{ notation}$$

Compare this to what I got above

$$\mathbf{e}'_n = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,R_z(\varphi)\,\mathbf{e}_n$$

This would imply that

$$\exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,R_z(\varphi) = R_z(-\psi)R_x(-\theta)R_z(-\varphi)$$

I don't like what happens on the right side, a mismatch.

But maybe we have the reversal idea and he is really saying that

$$\mathbf{e}'_n = [R_z(-\psi)R_x(-\theta)R_z(-\varphi)]^{-1}\,\mathbf{e}_i = R_z(\varphi)R_x(\theta)R_z(\psi)\,\mathbf{e}_i$$

In that case I have to show that

$$\exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,R_z(\varphi) = R_z(\varphi)R_x(\theta)R_z(\psi)$$

Is it possible that this is true?

Big Question:

Where does the following matrix **FIT** into the above picture ???

$$BCD = R_z(-\psi)\,R_x(-\theta)\,R_z(-\varphi) = \exp(-i\psi\hat{\mathbf{z}}\bullet\mathbf{J})\;\exp(-i\theta\hat{\mathbf{x}}\bullet\mathbf{J})\,\exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J})$$

[answer: it is the R of (G.5.17a)]

What I have above is this

$$\mathbf{e}'_i = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J})\,\mathbf{e}_i$$

which seems to have NO connection whatsoever!

So this is Paradox #1 and I find it hard to move in any direction until this is resolved.

Is it possible that this is true,

$$\exp(-i\theta\hat{\mathbf{x}}\bullet\mathbf{J}) = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})$$

NO, this is NOT true.

What if $\theta = \varphi = 0$. Then I have

$$\mathbf{e}'_i = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})$$

$$\hat{\boldsymbol{\zeta}}' = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\hat{\boldsymbol{\zeta}} = \hat{\boldsymbol{\zeta}} = \hat{\mathbf{z}}$$

Then I have

$$\mathbf{e}'_i = \exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\mathbf{e}_i = \exp(-i\psi\hat{\mathbf{z}}\bullet\mathbf{J})\mathbf{e}_i$$

and in this special case this is the same result you get from

$$\mathbf{e}'_i = R_{\mathbf{z}}(-\psi)\,R_{\mathbf{x}}(-\theta)\,R_{\mathbf{z}}(-\varphi)\,\mathbf{e}_i$$

What if $\theta = \psi = 0$? They trivially agree in this case.

In my Appendix G I would write (to be G compatible),

$$\mathbf{e}_n = R\,\mathbf{e}'_n \qquad R = R_{\mathbf{z}}(\psi)\,R_{\mathbf{x}}(\theta)\,R_{\mathbf{z}}(\varphi)$$

and therefore

$$\mathbf{e}'_n = R^{-1}\,\mathbf{e}_n \qquad R^{-1} = R_{\mathbf{z}}(-\varphi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\psi)$$

So I think I am trying to show that

$$\exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J}) = R_{\mathbf{z}}(-\varphi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\psi)$$

This could conceivably be true I suppose.

What if only $\theta = 0$? Can I show that

$$\exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})\exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J}) = R_{\mathbf{z}}(-\varphi)R_{\mathbf{z}}(-\psi)$$

In this case we have

$$\hat{\boldsymbol{\zeta}}' = \exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\,\hat{\boldsymbol{\zeta}} = \hat{\boldsymbol{\zeta}} = \hat{\mathbf{z}}$$

and YES in this case it IS true.

What if only $\psi = 0$. Can I show that

$$\exp(-i\theta\hat{\boldsymbol{\xi}}\bullet\mathbf{J})\exp(-i\varphi\hat{\mathbf{z}}\bullet\mathbf{J}) = R_z(-\varphi)R_x(-\theta) \quad ?$$

In this case.

$$\hat{\boldsymbol{\xi}} = R_z(\varphi)\,\hat{\mathbf{x}}$$

So I would have to show that

$$\exp(-i\theta\,[R_z(\varphi)\,\hat{\mathbf{x}}]\bullet\mathbf{J})\,R_z(\varphi) = R_z(-\varphi)R_x(-\theta)$$

or

$$R_z(-\varphi)\exp(-i\theta\,[R_z(\varphi)\,\hat{\mathbf{x}}]\bullet\mathbf{J})\,R_z(\varphi) = R_x(-\theta)$$

This is a sandwich rule I have not worked with! I don't think I have ever pondered such a thing. I could have Maple run a quick check on this to see if it is way off base. I know that

$$[R_z(\varphi)\,\hat{\mathbf{x}}]\bullet\mathbf{J} = \cos\varphi\, J_1 + \sin\varphi\, J_2$$

Then I need to show that

$$R_z(-\varphi)\exp(-i\theta\,[\cos\varphi\, J_1 + \sin\varphi\, J_2]\,)\,R_z(\varphi) = R_x(-\theta)$$

This just looks impossible. Go back and write as

$$R_z(-\varphi)\exp(-i\theta\,\hat{\mathbf{x}}\bullet[R_z(-\varphi)\mathbf{J}])\,R_z(\varphi) = R_x(-\theta)$$

Just the same messy thing.

OK, what can I say in general about

$$R^{-1}\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\,R$$

where R is some rotation? I talk about this type of thing in "confusion about rotation... doc".

Here is a **conjectured theorem**: [this is now Theorem 2 of (G.4.7)]

$$R\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\,R^{-1} = \exp(-i\theta[R\mathbf{n}]\bullet\mathbf{J})$$

What else could the result be??

I do know this much

$$\exp(-i\theta \mathbf{n}\bullet\mathbf{J}) \;=\; 1 + \;(\cos\theta - 1)\, T \;+ \sin\theta[\, -i(\mathbf{n}\bullet\mathbf{J})]$$

and therefore

$$R\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1} \;= 1 + \;(\cos\theta - 1)RTR^{-1} \;\; -i\sin\theta\, R\,(\mathbf{n}\bullet\mathbf{J})R^{-1}$$

which is at least something one can ponder. $T_{ab} = (\delta_{ab} - n_a n_b)$. then

$$[RTR^{-1}]_{ad} \;= R_{ab}T_{bc}R^{T}_{cd} \;= \delta_{ad} - \;R_{ab}n_b n_c R_{dc} = \delta_{ad} \; - (R\mathbf{n})_a(R\mathbf{n})_d \;= \delta_{ad} - n'_a n'_d = T'_{ad}$$

Next

$$R\,(\mathbf{n}\bullet\mathbf{J})R^{-1} = n_i(RJ_iR^{-1})$$

I have a feeling that this might be true

$$RJ_iR^{-1} = R^{-1}{}_{ij}J_j \qquad\qquad [\text{ this is now Theorem 1 of (G.4.2) }]$$

If so, then we have

$$R\,(\mathbf{n}\bullet\mathbf{J})R^{-1} = n_i(RJ_iR^{-1}) = \; n_i\, R^{-1}{}_{ij}J_j = n_i\, R_{ji}J_j \; = J_j\,[R_{ji}n_i] = \; J_j\,[R\mathbf{n}]_j \; = J_j\, n'_j = (\mathbf{n}'\bullet\, J)$$

Then we have

$$R\exp(-i\theta \mathbf{n}\bullet\mathbf{J})\, R^{-1} \;= 1 + \;(\cos\theta - 1)RTR^{-1} \;\; -i\sin\theta\, R\,(\mathbf{n}\bullet\mathbf{J})R^{-1}$$

$$= \; 1 + \;\; (\cos\theta - 1)\; T' \;\; - \;\; -i\sin\theta \;\; (\mathbf{n}'\bullet\, \mathbf{J})$$

$$= \exp(-i\theta \mathbf{n}'\bullet\mathbf{J})$$

and this more or less proves the conjectured theorem!!! [yup]

Let's look at the might be true conjecture

$$RJ_iR^{-1} = R^{-1}{}_{ij}J_j \qquad\qquad R\mathbf{J}R^{-1} = [R^{-1}\mathbf{J}]$$

Do a quick preliminary check

$$\det(RJ_iR^{-1}) \;= \det(J_i) = 0$$

$$\det(R^{-1}{}_{ij}J_j) = R^{-1}{}_{ij}\,\det(J_j) = 0 \qquad\qquad \text{OK}$$

Sandwich rule please! [these are now in section G.3]

$$\exp(-i\theta \mathbf{n}\bullet \mathbf{J})\ J_k \exp(+i\theta \mathbf{n}\bullet \mathbf{J}) = \cos\theta J_k + \sin\theta[\mathbf{J}\times\mathbf{n}]_k + (1-\cos\theta)\, n_k (\mathbf{n}\bullet \mathbf{J})$$

Then I am claiming *yet another* new fact, namely

$$\cos\theta J_i + \sin\theta[\mathbf{J}\times\mathbf{n}]_i + (1-\cos\theta)\, n_i (\mathbf{n}\bullet \mathbf{J}) = R^{-1}{}_{ij} J_j$$

Write out both sides in components

$$\cos\theta J_i + \sin\theta \varepsilon_{ijk} n_k J_j + (1-\cos\theta)\, n_i n_j J_j = R^{-1}{}_{ij} J_j \quad ?$$

$$\delta_{ij}\cos\theta J_j + \sin\theta \varepsilon_{ijk} n_k J_j + (1-\cos\theta)\, n_i n_j J_j = R^{-1}{}_{ij} J_j \quad ?$$

$$\delta_{ij}\cos\theta + \sin\theta \varepsilon_{ijk} n_k + (1-\cos\theta)\, n_i n_j = R^{-1}{}_{ij} \quad ?$$

Is this possibly true? We know that

$$\exp(i\theta \mathbf{n}\bullet\mathbf{J}) = 1 + (\cos\theta - 1)\, T + \sin\theta[\, i(\mathbf{n}\bullet\mathbf{J})] = R^{-1}$$

Then

$$R^{-1}{}_{ij} = \delta_{ij} + (\cos\theta - 1)\, T_{ij} + i \sin\theta\, n_k (J_k)_{ij}$$

$$R^{-1}{}_{ij} = \delta_{ij} + (\cos\theta - 1)(\delta_{ij} - n_i n_j) + \sin\theta\, n_k (i J_k)_{ij}$$

$$R^{-1}{}_{ij} = \delta_{ij} + (\cos\theta - 1)(\delta_{ij} - n_i n_j) + \sin\theta\, n_k \varepsilon_{kji}$$

$$R^{-1}{}_{ij} = \delta_{ij}\cos\theta - (\cos\theta - 1)\, n_i n_j + \sin\theta\, n_k \varepsilon_{kji}$$

How does a tensor transform?

$$M'_{ab} = R_{aa'} R_{bb'} M_{a'b'} \qquad \text{// contravariant rank-2 tensor}$$

$$M' = R\, M\, R^{T} \qquad \text{tensor doc (5.7.1)}$$

$$M' = R\, M\, R^{-1} \qquad \text{for rotation}$$

But there are no generators here, it is just a matrix. So this does not really help.

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[Here now is my proof of Theorem 3, first written down]

I have now written up two powerful matrix theorems in Appendix G.4. Let's now return to our conjecture

$$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ R_z(\varphi) \;=\; R_z(\varphi)R_x(\theta)R_z(\psi) \qquad ??$$

where we know that

$$\hat{\zeta}' = \exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ \hat{\zeta} \qquad = \exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ R_z(\varphi)\ \hat{\mathbf{z}}$$

$$\hat{\xi} = R_z(\varphi)\ \hat{\mathbf{x}}$$

$$\hat{\zeta} = R_z(\varphi)\ \hat{\mathbf{z}}\ .$$

Let's try to apply Theorem **2**

Theorem 2: $R\exp(-i\theta\mathbf{n}\bullet\mathbf{J})\ R^{-1} = \exp(-i\theta\mathbf{n}'\bullet\mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$ (G.4.7)

Consider $\hat{\xi} = R_z(\varphi)\ \hat{\mathbf{x}}$ as being $\mathbf{n}' = R\mathbf{n}$. Then the theorem says

$$R_z(\varphi)\exp(-i\theta\hat{\mathbf{x}}\bullet\mathbf{J})\ R_z(-\varphi) = \exp(-i\theta\hat{\xi}\bullet\mathbf{J})$$

or

$$R_z(\varphi)\ R_x(\theta)\ R_z(-\varphi) = \exp(-i\theta\hat{\xi}\bullet\mathbf{J}) \qquad \text{// at least it's interesting....}$$

and this is indeed one of our matrices of interest. Let's shoot this in and see what we get

$$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta\hat{\xi}\bullet\mathbf{J})\ R_z(\varphi) \;=\; R_z(\varphi)R_x(\theta)R_z(\psi) \qquad ??$$

$$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})[\ R_z(\varphi)\ R_x(\theta)\ R_z(-\varphi)]\ R_z(\varphi) \;=\; R_z(\varphi)R_x(\theta)R_z(\psi) \qquad ??$$

$$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})R_z(\varphi)\ R_x(\theta) \;=\; R_z(\varphi)R_x(\theta)R_z(\psi) \qquad ??$$

You would have to show that

$$R_z(-\varphi)\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})R_z(\varphi)\ R_x(\theta) = R_x(\theta)R_z(\psi) \qquad ??$$

or

$$R_x(-\theta)R_z(-\varphi)\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})R_z(\varphi)R_x(\theta) = R_z(\psi) \qquad ??$$

We do have these facts from above

$\hat{\zeta}' = \exp(-i\theta\hat{\xi}\bullet\mathbf{J})\,\hat{\zeta} = \exp(-i\theta\hat{\xi}\bullet\mathbf{J})\,\hat{\mathbf{z}}$

or

$\hat{\zeta}' = R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi)\hat{\zeta} = R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi)\hat{\mathbf{z}}$

$\mathbf{n}' = R\,\mathbf{n}$ $\quad\quad$ $\mathbf{n}' = \hat{\zeta}'$ $\quad\quad$ $R = R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi)$ $\quad$ $\mathbf{n} = \hat{\mathbf{z}}$

Theorem 2 then says,

$R\exp(-i\psi\mathbf{n}\bullet\mathbf{J})\,R^{-1} = \exp(-i\psi\mathbf{n}'\bullet\mathbf{J})$ $\quad$ where $\mathbf{n}' = R\mathbf{n}$

$R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi)\exp(-i\psi\hat{\mathbf{z}}\bullet\mathbf{J})\,R_z(\varphi)R_x(-\theta)R_z(-\varphi) = \exp(-i\psi\,\hat{\zeta}'\bullet\mathbf{J})$

or

$R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi)R_z(\psi)R_z(\varphi)R_x(-\theta)R_z(-\varphi) = \exp(-i\psi\,\hat{\zeta}'\bullet\mathbf{J})$

or

$R_z(\varphi)\,R_x(\theta)R_z(\psi)R_x(-\theta)R_z(-\varphi) = \exp(-i\psi\,\hat{\zeta}'\bullet\mathbf{J})$ $\quad\quad$ // result of Thm 2

OK, now here is our conjecture

$R_x(-\theta)R_z(-\varphi)\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})R_z(\varphi)R_x(\theta) = R_z(\psi)$ $\quad\quad$??

So to test this we insert our just-found result

$R_x(-\theta)R_z(-\varphi)R_z(\varphi)\,R_x(\theta)R_z(\psi)R_x(-\theta)R_z(-\varphi)R_z(\varphi)R_x(\theta) = R_z(\psi)$ $\quad\quad$??

$R_x(-\theta)R_x(\theta)R_z(\psi)R_x(-\theta)R_x(\theta) = R_z(\psi)$ $\quad\quad$??

$R_z(\psi) = R_z(\psi)$ $\quad\quad$?? $\quad\quad$ YES!!~!!!!!!!!

So I have these interesting results along the way

$R_z(\varphi)\,R_x(\theta)\,R_z(-\varphi) = \exp(-i\theta\hat{\xi}\bullet\mathbf{J}) = R_\xi(\theta)$

$R_z(\varphi)\,R_x(\theta)R_z(\psi)R_x(-\theta)R_z(-\varphi) = \exp(-i\psi\,\hat{\zeta}'\bullet\mathbf{J}) = R_{\hat{\zeta}'}(\psi)$

These have an interpretation that I think is simple, but I will have to fiddle.

I have finally proven that

$\exp(-i\psi\hat{\zeta}'\bullet\mathbf{J})\exp(-i\theta\hat{\xi}\bullet\mathbf{J})\,R_z(\varphi) = R_z(\varphi)R_x(\theta)R_z(\psi)$ $\quad\quad$ [proof completed]

[ignore everything below, just ranting and raving, I do something wrong below, don't care now]

RESUME HERE!!!

This is not looking too good. Suppose it is true that

$$\hat{\mathbf{z}}' = \hat{\boldsymbol{\zeta}}' = R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)\,\hat{\mathbf{z}} = R\,\hat{\mathbf{z}}$$

What then does Theorem 2 say?

Theorem 2: $R\exp(-i\psi\mathbf{n}\bullet\mathbf{J})\,R^{-1} = \exp(-i\psi\mathbf{n}'\bullet\mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$ (G.4.7)

Consider $\hat{\boldsymbol{\zeta}}' = R\,\hat{\mathbf{z}}$ as $\mathbf{n}' = R\mathbf{n}$

Then

$$R\exp(-i\psi\hat{\mathbf{z}}\bullet\mathbf{J})\,R^{-1} = \exp(-i\psi\,\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})$$

or

$$R\,R_{\mathbf{x}}(\psi)\,R^{-1} = \exp(-i\psi\,\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})$$

or

$$R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)R_{\mathbf{x}}(\psi)\,R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi) = \exp(-i\psi\,\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})$$

Then we have

$$R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)\exp(-i\psi\hat{\boldsymbol{\zeta}}'\bullet\mathbf{J})R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta) = R_{\mathbf{z}}(\psi) \quad ?$$

or

$$R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)R_{\mathbf{x}}(\psi)\,R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta) = R_{\mathbf{z}}(\psi)$$

or

$$R_{\mathbf{z}}(\psi)R_{\mathbf{x}}(\psi)\,R_{\mathbf{z}}(-\psi) = R_{\mathbf{z}}(\psi) \quad ?$$

$R_{\mathbf{x}}(\psi)\,R_{\mathbf{z}}(-\psi) = 1$?? close.... maybe I goofed above with a subscript.

Oy!

$$R_{\mathbf{z}}(-\varphi)\,R_{\mathbf{x}}(-\theta)\,R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(\psi)\,R_{\mathbf{z}}(\psi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\varphi)\;R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta) = R_{\mathbf{z}}(-\varphi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\psi)\;\text{??}$$

I think I see some new Theorems that should be written down maybe. Or special cases of Theorem 2:

Theorem 2: $R\exp(-i\psi\mathbf{n}\bullet\mathbf{J})\,R^{-1} = \exp(-i\psi\mathbf{n}'\bullet\mathbf{J})$ where $\mathbf{n}' = R\mathbf{n}$ (G.4.7)

Suppose $\mathbf{n} = \hat{\mathbf{x}}$ and $R = R_{\mathbf{z}}(\varphi)$ just as an example. Then

$\mathbf{n}' = R\mathbf{n} = R_z(\varphi)\,\hat{\mathbf{x}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$ I think

Then we get

$$R_z(\varphi)\exp(-i\psi\hat{\mathbf{x}}\bullet\mathbf{J})\,R_z(-\varphi) = \exp(-i\psi[\cos\varphi\, J_1 + \sin\varphi J_2)$$

or

$$R_z(\varphi)R_x(\psi)\,R_z(-\varphi) = \exp(-i\psi[\cos\varphi\, J_1 + \sin\varphi J_2))$$

which is really weird but I think correct. Things like this might be useful. I think for example earlier above I had

$$\hat{\xi} = \hat{\xi}' = \cos\psi\,\hat{\mathbf{x}}' - \sin\psi\,\hat{\mathbf{y}}'$$

But these are primed unit vectors, I don't know how to do that yet.