A rotation matrix problem v2 PhL 2.12.17

This is where I do my first attempt at computing Goldstein's ω in Frame S' . I got the right answer after some fiddling around.

Here I am doing a bit of a rewrite on the previous version. First, note that :

Backwards Rule. Consider: for some vector **v**,

$$(v)'_n = \mathbf{v} \bullet \mathbf{e}'_n = \mathbf{v} \bullet [R\mathbf{e}_n] = [R^{-1}\mathbf{v}] \bullet \mathbf{e}_n = [R^{-1}\mathbf{v}]_n = (R^{-1})_{ni}\,(v)_i$$

This is my "famous fact" that if you rotate the basis vectors one way, you have to rotate the components of a [kinematic] vector the other way. In frames doc I start with $\mathbf{e}_n = R\,\mathbf{e}'_n$ which says $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$ so there I rotate the basis vectors backwards, and I then end up with

$$(v)'_n = \mathbf{v} \bullet \mathbf{e}'_n = \mathbf{v} \bullet [R^{-1}\mathbf{e}_n] = [R\mathbf{v}] \bullet \mathbf{e}_n = [R\mathbf{v}]_n = (R)_{ni}\,(v)_i$$

and so $\mathbf{v}' = R\mathbf{v}$ gives the components of **v** in Frame S'. An example would be $\boldsymbol{\omega}' = R\boldsymbol{\omega}$. [ok]

Question: Recall in frames doc that I show that

$$\mathbf{a}(t+dt) = R(d\boldsymbol{\varphi})\,\mathbf{a}(t) \qquad (1.5.7)$$

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \qquad (1.5.8)$$

Is this valid for both regular vectors and for basis vectors? [a reasonable question] In the first line above **a**(t) I think is ANY vector you want, so yes it could be a basis vector like $\mathbf{e}_n$ or $\mathbf{e}'_n$ or a regular vector like **v** . [yes]

Statement of the Problem: Suppose you have the following, *consistent* as noted above with frames doc,

$$\mathbf{e}'_i = R^{-1}(\boldsymbol{\Phi})\mathbf{e}_i = R(-\boldsymbol{\Phi})\mathbf{e}_i \qquad \text{where } R(\boldsymbol{\Phi}) = R_z(\psi)\,R_x(\theta)\,R_z(\varphi)$$

$$R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)R_z(-\psi)$$

$$\mathbf{e}_i = R(\boldsymbol{\Phi})\mathbf{e}'_i$$

At this point, $R_i(\alpha)$ could refer to either my active convention or Goldstein's passive convention. Only when you write out specific matrices do you have to choose!

Here $\mathbf{e}_i$ are the usual Cartesian unit vectors of Frame S and $\mathbf{e}'_i$ are some unit vectors of Frame S'.

In frames doc I wrote $R(\boldsymbol{\varphi}) = \exp(-i\,\boldsymbol{\varphi} \bullet \mathbf{J})$ but here I am writing $R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi} \bullet \mathbf{J})$ to not overload the symbol φ. [here **Φ** is meant to be a completely arbitrary rotation]

Following the path of frames doc, we know that for any vector **a**

$$d\mathbf{a} = d\mathbf{\Phi} \times \mathbf{a} \qquad (1.5.8)$$

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} \quad \text{where } \boldsymbol{\omega} \equiv d\mathbf{\Phi}/dt \quad . \qquad (1.6.1)$$

$$(d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e'}_n \quad . \quad \text{// selecting } \mathbf{a} = \mathbf{e'}_n \qquad (1.7.1)$$

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \quad . \quad \text{// "the other one"} \qquad (1.7.4)$$

The Problem is to find an expression for $\boldsymbol{\omega}$ which involves $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$. Hopefully this expression will look something like Goldstein page 134 (4-103).

• Note that $\mathbf{\Phi} \neq (\psi,\theta,\varphi)$ or anything simple like that. [ie, these are not the Cartesian comps of $\mathbf{\Phi}$]

• A separate Problem would be to compute the function $\mathbf{\Phi}(\psi,\theta,\varphi)$ which I am not doing right now.
[Here I must have been thinking of $\mathbf{\Phi}$ as indicating an Euler angle rotation with those params.]

Solution: I am not sure how to start, so I will try several Plans.

[I think this is the very first time I went down this path]

$\mathbf{e}_n = R(\mathbf{\Phi})\, \mathbf{e'}_n$ [This is my standard Section 1.1 formalism R]

$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'}_n - R(\mathbf{\Phi})\mathbf{e'}_n$

$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi)\, \mathbf{e'}_n - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\, \mathbf{e'}_n$

[above I just made the assumption that $R(\mathbf{\Phi}) = R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$. If I instead use (G.5.19) for R, then all three angles would be negated and that would make the result come out exactly right!]

$= [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,]\mathbf{e'}_n$

$= dR\, \mathbf{e'}_n\,, \qquad dR \equiv [\, R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)\,]$

Now process dR' as follows

$dR = [\, R_z(\psi)R_z(d\psi)\, R_x(\theta)\, R_x(d\theta)R_z(\varphi)R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$

$= [\, R_z(\psi)\{1-id\psi J_3\}\, R_x(\theta)\{1-id\theta J_1\}R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]$

$\approx [\, -id\psi\ R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) - id\theta\ R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]$

Then to first order,

$(dR/dt)_{S'} = [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta)R_z(\varphi) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3\,]\ .$

Claim: I think the matrix elements of R have time derivatives which do not need S or S' Frame label, recall that any component of any tensor has this property of not needing a specifier. But any symbol representing a tensor without components *does* need a specifier if you try to take a time derivative

$$d\mathbf{e}_n = (dR')\,\mathbf{e}'_n$$

$$(d\mathbf{e}_n/dt)_{S'} = (dR'/dt)_{S'}\mathbf{e}'_n$$

This last equation is true because in Frame S' the vector $\mathbf{e}'_n$ is a constant. We need the S' specifier on the both left and right since $\mathbf{e}_n$ is a rank-1 tensor and dR' is a rank-2 tensor. Now replace $\mathbf{e}'_n = R^{-1}(\mathbf{\Phi})\mathbf{e}_n$ in *** to get

$$(d\mathbf{e}_n/dt)_{S'} = (dR/dt)_{S'}\mathbf{e}'_n = (dR/dt)_{S'}[R^{-1}(\mathbf{\Phi})\mathbf{e}_n]$$

$$= [\,(dR/dt)_{S'}[R^{-1}(\mathbf{\Phi})\,]\,\mathbf{e}_n$$

$$\equiv A'\,\mathbf{e}_n \qquad\qquad A' = (A)_{S'} = [\,(dR/dt)_{S'}[R^{-1}(\mathbf{\Phi})\,]$$

Since A' is a rank-2 tensor which involves time derivatives within Frame S', it really should have an S' specifier. We abbreviate this by writing it as A'.

Now we want to compute the matrix A'.

Commutator: Define this operation for square matrices X and Y,

$$[X,Y] \equiv XY-YX.$$

Then consider ,

$$[\,R_i(\alpha)\,,\,J_i\,] = [\,\exp(-i\alpha J_i)\,,\,J_i\,] = 0 \qquad \text{since } [J_i,J_i] = 0$$

Therefore,

$$R_i(\alpha)\,J_i\,R_i(-\alpha) = R_i(\alpha)\,R_i(-\alpha)J_i = 1 * J_i = J_i$$

Calculate the matrix A

$$A' = (dR/dt)_{S'}R^{-1}(\mathbf{\Phi})$$

$$= [\,-i\dot{\psi}\;R_z(\psi)J_3\,R_x(\theta)R_z(\varphi) \;-\; i\dot{\theta}\;R_z(\psi)\,R_x(\theta)J_1R_z(\varphi) \;-\; i\dot{\varphi}\,R_z(\psi)\,R_x(\theta)R_z(\varphi)J_3\,]\,R_z(-\varphi)\,R_x(-\theta)\,R_z(-\psi)$$

$$= [\,-i\dot{\psi}\;R_z(\psi)J_3\,R_x(\theta) \;-\; i\dot{\theta}\;R_z(\psi)\,R_x(\theta)J_1 \;-\; i\dot{\varphi}\,R_z(\psi)\,R_x(\theta)R_z(\varphi)J_3\,R_z(-\varphi)]\,R_x(-\theta)\,R_z(-\psi)$$

$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ R_x(\theta)\ -\ i\dot{\theta}\ R_z(\psi)\ R_x(\theta)J_1\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3]\ R_x(-\theta)\ R_z(-\psi)\ \ //\ ***$

$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ -\ i\dot{\theta}\ R_z(\psi)\ R_x(\theta)J_1R_x(-\theta)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)]\ R_z(-\psi)$

$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ -\ i\dot{\theta}\ R_z(\psi)\ J_1\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)]\ R_z(-\psi)$

$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ R_z(-\psi)\ -\ i\dot{\theta}\ R_z(\psi)\ J_1R_z(-\psi)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi)]$

$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ \{R_z(\psi)\ J_1R_z(-\psi)\ \}\ -\ i\dot{\varphi}\ \{R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi)\ \}]$

$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ M_1\ -\ i\dot{\varphi}\ M_2\ ]$

where

$M_1 \equiv R_z(\psi)\ J_1R_z(-\psi) = R_3(\psi)\ J_1R_3(-\psi)$
$M_2 \equiv R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi) = R_3(\psi)\ R_1(\theta)J_3R_1(-\theta)R_3(-\psi)$

Now we evaluate M_1 and M_2 using the following results which are derived in a Footnote at the end of this section,

$R_1(\theta)\ J_2R_1(-\theta) = \cos\theta\ J_2 + \sin\theta\ J_3$
$R_1(\theta)\ J_3R_1(-\theta) = \cos\theta\ J_3 - \sin\theta\ J_2$

$R_2(\theta)\ J_1R_2(-\theta) = \cos\theta\ J_1 - \sin\theta\ J_3$
$R_2(\theta)\ J_3R_2(-\theta) = \cos\theta\ J_3 + \sin\theta\ J_1$

$R_3(\theta)\ J_1R_3(-\theta) = \cos\theta J_1 + \sin\theta\ J_2$
$R_3(\theta)\ J_2R_3(-\theta) = \cos\theta J_2 - \sin\theta\ J_1$

Therefore

$M_1 = R_3(\psi)\ J_1R_3(-\psi) = \cos\psi J_1 + \sin\psi\ J_2$

$M_2 = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi) = R_3(\psi)\ [\ \cos\theta\ J_3 - \sin\theta\ J_2\ ]\ R_3(-\psi)$

$= \cos\theta\ [\ R_3(\psi)\ J_3\ R_3(-\psi)]\ - \sin\theta\ [R_3(\psi)J_2R_3(-\psi)]$

$= \cos\theta\ J_3\ - \sin\theta\ (\cos\psi J_2 - \sin\psi\ J_1)$

$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3$

Our calculation of matrix A may now be concluded,

$$A' = [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\ M_1 \ - i\dot{\varphi}\, M_2\,]$$

$$= [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\,(\cos\psi J_1 + \sin\psi\, J_2) \ - i\dot{\varphi}\,(\sin\theta\csin\psi J_1 - \sin\theta\cos\psi J_2 + \ \cos\theta\, J_3)\,]$$

$$= [\, -i\dot{\psi}\ J_3 \ - \ i\dot{\theta}\,\cos\psi J_1 \ - \ i\dot{\theta}\,\sin\psi\, J_2 \ - i\dot{\varphi}\,\sin\theta\sin\psi J_1 + \ i\dot{\varphi}\sin\theta\cos\psi J_2 - i\dot{\varphi}\ \cos\theta\, J_3\,]$$

$$= (-i)[\, \dot{\psi}\ J_3 \ + \dot{\theta}\,\cos\psi J_1 \ + \dot{\theta}\,\sin\psi\, J_2 \ + \dot{\varphi}\,\sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2 \ + \dot{\varphi}\ \cos\theta\, J_3\,]$$

$$= (-i)[\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)J_1 \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)J_2 + (\dot{\psi} + \dot{\varphi}\,\cos\theta)J_3\,]$$

$$- [\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)(iJ_1) \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\,\cos\theta)(iJ_3)\,]$$

We now take the ij element of this matrix using the fact (1.5.2) that $(iJ_k)_{ij} = \varepsilon_{kij}$:

$$A'_{ij} = \ - [\, (\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)\varepsilon_{1ij} \ + (\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot{\psi} + \dot{\varphi}\,\cos\theta)\varepsilon_{3ij}\,]$$

Notice from the properties of ε shown in (1.5.3) that ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A' is antisymmetric and thus has only three distinct matrix elements. Then are:

$$A'_{23} = -\,(\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)\varepsilon_{123} \ = \ -\,(\,\dot{\theta}\,\cos\psi + \dot{\varphi}\,\sin\theta\sin\psi)$$

$$A'_{31} = -\,(\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)\varepsilon_{231} = -\,(\dot{\theta}\,\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$A'_{12} = -\ (\dot{\psi} + \dot{\varphi}\,\cos\theta)\,\varepsilon_{312} \ = \ -\ (\dot{\psi} + \dot{\varphi}\,\cos\theta)$$

The work is now nearly done. Recall from ** and ** that

$$(d\mathbf{e}_n/dt)_{S'} \ = \ A'\,\mathbf{e}_n \qquad\qquad // \text{ LHS } = \partial_{S'}(\mathbf{e}_n)$$

$$(d\mathbf{e}_n/dt)_{S'} \ = -\,\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_n$$

Therefore

$$A'\,\mathbf{e}_n = -\,\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_n$$

Since the tensor elements A'_{ij} as shown above are associated with Frame S', we take the Frame S' component of each side of this equation to get

$$[A'\,\mathbf{e}_n]'_i = -\,[\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_n]'_i$$

or

$$A'_{ij}\,(\mathbf{e}_n)'_j \;=\; -\,\varepsilon_{ikj}\,(\omega)'_k(\mathbf{e}_n)'_j$$

or

$$A'_{ij} \;=\; -\,\varepsilon_{ikj}\,(\omega)'_k$$

or

$$A'_{ij} \;=\; \varepsilon_{ijk}\,(\omega)'_k$$

or

$$A'_{12} = \varepsilon_{123}(\omega)'_3 = (\omega)'_3$$
$$A'_{23} = \varepsilon_{231}(\omega)'_1 = (\omega)'_1$$
$$A'_{31} = \varepsilon_{312}(\omega)'_2 = (\omega)'_2$$

Therefore,

$$(\omega)'_1 = A'_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi) = -\dot\varphi\sin\theta\sin\psi - \dot\theta\cos\psi$$
$$(\omega)'_2 = A'_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi) = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$(\omega)'_3 = A'_{12} = -(\dot\psi + \dot\varphi\cos\theta) = -\dot\varphi\cos\theta - \dot\psi$$

[this result agrees with G if you negate all three angles, I could make connection to his ABC]

In our specification at the start of the Euler angle rotation we wrote

$$R(\mathbf{\Phi}) = R_z(\psi)\,R_x(\theta)\,R_z(\varphi) = R_z{}^{act}(\psi)\,R_x{}^{act}(\theta)\,R_z{}^{act}(\varphi)$$

where the three rotations on the right are represented by their *active* rotation matrices.

$$R^{act}{}_i(\alpha) = \exp(-i\alpha J_i) \quad \text{with } J_i \text{ as in (1.5.2)} \qquad \text{eg,} \qquad R^{act}{}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Such a matrix rotates a vector **v** "actively" in the usual counterclockwise right-hand-rule direction to get $\mathbf{v}' = R^{act}{}_z(\theta)\,\mathbf{v}$. As a quick example

$$R^{act}{}_z(\theta)\,\hat{\mathbf{x}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \text{vector in quadrant I of the z=0 plane.}$$

Goldstein (G p ** and GPS p ***) instead use passive rotation matrices, so for them

$$R^{pas}{}_i(\alpha) = \exp(+i\alpha J_i) \qquad \text{with } J_i \text{ as in (1.5.2)} \qquad \text{eg,} \qquad R^{pas}{}_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since Goldstein is then using

$$R(\mathbf{\Phi})_G = R_z^{pas}(\psi)\, R_x^{pas}(\theta)\, R_z^{pas}(\varphi) = R_z^{act}(-\psi)\, R_x^{act}(-\theta)\, R_z^{act}(-\varphi)$$

in order to obtain the Goldstein result we have to negate all the angles ψ,θ and φ (and therefore also their time derivatives) . Doing so in ** above gives

$$(\omega)'_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$(\omega)'_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$(\omega)'_3 = \dot{\varphi}\cos\theta + \dot{\psi} \qquad \text{// Goldstein}$$

which we compare to GPS page 174,

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}. \qquad (4.87)$$

They agree! This was the first time

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Goldstein addresses this same problem but uses passive rotation matrices instead of our active rotation matrices (G p 109, GPS p 153), so we may convert our results to their his by negating all three Euler angles θ,φ,ψ with this result

$$\omega_1 = A_{23} = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = A_{31} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = A_{12} = \dot{\varphi}\cos\theta + \dot{\psi} \qquad \text{// G and GPS} \qquad (*)$$

We compare this result with that shown on G page 134 and GPS page 174,

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\phi}\cos\theta + \dot{\psi}. \qquad (4.87)$$

Paradox: when I compute the components of **ω** in Frame S (using Goldstein's rotation matrices) I end up with what he says are components of **ω** in Frame S' which is his body frame (non-inertial).

[yes, this is a crucial issue. You have to say you are working in Frame S' with the same R and you put a prime on A' as I have done now above and that made things work.]

Here I sort of flail away at this question thru the end of this doc.

Can I find some simple examples to see who is right and who is wrong here?

$\theta = 0$ and $\dot{\theta} = 0$: MY result in this case is

$$\omega_1 = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = 0$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = 0$$
$$\omega_3 = -\dot{\varphi}\cos\theta - \dot{\psi} = -\dot{\varphi} - \dot{\psi} \qquad \text{// us}$$

In this case $R = R_z(\psi)\, R_x(0)\, R_z(\varphi) = R_z(\varphi+\psi)$, pretty simple. I manually compute that

$$A = \begin{pmatrix} 0 & -\dot{\varphi} - \dot{\psi} & 0 \\ \dot{\varphi} + \dot{\psi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then suppose to have

$$(d\mathbf{e}_n/dt)_{S'} = A\,\mathbf{e}_n$$

so that

$$(d\mathbf{e}_1/dt)_{S'} = A\,\mathbf{e}_1 = \dot{\varphi} + \dot{\psi}$$
$$(d\mathbf{e}_2/dt)_{S'} = A\,\mathbf{e}_2 = -\dot{\varphi} - \dot{\psi}$$
$$(d\mathbf{e}_3/dt)_{S'} = A\,\mathbf{e}_3 = 0$$

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Question: Consider:

$$\mathbf{a}(t+dt) = R_z(d\boldsymbol{\varphi})\,\mathbf{a}(t)$$

Which of the following is true:

1. This equation is valid only in Frame S
2. This equation is valid only in Frame S'
3. This equation is valid in both Frame S and Frame S'

Answer: I start off in Frame S which has a clear z axis and I take Frame S components to get

$$[\mathbf{a}(t+dt)]_i = [R_z(d\boldsymbol{\varphi})\,\mathbf{a}(t)]_i = [R_z(d\boldsymbol{\varphi})]_{ij}\,[\mathbf{a}(t)]_j$$

This seems to be completely well defined and I know the matrix $[R_z(d\boldsymbol{\varphi})]_{ij}$.

Here is another way to approach this claim:

$$\mathbf{a}(t+dt) \bullet \mathbf{e}_i = [R_z(d\varphi)\ \mathbf{a}(t)] \bullet \mathbf{e}_i$$

$$[\mathbf{a}(t+dt)]_i = [R_z(d\varphi)\ \mathbf{a}(t)]_i$$

$$= [R_z(d\varphi)^{(S)}]_{ij}\ [\mathbf{a}(t)]_j$$

where I now indicate that the tensor object $[R_z(d\varphi)^{(S)}]_{ij}$ is "in Frame S". Remember that components of vectors have different values in different frames, and the same is true of a rank-2 tensor.

The question this brings up is the following: Is $[R_z(d\varphi)^{(S)}]_{ij}$ a true tensor, or is it just a matrix of constants like π and 2 ? I think it really is a tensor, and you might say

$$R\ [R_z(d\varphi)^{(S)}]\ R^{-1} = [R_z(d\varphi)^{(S')}]$$

or something like that. You want $\mathbf{a}(t+dt) = R_z(d\varphi)\ \mathbf{a}(t)$ to be a true vector equation, so R_z has to then be a true rank-2 tensor.

Now go back to

$$\mathbf{a}(t+dt) = R_z(d\varphi)\ \mathbf{a}(t)$$

In Frame S' components we have

$$[\mathbf{a}(t+dt)]'_i = [R_z(d\varphi)\ \mathbf{a}(t)]'_i = [R_z(d\varphi)]'_{ij}\ [\mathbf{a}(t)]'_j = [R_z(d\varphi)^{(S')}]_{ij}\ [\mathbf{a}(t)]'_j$$

So I think this would be a different matrix! Do it another way

$$\mathbf{a}(t+dt) = R_z(d\varphi)\ \mathbf{a}(t)$$

$$R\mathbf{a}(t+dt) = RR_z(d\varphi)R^{-1}R\ \mathbf{a}(t) \qquad \mathbf{a}' = R\mathbf{a} \quad \text{for general vector}$$

$$\mathbf{a}'(t+dt) = [RR_z(d\varphi)R^{-1}]\mathbf{a}'(t)$$

So my answer I think is this. The equation

$$\mathbf{a}(t+dt) = R_z(d\varphi)\ \mathbf{a}(t)$$

with the usual matrix associated with $R_z(d\varphi)$ is valid only in Frame S.

Similarly, consider

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} = (d\varphi\hat{\mathbf{n}}) \times \mathbf{a}$$

For a specific $\hat{\mathbf{n}}$ specified in Frame S, this equation is valid only in Frame S. I then get to

$(d\mathbf{a}/dt) = \boldsymbol{\omega} \times \mathbf{a}$ where $\boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt.$ $d\boldsymbol{\varphi} = d\varphi\hat{\mathbf{n}}$ (1.6.1)

I write this as

$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a}$ $\boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt.$ $d\boldsymbol{\varphi} = d\varphi\hat{\mathbf{n}}$

Is this valid only in Frame S? I certainly can compute components of $\boldsymbol{\omega} \times \mathbf{a}$ in any frame of reference since it is just a vector. But the equation itself may only be true in Frame S ! But $(d\mathbf{a}/dt)_S$ is also a vector, so why can't we get its components in Frame S' ?

$(d\mathbf{a}/dt)_S = (\dot{\varphi}\hat{\mathbf{n}}) \times \mathbf{a}$

Back up:

$\mathbf{a}(t+dt) = R_{\mathbf{n}}(d\boldsymbol{\varphi})\, \mathbf{a}(t)$ $\mathbf{n}$ = a vector in Frame S

I guess this equation in Frame S' might be

$\mathbf{a}'(t+dt) = R_{\mathbf{n}'}(d\boldsymbol{\varphi})\, \mathbf{a}'(t)$ $\mathbf{n}' = R\mathbf{n}$

and this is definitely a different matrix. So,

$\mathbf{q} = (d\mathbf{a}/dt)_S = (\dot{\varphi}\hat{\mathbf{n}}) \times \mathbf{a}$

$\mathbf{q}' = [(d\mathbf{a}/dt)_S]' = (\dot{\varphi}\hat{\mathbf{n}}') \times \mathbf{a}'$