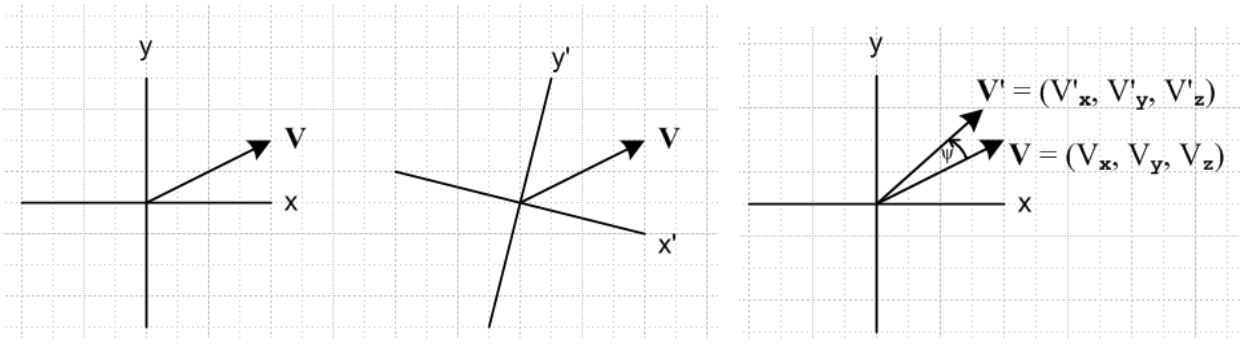
Interpreting Goldstein's Euler Angle Picture PhL 2.22.17

Note added 2.24.17. Here I dive into the active/passive business for the first time in this fortnight. I think this is well written up now in Section 1.3 and the pictures were not needed though I could add them at some point. Other details below are cleared up in App G.5. No need to read this again ever.

First, let's go find once again and review my active/passive doc with the pictures. I did this maybe a week ago. Got it, just search on passive. It is in the curvilinear systems folder. My usual picture is this



a Frame S b Frame S' c Frame S active

The axes in b are right-hand-rule *back-rotated* by some angle positive α around 15 degrees.
The coordinates (components) of vector **V** are different in Frame S versus Frame S' as you can see.
In the **passive** view, I can get the Frame S' coordinates from the Frame S ones by this rule: $\mathbf{V}' = R\mathbf{V}$. I would write in my frames doc notation,

$$(V)'_i = R_{ij}(\alpha)\,(V)_i \qquad (1) \qquad \mathbf{V}' = R_z(\alpha)\mathbf{V} \quad \text{and } \mathbf{e}'_i = R_z(-\alpha)\mathbf{e}_i \text{ in Frame S}$$

In the **active** view all within Frame S I would again write $\mathbf{V}' = R\mathbf{V}$ but I would express this as

$$(V')_i = R_{ij}(\alpha)\,(V)_i \qquad (2) \qquad R = R_z(\alpha)$$

In frames doc, it is item (1) that you normally want to do: express Frame S' components in terms of Frame S components, and maybe vice versa. You think "passive".

Now suppose we set $\alpha = -\beta$ with $\beta > 0$. Then write the above equations:

$$(V)'_i = R_{ij}(-\beta)\,(V)_j \qquad (1) \qquad \mathbf{V}' = R_z(-\beta)\mathbf{V} \quad \text{and } \mathbf{e}'_i = R_z(\beta)\mathbf{e}_i$$

In the active view all within Frame S I would again write $\mathbf{V}' = R\mathbf{V}$ but I would express this as

$$(V')_i = R_{ij}(-\beta)\,(V)_j \qquad (2) \qquad R = R_z(-\beta)$$

Now we front-rotate the basis vectors, and to get the V' components we have to back-rotate V.

I don't think this is rocket science. I prefer to think the first way, but the second is OK too.

What is Goldstein doing? He forward-rotates the $\mathbf{e}_n$ basis vectors to get the $\mathbf{e}'_n$ ones.

$$\mathbf{e}'_n = \text{"}R_z(\beta)\text{"}\mathbf{e}_i \qquad \text{"}R_z(\beta)\text{"} = R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)$$

I show this as

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)\, (\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$$

and in particular

$$\begin{aligned}
\hat{\mathbf{x}}' &= R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)\, \hat{\mathbf{x}} \equiv Q\hat{\mathbf{x}} \qquad Q = \text{"}R_z(\beta)\text{"} \qquad \mathbf{e}'_i = \text{"}R_z(\beta)\text{"}\mathbf{e}_i \\
\hat{\mathbf{y}}' &= R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)\, \hat{\mathbf{y}} \equiv Q\hat{\mathbf{y}} \\
\hat{\mathbf{z}}' &= R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)\, \hat{\mathbf{z}} \equiv Q\hat{\mathbf{z}}
\end{aligned} \qquad \text{(G.5.4)}$$

These equations make complete sense in terms of his picture, no problemo.

Now, if I know the components of $\mathbf{V}$ in Frame S and I want to know them in Frame S', I do this

$$(V)'_i = R_{ij}(-\beta)\,(V)_j = [R_z(\beta)]^{-1}{}_{ij} V_j$$

So then in Goldstein's case I would have

$$(V)'_i = Q^{-1}{}_{ij} V_j$$

So $Q = R_{\zeta'}(\psi)\, R_\xi(\theta) R_z(\varphi)$

$$Q^{-1} = R_z(-\varphi) R_\xi(-\theta) R_{\zeta'}(-\psi)$$

But I know that

$$R_{\zeta'}(\psi)\, R_\xi(\theta)\, R_z(\varphi) = R_z(\varphi) R_x(\theta) R_z(\psi) = Q$$

and therefore Q^{-1} can also be written $R_z(-\psi)R_x(-\theta)R_z(-\varphi)$.

So to get Frame S' components I do this

$$(V)'_i = [R_z(-\psi)R_x(-\theta)R_z(-\varphi)]_{ij}(V)_j$$

That is my main point so far! This thing is my traditional "R" and that is what I show in

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi)\ . \qquad \text{(G.5.25)}$$

The rule for transformation of a normal vector is given by (1.3.2) as

$$\boldsymbol{\omega}' = R\boldsymbol{\omega} \qquad \text{(G.5.26)}$$

I need to get this stated somewhere in G.5 !

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