Euler unit vector exercise PhL 3.14.17

After looking at the Euler Angle unit vectors, I show (Question 3) that

$(L)_z = p_\varphi$
$(L)'_z = p_\psi$

which is something Goldstein claims. This is installed now in Appendix H.

Based on (E.5.3), I think the following equations are valid

$\hat{\boldsymbol{\varphi}} = -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}}$

$\hat{\boldsymbol{\psi}} = -\sin\psi\ \hat{\mathbf{x}}' + \cos\psi\ \hat{\mathbf{y}}'$

$\hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{\mathbf{z}} + \cos\theta\ \hat{\mathbf{y}}$

These unit vectors and equations do not appear anywhere in my Section G.5 (for some reason). Probably yes, but they are never really used as a basis by me.

Question 1: do these vectors form a basis for Frame S or for Frame S' ? **Probably yes.**

I could write them all out in terms of $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ or $(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}')$.

Question 2: Do $\hat{\mathbf{r}},\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ form a basis for E^3 ? Well, basically these are the tangent base vectors which I write there as $\mathbf{e}_n$ apart from scaling factors, and the $\mathbf{e}_n$ do form basis for x-space. **Yes**.

Question 3: In the top analysis, can I make these claims ?

L_φ = constant because φ is cyclic

L_ψ = constant because ψ is cyclic

Using the above dot products I could write for example,

$L_\psi = \mathbf{L} \bullet \hat{\boldsymbol{\psi}} = -\sin\psi\ (L)'_x + \cos\psi\ (L)'_y$

using body frame Frame S' evaluation. Is this related to Goldstein's

$p_\psi = \partial \mathcal{L}/\partial\dot{\psi} = (\dot{\varphi}\cos\theta + \dot{\psi})I'_3 \equiv aI'_1 = (\omega)'_3 I'_3$?

Recall that $\mathbf{L} = I\boldsymbol{\omega}$ so for sure,

$(L)'_i = (I)'_{ij}(\omega)'_j = I'_i(\omega)'_i$

Then

$(L)'_x = (L)'_1 = I'_1(\omega)'_1$

$(L)'_y = (L)'_2 = I'_2(\omega)'_2$

Now recall that

$(\omega)'_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$

$(\omega)'_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$

$(\omega)'_3 = \dot{\varphi}\cos\theta + \dot{\psi}$ // Frame S' *(G.6.11)*

So I then know thaqt

$(L)'_x = (L)'_1 = I'_1(\omega)'_1 = I'_1 [\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]$

$(L)'_y = (L)'_2 = I'_2(\omega)'_2 = I'_2 [\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]$

Then

$L_\psi = -\sin\psi\,(L)'_x + \cos\psi\,(L)'_y$

$= -\sin\psi [\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi] I'_1 + \cos\psi [\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi] I'_2$

But for the top we have $I'_2 = I'_1$ so write this as

$L_\psi = -\sin\psi [\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi] I'_1 + \cos\psi [\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi] I'_1$

$= \dot{\varphi} I'_1 [-\sin\theta\sin^2\psi + \sin\theta\cos^2\psi] + \dot{\theta} I'_1 [-\sin\psi\cos\psi - \cos\psi\sin\psi]$

Compare this to

$p_\psi = \partial\mathcal{L}/\partial\dot{\psi} = (\dot{\varphi}\cos\theta + \dot{\psi})I'_3$

I don't see any connection!!! So it looks like my conjecture that $p_\psi = L_\psi$ is no good.

Go back and review what has happened here. I wrote

$L_\psi = \mathbf{L} \bullet \hat{\boldsymbol{\psi}} = -\sin\psi\,(L)'_x + \cos\psi\,(L)'_y$

$= \dot{\varphi}\, I'_1 [\, -\sin\theta \sin^2\psi + \sin\theta \cos^2\psi \,] + \dot{\theta} I'_1 [\, -\sin\psi \cos\psi - \cos\psi \sin\psi \,]$

It seems more that, associated with angle ψ, we should have $(L)'_z$ being a constant of the motion, and not the quantity L_ψ which really is orthogonal to the $\hat{z}'$ axis. You look for the axis about which your angle is a rotation. So now I think:

$p_\psi = (L)'_z$???

OK, write this out

$(L)'_z = (L)'_3 = I'_3 (\omega')_3$ and yes this = p_ψ exactly.

Next conjecture that

$(L)_z$ is associated with φ, so should have $(L)_z$ = constant. Then

$(L)_z = \mathbf{L} \bullet \hat{\mathbf{z}}$

Now how can I relate $\hat{\mathbf{z}}$ to the body frame basis vectors? I have failed to write this down anywhere in G.5. // But I have now added this and the result is

$\hat{\mathbf{z}} = (\sin\psi\sin\theta)\hat{\mathbf{x}}' + (\cos\psi\sin\theta)\hat{\mathbf{y}}' + (\cos\theta)\,\hat{\mathbf{z}}'$

Therefore

$(L)_z = \mathbf{L} \bullet \{ (\sin\psi\sin\theta)\hat{\mathbf{x}}' + (\cos\psi\sin\theta)\hat{\mathbf{y}}' + (\cos\theta)\,\hat{\mathbf{z}}' \}$

$= \sin\psi\sin\theta\,(L)'_1 + \cos\psi\sin\theta\,(L)'_2 + \cos\theta\,(L)'_3$

$= \sin\psi\sin\theta\, I'_1(\omega)'_1 + \cos\psi\sin\theta\, I'_2(\omega)'_2 + \cos\theta\, I'_3(\omega)'_3$

$= \sin\psi\sin\theta\, I'_1[\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi] + \cos\psi\sin\theta\, I'_2[\,\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi] + \cos\theta\, I'_3[\,\dot{\varphi}\cos\theta + \dot{\psi}\,]$

$= \sin\psi\sin\theta\, I'_1[\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi] + \cos\psi\sin\theta\, I'_1[\,\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi] + \cos\theta\, I'_3[\,\dot{\varphi}\cos\theta + \dot{\psi}\,]$

$= \sin\theta\, I'_1 \{\, \dot{\varphi}\sin\theta\sin^2\psi + \dot{\theta}\cos\psi\sin\psi \,\} + \sin\theta\, I'_1 \{\, \dot{\varphi}\sin\theta\cos^2\psi - \dot{\theta}\sin\psi\cos\psi)$
$+ \cos\theta\, I'_3[\,\dot{\varphi}\cos\theta + \dot{\psi}\,]$

$= \sin\theta\, I'_1 \{\, \dot{\varphi}\sin\theta \,\} + \cos\theta\, I'_3[\,\dot{\varphi}\cos\theta + \dot{\psi}\,]$

$= \sin^2\theta\dot{\varphi} I'_1 + \cos^2\theta I'_3\dot{\varphi} + \cos\theta\, I'_3\dot{\psi}$

$$= [\sin^2\theta I'_1 + \cos^2\theta I'_3]\,\dot{\phi} + \cos\theta\, I'_3\dot{\psi}$$

Compare this with

$$p_\phi = (\sin^2\theta\, I'_1 + \cos^2\theta\, I'_3)\dot{\phi} + \cos\theta\, I'_3\dot{\psi}$$

So I am now sure that

$$(L)_z = p_\phi$$
$$(L)'_z = p_\psi$$

Clarity keeps improving !