α

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Interpretation of the Goldstein's Triple Concatenation

In our Section 1 formalism, we discuss the idea of three concatenated transformations near (1.1.41). We can compare the equations there to those of (G.5.6),

$\mathbf{e'''}_n = U^{-1}\mathbf{e''}_n$ $\qquad$ $\mathbf{e''}_n = S^{-1}\mathbf{e'}_n$ $\qquad$ $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$

$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\,(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}')$ $\qquad$ $(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_{\xi}(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}})$ $\qquad$ $(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_{z}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$.

It follows that the three "back-rotations" are

$U^{-1} = R_{\zeta'}(\psi)$ $\qquad$ $S^{-1} = R_{\xi}(\theta)$ $\qquad$ $R^{-1} = R_{z}(\varphi)$

In Fig (1.3.4) we show a back-rotation example where $R^{-1} = R_z(-\alpha)$ = "back-rotation" and we drew the figure for some small $\alpha > 0$. In Goldstein's back rotations, the role of α is played by -ψ ,-θ and -φ. Figure (G.5.5) shows that the basis vector "back rotations" are really a forward rotations by ψ, θ and φ.

Doing the concatenation one gets,

$$\mathbf{e'''}_n = U^{-1}S^{-1}R^{-1}\mathbf{e}_n = R_{\zeta'}(\psi)R_{\xi}(\theta)\,R_{z}(\varphi)\mathbf{e}_n$$

or

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_{\zeta'}(\psi)\,R_{\xi}(\theta)\,R_{z}(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$$

in agreement with (G.5.7).

Transformation of Kinematic Vectors

The corresponding Passive View transformations of Kinematic Vectors for the three concatenations are,

$(\mathbf{V})''' = U\mathbf{V}''$ $\qquad$ $(\mathbf{V})'' = S\mathbf{V}'$ $\qquad$ $(\mathbf{V})' = R\mathbf{V}$.

Doing the concatenation and then changing the Section 1 triple-prime to Goldstein's single-prime, we get

$$(\mathbf{V})' = USR\,\mathbf{V}$$

$$= R_{\zeta'}(-\psi)R_{\xi}(-\theta)R_{z}(-\varphi)\,\mathbf{V} \quad .$$

If we now redefine R to be our notation for Goldstein's *overall* transformation, we then have

$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R^{-1}(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ $\qquad$ $R^{-1} = R_{\zeta'}(\psi)\,R_{\xi}(\theta)\,R_{z}(\varphi)$

$$= R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{z}(\varphi)\, (\hat{x},\hat{y},\hat{z})$$

$$(\mathbf{V})' = R\,\mathbf{V} \qquad\qquad R = [R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{z}(\varphi)]^{-1}$$
$$= [R_{\zeta'}(\psi)\, R_{\xi}(\theta)\, R_{z}(\varphi)]^{-1}\,\mathbf{V}$$

We can rewrite the above lines making use of Theorem 3 item (3) to get

$$(\hat{x}',\hat{y}',\hat{z}') = R^{-1}(\hat{x},\hat{y},\hat{z}) \qquad\qquad R^{-1} = R_{z}(\varphi)R_{x}(\theta)R_{z}(\psi)$$
$$= R_{z}(\varphi)R_{x}(\theta)R_{z}(\psi)\,(\hat{x},\hat{y},\hat{z})$$

$$(\mathbf{V})' = R\,\mathbf{V} \qquad\qquad R = [\,R_{z}(\varphi)R_{x}(\theta)R_{z}(\psi)]^{-1}$$
$$= [\,R_{z}(\varphi)R_{x}(\theta)R_{z}(\psi)]^{-1}\,\mathbf{V}$$

$$= R_{z}(-\psi)R_{x}(-\theta)R_{z}(-\varphi)\,\mathbf{V}$$

This last item is the rule for finding the Frame S' components $(V)'_i$ of a vector **V** in terms of the Frame S components V_i .

Figure (G.5.5) shows the Kinematic Vector **r** which is the position of some point in Frame S .

Exercise 1:

Compute the Frame S' components of the vector **r** which in Frame S has components (x,y,z) .

$$(\mathbf{r})' = R\,\mathbf{r} = R_{z}(-\psi)R_{x}(-\theta)R_{z}(-\varphi)\,\mathbf{r}$$

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R := evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi));
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$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

Therefore

$$(\mathbf{r})' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and so

$$x' = (\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)\, x + (\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)\, y + \sin\psi\sin\theta\, z$$

$$y' = (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)\, x + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)\, y + \cos\psi\sin\theta\, z$$

$$z' = \sin\theta\sin\varphi\, x - \sin\theta\cos\varphi\, y + \cos\theta\, z \quad .$$

Application: What are the Frame S' components of $\hat{\mathbf{x}}$?

Apply the previous equation to (x,y,z) = (1,0,0):

$\hat{\mathbf{x}}_{x'} = \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi \quad = (\hat{\mathbf{x}})'_1$

$\hat{\mathbf{x}}_{y'} = -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi \quad = (\hat{\mathbf{x}})'_2$

$\hat{\mathbf{x}}_{z'} = \sin\theta\sin\varphi \quad = (\hat{\mathbf{x}})'_3$

We now reverse the above Exercise.

Exercise 2:

Compute the Frame S components of the vector **r** which in Frame S' has components (x',y',z') .

$$\mathbf{r} = R^{-1}(\mathbf{r})' = R_z(\varphi)R_x(\theta)R_z(\psi)(\mathbf{r})'$$

The matrix $R^{-1} = R^T$ is just the transpose of the matrix shown above; Maple computes it anyway :

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RINV := evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
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$$RINV := \begin{bmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\phi)\sin(\psi) - \cos(\theta)\sin(\phi)\cos(\psi) & \sin(\theta)\sin(\phi) \\ \sin(\phi)\cos(\psi) + \cos(\theta)\cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\theta)\cos(\phi)\cos(\psi) & -\sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{bmatrix}$$

Therefore

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R^{-1}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

and so

$x = (\cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi)\, x' + (-\cos\varphi\sin\psi - \cos\theta\sin\varphi\cos\psi)\, y' + \sin\theta\sin\varphi\, z'$

$y = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\, x' + (-\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi)\, y' - \sin\theta\cos\varphi\, z'$

$z = \sin\theta\sin\psi\, x' + \sin\theta\cos\psi\, y' + \cos\theta\, z'$

Application: What are the Frame S components of $\hat{\mathbf{x}}'$?

Apply the previous equation to (x',y',z') = (1,0,0):

$\hat{\mathbf{x}}'_x = \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi \quad = (\hat{\mathbf{x}}')_1$

$\hat{\mathbf{x}}'_y = \sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi \quad = (\hat{\mathbf{x}}')_2$

$\hat{\mathbf{x}}'_z = \sin\theta\sin\psi \quad = (\hat{\mathbf{x}}')_3$