archive off olds stuff from Appendix G.5 PhL 1.11.15

The Meaning of the Euler Angles in Fig (G.5.5)

In our Section 1 formalism, we discuss the idea of three concatenated transformations near (1.1.41). We can compare the language there to that of our current section's (G.5.6).

$\mathbf{e'''}_n = U^{-1}\mathbf{e''}_n$ $\quad$ $\mathbf{e''}_n = S^{-1}\mathbf{e'}_n$ $\quad$ $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$

$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\,(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}')$ $\quad$ $(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\xi}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}})$ $\quad$ $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{z}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$

$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R^{-1}_{\zeta'}(-\psi)\,(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}')$ $\quad$ $(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R^{-1}_{\xi}(-\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}})$ $\quad$ $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R^{-1}_{z}(-\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$

We speak of U^{-1} and S^{-1} and R^{-1} as "back rotations" and these then accompany Passive View "forward rotations" by U, S and R of a Kinematic Vector ,

$(\mathbf{V})''' = U\mathbf{V}''$ $\quad$ $(\mathbf{V})'' = S\mathbf{V}'$ $\quad$ $(\mathbf{V})' = R\mathbf{V}$

To make this connection, we must make these identifications

$U = R_{\zeta'}(-\psi)$ $\quad$ $S = R_{\xi}(-\theta)$ $\quad$ $R = R_{z}(-\varphi)$

and so it happens that we have negative angles in our transformations of Kinematic vectors

$(\mathbf{V})''' = R_{\zeta'}(-\psi)\,\mathbf{V}''$ $\quad$ $(\mathbf{V})'' = R_{\xi}(-\theta)\mathbf{V}'$ $\quad$ $(\mathbf{V})' = R_{z}(-\varphi)\mathbf{V}$

When concatenated, and when we then replace the triple-prime of our Section 1.3 with the single prime of Goldstein's picture, we get

$$(\mathbf{V})' = R_{\zeta'}(-\psi)R_{\xi}(-\theta)R_{z}(-\varphi)\mathbf{V}$$

for the overall "forward rotation", corresponding to these "back-rotated" basis vectors

$$\mathbf{e'}_n = [R_{\zeta'}(-\psi)R_{\xi}(-\theta)R_{z}(-\varphi)]^{-1}\,\mathbf{e}_n$$

or

How to get Frame S' components of a kinematic vector from its Frame S components

In our Section 1 formalism, we back-rotate the basis vectors according to $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$ and then a kinematic vector **V** has components **(V)'** = R**V** in Frame S' as in (1.3.3). Specifically,

$$(V)'_i = R_{ij}(V)_j \ . \qquad \text{(G.5.17)}$$

This gives the Frame S' components of **V** in terms of the Frame S components of **V**.

In our current context as shown in (G.5.8) we have

$$R^{-1} = R_{\zeta'}(\psi)\, R_{\xi}(\theta) R_z(\varphi) \ . \qquad \textit{(G.5.8)}$$

From Theorem 3 (3) this can be rewritten

$$R^{-1} = R_z(\varphi) R_x(\theta) R_z(\psi) \qquad \text{(G.5.18)}$$

and so our R is given by

$$R = R_z(-\psi) R_x(-\theta)\, R_z(-\varphi) \ . \qquad \text{(G.5.19)}$$

Finally we can write ** as

$$(V)'_i = [R_z(-\psi) R_x(-\theta)\, R_z(-\varphi)]_{ij}(V)_j = R_{ij}(V)_j \qquad \text{(G.5.20)}$$

As Fig (G.5.5) shows, Goldstein has right-hand-rule-back-rotated the basis vectors by negative angles $-\varphi$, $-\theta$ then $-\psi$ (that is, he has "forward-rotated" them by positive angles φ,θ,ψ) and that is why we have negative angles in ***.

Specifically, the matrix R is this:

```
evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
```

$$\begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

(G.5.21)

We shall apply this below to compute $(\omega)'_i$ in terms of $(\omega)_j$.

We now do two brief exercises in computing components of basis vectors in "the other" frame.

Exercise 2: Compute the Frame S' components of the basis vector $\hat{\mathbf{x}}$.

Before doing this calculation we can see the basis vector $\hat{\mathbf{x}}$ in the lower right drawing of Fig (G.5.5). For the small Euler angles used in the figure, $\hat{\mathbf{x}}$ appears to have positive x' and z' components, but a negative y' component. This will be confirmed in the result below.

METHOD A

The equation of interest is (G.5.14d) which says

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = R_{\mathbf{z}}(\varphi)R_{\mathbf{x}}(\theta)R_{\mathbf{z}}(\psi)\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} \qquad \textit{(G.5.14d)}$$

Maple computes the matrix shown (it is R^{-1} which we shall call Q),

```
Q := evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
```

$$Q := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \text{(G.5.22)}$$

Then $\hat{\mathbf{x}} = Q_{11}\,\hat{\mathbf{x}}' + Q_{12}\hat{\mathbf{y}}' + Q_{13}\,\hat{\mathbf{z}}'$, so it is the top row that is involved. Our conclusion is:

$$\begin{aligned} (\hat{\mathbf{x}})_{\mathbf{x}'} &= \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi && \text{positive} \\ (\hat{\mathbf{x}})_{\mathbf{y}'} &= -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi && \text{negative} \\ (\hat{\mathbf{x}})_{\mathbf{z}'} &= \sin\theta\sin\varphi && \text{positive} \end{aligned} \qquad \text{(G.5.23)}$$

and the signs of the components agree with the observations above.

METHOD B

Set (x,y,z) =(1,0,0) in ***.

Exercise 3: Compute the Frame S components of the basis vector $\hat{\mathbf{x}}'$.

Before doing this calculation we can see the basis vector $\hat{\mathbf{x}}'$ in the lower right drawing of Fig (G.5.5). For the small Euler angles used in the figure, $\hat{\mathbf{x}}'$ appears to have positive x,y and z components. This can be confirmed in the result below.

The equation of interest is (G.5.14c) which says

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \qquad \textit{(G.5.14c)}$$

Maple computes the matrix shown (which is R),

```
R := evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
```

$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix} \tag{G.5.24}$$

Then $\hat{\mathbf{x}}' = R_{11}\,\hat{\mathbf{x}} + R_{12}\hat{\mathbf{y}} + R_{13}\,\hat{\mathbf{z}}$, so it is the top row that is involved. Our conclusion is:

$$\begin{aligned} (\hat{\mathbf{x}}')_x &= \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi && \text{positive} \\ (\hat{\mathbf{x}}')_y &= \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi && \text{positive} \\ (\hat{\mathbf{x}}')_z &= \sin\psi\sin\theta && \text{positive} \end{aligned} \tag{G.5.25}$$

and the signs of the components agree with the observations above.